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The Stability of Order $m = 4$ Rational Integrator for the Solution of Ordinary Differential Equation

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Abstract

In this article, investigation is carried out on a rational interpolation scheme of order $m = 4$, through two major processes of converting the resulting rational function to a complex outlook, and then, transforming the complex function to a polar form, from which the stability region of the method is constructed in the form of Jordan Curve. The regions of stability and instability as well as the encroachment interval of the scheme is determined with the use of Maple-18 and Matlab packages.

1. Introduction

Numerical methods for evaluating systems of Ordinary Differential Equations (ODEs) have been attracting much attention because they proffer the solutions of problems arising from the mathematical formulation of physical situations such as those in chemical kinetics, population, economic, political and social models. Numerical solution of ordinary differential equations can be obtained using rational integrators, such as linear multistep, Runge-Kutta and exponential methods and many others.

The object of our study is the stability function of a general rational integrator reported in [1] and whose underlying interpolant is a rational function where are

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polynomial functions $P_L(x)\ Q_M(x)^{-1}$ where $P_L(x)$ and $Q_M(x)$ are polynomial functions of degrees L and M respectively. The stability function of any integrator is what is normally used in determining the Region of absolute stability of such an integrator. A common yardstick which is used in the determination of the Region of Absolute Stability (RAS) is the unit ball in $\mathbb{R}^n$. The stability function $S(\overline{h})$ is defined as the ratio $\frac{y_{n+1}}{y_n}$, where $\frac{y_{n+1}}{y_n} = S\left(\overline{h}\right) = R_{L,M}(\overline{h})$ and $\left|S\left(\overline{h}\right)\right| < 1 \Leftrightarrow |< 1|$. The overall aim of this study is to determine the stability polynomial, and establish the region of absolute stability of a rational integrator of order $m = 4$. Our computational experience as exemplified by the works of [2] & [3], along with the research work given by [4], [5] & [6], all give credence to the need for rational integrators. To go about this there is need to first establish the basic matrix of the method, which will give an investigative advantage to obtain result easily. The Crammer's Rule is implemented to enable the determination of the stability function of the method. Basically, there are two ways of approaching the expansion of the stability function; these will be through direct algebraic method using $F(u,v)$ and polar method of $F(R,\emptyset)$. According to [8] & [9], by employing the binomial expansion on $F(u,v)$, we established the complex integrating function and reduce the expansion to polar form at which point we introduce $F(R,\emptyset)$, where R represents the polynomial of the complex integrator function, while argument ($\emptyset$) is the angle of rotation, about which the roots of R, are determined. After the expansion, the integrator is subjected to a sequence of solvability analysis to establish the stability function for the rational integrator for the case $m = 4$. The roots of the complex polynomial are determined to enable the plotting of the Jordan curve, which will show both the Region of Absolute stability (RAS) and the Region of instability (RIS).

2. Stability Analysis of the Method

The validity of a method can be verified by analyzing the stability property of a rational interpolation scheme. The rational interpolant $U : \mathbb{R} \to \mathbb{R}$ is defined by

$$y_{n+1} = r + \frac{\sum_{i=0}^{m-1} p_i x_{n+1}^i}{1 + \sum_{i=1}^{m} q_i x_{n+1}^i} \tag{1}$$

at $m = 4$, we have

$$y_{n+1} = r + \frac{\sum_{i=0}^{3} p_i x_{n+1}^i}{1 + \sum_{i=1}^{4} q_i x_{n+1}^i} \tag{2}$$

where p_i, and q_i are called the integrator parameters, and r is an inhomogeneous constant. By obtaining the various integrator parameters and the inhomogeneous constant, our general integrator becomes:

$$(3)\quad y_{n+1} = \frac{\sum_{i=0}^{4} \frac{h^i y_n^{(i)}}{i!} + A\sum_{i=0}^{3} \frac{h^i y_n^{(i)}}{i!} + B\sum_{i=0}^{2} \frac{h^i y_n^{(i)}}{i!} + C\sum_{i=0}^{1} \frac{h^i y_n^{(i)}}{i!} + Dy_n}{1 + A + B + C + D}$$

Our new integrator is said to be Absolutely Stable if $\left|\zeta(\overline{h})\right| \leq 1$, From (3) we obtained the stability function of our scheme as:

$$(4)\qquad \zeta\left(\overline{h}\right) = \frac{1680 + 840\overline{h} + 180\overline{h}^2 + 20\overline{h}^3 + \overline{h}^4}{1680 - 840\overline{h} + 180\overline{h}^2 - 20\overline{h}^3 + \overline{h}^4} = \frac{\phi(\overline{h})}{\Psi(\overline{h})}$$

It implies $\left|\zeta(\overline{h})\right| \leq 1 \iff \left\|\frac{\phi(\overline{h})}{\Psi(\overline{h})}\right\| \leq 1$

To anaylse the formula, we set $\overline{h} = u + iv$ where $i^2 = -1$,

Hence $\left\|\phi(\overline{h})\right\| \leq \left\|\Psi(\overline{h})\right\| \therefore \left\|\phi(\overline{h})\right\| - \left\|\Psi(\overline{h})\right\| \leq 0$

where $\phi\left(\overline{h}\right) = 1680 + 840\overline{h} + 180\overline{h}^2 + 20\overline{h}^3 + \overline{h}^4$ and

$$\Psi\left(\overline{h}\right) = 1680 - 840\overline{h} + 180\overline{h}^2 - 20\overline{h}^3 + \overline{h}^4$$

Let $\phi\left(\mathrm{u,v}\right) = A\left(u,\ v\right) + iB\left(u,\ v\right)$ and set $\overline{h} = u + iv$ where $i^2 = -1\Psi\left(\mathrm{u,v}\right) = C\left(u,\ v\right) + iD\left(u,\ v\right)$

$$\Rightarrow\ \phi\left(\mathrm{u,v}\right) = 1680 + 840(u+iv) + 180(u+iv)^2 + 20(u+iv)^3 + (u+iv)^4$$

$$\phi\left(\mathrm{u,v}\right) = 1680 + 840\left(u+iv\right) + 180(u^2 + 2uvi + v^2i^2) + 20(u^3 + 3u^2vi + 3uv^2i^2 + v^3i^3) + \left(u^4 + 4u^3vi + 6u^2v^2i^2 + 4uv^3i^3 + v^4i^4\right)$$

On expansion, we get

$$A(u,v) = u^4 - 6u^2v^2 + v^4 + 20u^3 - 60uv^2 + 180u^2 - 180v^2 + 840u + 1680$$

$$B(u,v) = 4u^3v - 4uv^3 + 60u^2v - 20v^3 + 360uv + 840v$$

Hence,

$$A^2 = u^8 - 12\,u^6v^2 + 38u^4v^4 - 12u^2v^6 + v^8 + 40u^7 - 360u^5v^2 + 760u^3v^4 - 120uv^6 + 760u^6 - 4920u^4v^2 + 6120u^2v^4 - 360v^6 + 8880u^5 - 38880u^3v^2 + 23280uv^4 + 69360u^4 - 185760u^2v^2 + 35760v^4 + 539600u^3 - 504000uv^2 + 1310400u^2 - 604800v^2 + 2822400u + 2822400$$

$$B^2 = 16u^6v^2 - 32u^4v^4 + 16u^2v^6 + 480u^5v^2 - 640u^3v^4 + 160uv^6 + 6480u^4v^2 - 5280u^2v^4 + 400v^6 + 49920u^3v^2 - 21120uv^4 + 230400u^2v^2 - 33600v^4 + 604800uv^2 + 705600v^2$$

$$A^2{+}B^2 = u^8{+}4u^6v^2{+}4u^2v^6{+}v^8{+}40u^7{+}120u^5v^2{+}40uv^6{+}760u^6{+}1560u^4v^2{+}840u^2v^4$$
$$+40v^6+8880u^5+11040u^3v^2+2160uv^4+69360u^4+4460u^2v^2+2160v^4+369600u^3$$
$$+100800uv^2+1310400u^4+100800v^2+2822400u+2822400$$
$$\Psi\left(\overline{h}\right)=1680-840\overline{h}+180\overline{h}^2-20\overline{h}^3+\overline{h}^4$$

Set $\overline{h}=u+iv$ where $i^2=-1$.
Let $\Psi\left(\mathrm{u,v}\right)=C\left(u,\ v\right)+iD\left(u,\ v\right)$

$$\Rightarrow\Psi\left(\mathrm{u,v}\right)=1680-840\left(u+iv\right)+180(u+iv)^2-20(u+iv)^3+(u+iv)^4$$
$$\Psi\left(\mathrm{u,v}\right)=1680{-}840\left(u+iv\right){+}180\left(u^2+2uvi+v^2i^2\right){-}20(u^3{+}3u^2vi{+}3uv^2i^2{+}v^3i^3)$$
$$+\left(u^4+4u^3vi+6u^2v^2i^2+4uv^3i^3+v^4i^4\right)$$

On expansion, we get

$$\|\Psi(u,v)\|=C^2\left(u,v\right)+D^2(u,v)$$
$$C(u,v)=u^4-6u^2v^2+v^4-20u^3+60uv^2+180u^2-180v^2-840u+1680$$
$$C^2=u^8{-}12\,u^6v^2{+}38u^4v^4{-}12u^2v^6{+}v^8{-}40u^7{+}360u^5v^2{-}760u^3v^4{+}120uv^6{+}760u^6$$

$$-4920u^4v^2+6120u^2v^4-360v^6-8880u^5+38880u^3v^2-23280uv^4+69360u^4-$$

$$185760u^2v^2+35760v^4-369600u^3+504000uv^2+1310400u^2-604800v^2$$

$$-2822400u+2822400$$

$$D(u,v)=4u^3v-4uv^3-60u^2v+20v^3+360uv-840v$$

$$D^2=16u^6v^2{-}32u^4v^4{+}16u^2v^6{-}480u^5v^2{-}640u^3v^4{-}160uv^6{+}6480u^4v^2{-}5280u^2v^4$$
$$+400v^6-49920u^3v^2+21120uv^4+230400u^2v^2-33600v^4-604800uv^2+705600v^2$$
$$C^2{+}D^2=u^8{+}4u^6v^2{+}6u^4v^4{+}4u^2v^6{+}v^8{-}40u^7{-}120u^5v^2{-}120u^3v^4{-}40uv^6{+}760u^6{+}$$
$$1560u^4v^2+840u^2v^4+40v^6-8880u^5-11040u^3v^2-2160uv^4+69360u^4+4460u^2v^2+$$
$$2160v^4-369600u^3-100800uv^2+1310400u^4+100800v^2-2822400u+2822400$$
$$\left|\left|\zeta\left(\overline{h}\right)\right|\right|\leq 1\quad\Longleftrightarrow\|\phi(u,v)\|-\|\Psi(u,v)\|\leq 0$$
$$\Longleftrightarrow A\left(u,v\right)^2+B\left(u,v\right)^2-C\left(u,v\right)^2-D(u,v)^2\leq 0$$

$$A^2{+}B^2{-}\left(C^2 + D^2\right) = 80u^7{+}240u^5v^2{+}240u^3v^4{+}80uv^6{+}17760u^6{+}22080u^3v^2{+}4320uv^4$$
$$+739200u^3 + 201600uv^2 + 5644800u \leq 0.$$

Which shows that the integrator is $A-$Stable. If and only if $u < 0$.

To obtain our Region of Absolute Stability (RAS) we shall employ the polar form where $u = Rcos\theta$ and $v = Rsin\theta$

$$80R^7cos(\theta)^7{+}240R^7cos(\theta)^7sin(\theta)^2{+}240R^7cos(\theta)^7sin(\theta)^4{+}80R^7\cos(\theta)sin(\theta)^6{+}17760R^5cos(\theta)^5$$
$$+22080R^5cos(\theta)^3sin(\theta)^2{+}4320R^5\cos(\theta)\ sin(\theta)^4{+}739200R^3cos(\theta)^3{+}201600R^3\cos(\theta)\ sin(\theta)^2$$
$$+5644800R\cos(\theta)$$
$$\Rightarrow 80R^7cos(\theta)^7{+}240R^7cos(\theta)^7cos\theta(1 - cos^2\theta){+}240R^7cos(\theta)^7cos\theta(1 - cos^2\theta)^2{+}80R^7\cos(\theta)$$
$$cos\theta(1 - cos^2\theta)^3{+}17760R^5cos(\theta)^5{+}22080R^5cos(\theta)^3cos\theta(1 - cos^2\theta){+}4320R^5cos\theta(1 - cos^2\theta)^2$$
$$+739200R^3cos(\theta)^3 + 201600R^3\cos(\theta)\ cos\theta(1 - cos^2\theta)^2 + 5644800R\cos(\theta) = 0$$
$$\Rightarrow 80R^7cos(\theta)^7{-}20800R^5cos(\theta)^5{+}13440R^5cos(\theta)^3{+}22080R^5\cos(\theta)^7{+}4320R^5cos\theta$$
$$+3360000R^3cos(\theta)^3 + 201600R^3cos\theta + 2016000R^3cos(\theta)^5 + 5644800R\cos(\theta)$$

3. The Numerical Analysis of the Jordan Curve $m = 4$

The Jordan curve is the equation governing the region of absolute stability and the region of instability.
We take various degrees of θ from 0^0 to 360^0 and get a corresponding values of R.

Numerical Analysis of the Jordan Curve at $m = 4$.

θ^0	R	Approximation for Curve Sketch	θ^0	R	Approximation for Curve Sketch
0	13.0431937230128	13.0	195	-11.5878310205734	-11.6
15	11.5878310205734	11.6	210	-8.29739219069527	-8.3
30	8.29739219069527	8.3	225	-7.14298248626244	-7.5
45	7.14298248626244	7.1	240	-5.90258138115783	-5.9
60	5.90258138115783	5.9	255	-5.26838235030388	-5.3
75	5.26838235030388	5.3	270	0.000000000000000	0
90	0.00000000000000	0	285	5.26838235030388	5.3
105	-5.26838235030388	-5.3	300	5.90258138115783	5.9
120	-5.90258138115783	-5.9	315	7.14298248626244	7.1
135	-7.14298248626244	-7.1	330	8.29739219069527	8.3
150	-8.29739219069527	-8.3	345	11.5878310205734	11.6
165	-11.5878310205734	-11.6	360	13.0431937230128	13.0
180	-13.0431937230128	-13.0			

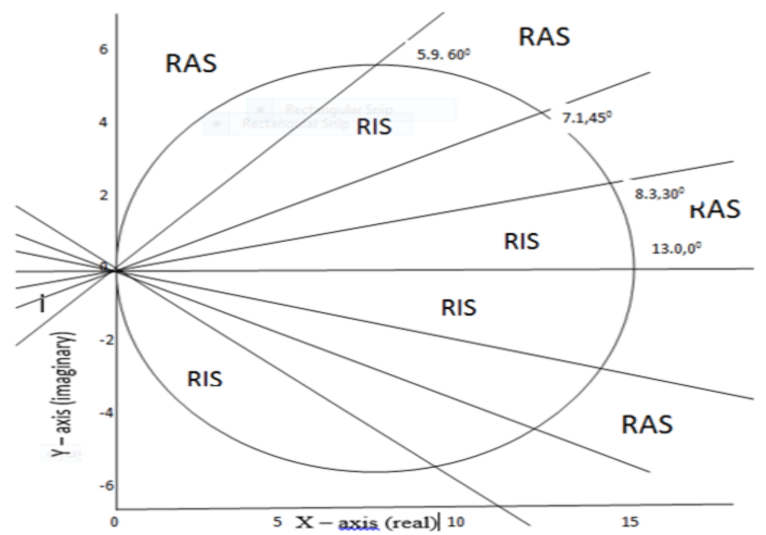

Figure 1: The Jordan curve of imaginary axis against real axis

Remark: The Jordan curve above shows that the integrator is A-stable and so the Region of Absolute Stability of the integrator is the entire left-half of the complex plane and the space outside the diagram.

4. Comparisms of the numerical experiments

We shall be concerned with the application of our new integrator in solving a number of problems and determine the convergence, consistency and computational stability structure of the method.
We consider the work of Esekhaigbe in [7].

$$y' = -y; \qquad y(0) = 0.5; \quad 0 \le x \le 1, \qquad h = 0.1$$

The exact solution is $y(x_n) = \frac{1}{e^{x_n}}$.

Numerical Integration

XN	TSOL	YN(4^{th}-stage)	$m = 4$
0.1D+00	0.90D+00	0.9048D+00	0.904e+00
0.2D+00	0.81D+00	0.8187D+00	0.818e+00
0.3D+00	0.74D+00	0.7408D+00	0.740e+00
0.4D+00	0.67D+00	0.6703D+00	0.670e+00
0.5D+00	0.60D+00	0.6065D+00	0.606e+00
0.6D+00	0.54D+00	0.5488D+00	0.548e+00
0.7D+00	0.49D+00	0.4965D+00	0.496e+00
0.8D+00	0.44D+00	0.4493D+00	0.449e+00
0.9D+00	0.40D+00	0.4065D+00	0.406e+00
1D+00	0.36D+00	0.3678D+00	0.367e+00

Error in Numerical Integration

XN	TSOL	YN(4^{th}-stage)	$m = 4$
0.1D+00	0.90D+00	-0.814D-07	1.864e-08
0.2D+00	0.81D+00	-0.143D-06	3.374e-08
0.3D+00	0.74D+00	-0.209D-06	4.579e-08
0.4D+00	0.67D+00	-0.240D-06	5.525e-08
0.5D+00	0.60D+00	-0.271D-06	6.249e-08
0.6D+00	0.54D+00	-0.294D-06	6.785e-08
0.7D+00	0.49D+00	-0.318D-06	7.163e-08
0.8D+00	0.44D+00	-0.323D-06	7.407e-08
0.9D+00	0.40D+00	-0.338D-06	7.540e-08
1D+00	0.36D+00	-0.334D-06	7.581e-08

The table above shows the performance of our new numerical integrator of order m = 4. Our new integrators compete favorably well with the explicit fourth-stage fourth-order Runge-Kutta methods of [7] with a very high rate of convergence at $m = 4$.

Conclusion. After a successful expansion of the rational integrator, we represented the stability function in the form of polar curve using MATLAB and MAPLE-18 packages. Thus, the following findings are visible:
(i) It shows the region of absolute stability; (ii) The exterior (outside) of the curve represent the region of absolute stability (RAS) of the integrator; (iii) The Region of instability (RIS) is on the positive side of the complex plane, which is within the stability (polar) curve; (iv) Our new integrators compete favorably well with the explicit fourth-stage fourth-order Runge-Kutta methods of [7] with a very high rate of convergence at $m = 4$; and (v) The rational integrator τ is within the interval ± 13.0 where τ is the encroachment point. It can be concluded that the Region of Absolute Stability (RAS) of the rational Interpolation method lies entirely on the Left-half of the complex plane. At the same time, the region of instability (RIS), lies within the Jordan curve as seen from the figure 1 above. This work revealed that, the Region of Absolute Stability (RAS) for $m = 4$ is a superset of the entire left-half of the complex plane. Furthermore, the rational interpolation scheme is not only A-stable, but also L- stable.

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