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# CONSTRUCTION OF POLYNOMIAL BASIS AND ITS APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The study identifies the versatility of basis functions in expansionary method by constructing basis functions of finite order, which satisfy some smoothness and differentiability conditions. Effort was intensified towards solving empirical problems via the finite element method.

## 1. Introduction

The inappropriateness, theoretically of the usage of $C^0$ elements in solving problems of mathematical physics, was first identified by Zienkiewicz [1]. Such elements were observed not differentiable at certain inter-element boundary points in the domain over which the problems are defined. It was however discovered by Bamigbola [3] that accurate results can be obtained with $C^0$ elements using the identified basis functions. We note that a basis function is an element of a particular basis for a function space. In fact, every continuous function in a function space can be represented as a linear combination of a basis function. It helps in giving mathematical description of a curve or any data distributed over space, time and any other type of continuum.

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## 2. Methodology and Results

**BASIS FUNCTION**

The set $\phi_n(x)$ of some given functions usually piecewise polynomials defined over a given domain $D$ is called basis functions when used for an expansion of the form

$$p(x) = \sum_{i=1}^{N} a_i \phi_i(x) \tag{1}$$

where $a_i$ , $i = 0, 1, \ldots, N$ are parameters of the approximation method. It is pertinent to note that the choice of the set of basis functions is essential to the expansion method for various reasons; among which is the facilitation of computational ease and accuracy of the resulting solution in [2], [3] and [4] polynomial basis functions up to cubic power were constructed with the zeros of the chebyshev polynomials of the first kind and applied using the finite element method to solve two points boundary value problems.

In [7] the zeroes of the legendre polynomial was employed to obtain same. It was in [8 ] that a comparative study of the computational efficiency of the above mentioned construction with some other polynomial basis functions were considered with a view to identifying the optimal choice among them which could be used as a better approximating tool in the expansion method. The result of the experiment is being generalized in this present work.

We reviewed the derivation of basis function of the nth Order and use MATLAB to obtain the inverse of stiffness matrix at each step of the construction. With the use of Garlarkin formulation in [3 ] we obtain solution to problems capped in differential equations.

**DERIVATION OF BASIS FUNCTIONS**

We denote by $C_r^n(\alpha)$ the space of polynomial of finite order defined over a closed interval $\infty$ which are n-times continuously differentiable in the open interval $\delta$ .
We note that n and r are integral values in which $n \geq 0$ , $r \geq 1$
We select mesh points $x_i$ in the real interval [a,b] as $x_i = x + ih$ $i = 1, 2, 3, \ldots, m$ where $h = \frac{(b-a)}{m}$
The appropriate form of a function $p(x)$ in the sub interval $x_i,\ x_{i+1}$ in line with [2 ] is

$$p(x) = \sum_{k=1}^{(n+1)(r+1)} a_i \left(\frac{x}{h}\right)^{k-1},\ x \in [a, b] \tag{2}$$

The $m^{th}$ derivatives of $p(x)$ is given as

$$p^m(x) = \left(\frac{1}{h}\right)^{(1-m)} \sum_{k=1}^{n+1,r+1} \frac{(k-1)!}{((k-1-m)!)} a_i \left(\frac{x}{h}\right)^{k-1-m}, \; x \in [0,h] \tag{3}$$

The process of deriving the set of basis functions involves the interpolation of the expression in (3) at the nodal points$x_k$, $(k = 1, 2, 3, \ldots, r-1)$ and solving for the parameters $a_i$ resulting there from. Adopting the usual notation $p_k^m = p^m(x_k)$ , and interpolating (5) at the nodes $x_k$ we have the matrix equation

$$p = Aa \tag{4}$$

**DERIVATION OF GENERALIZED POLYNOMIAL BASIS FUNCTIONS**

Consider a linear polynomial function

$$p(x) = a_1 + a_2\frac{x}{h}, \; C_1^0 : [0,h] \tag{5}$$

Interpolating (5) at the nodal points 0 and h, we have
$P_1 = P(0) = a_1$
$P_2 = P(h) = a_1 + a_2$
Which implies

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{6}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Then

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Thus, the basis functions are:

$$\phi_i = \begin{pmatrix} 1 & x/h \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \; i = 1, 2 \tag{7}$$

$$\phi_1 = 1 - \frac{x}{h}, \Phi_2 = \frac{x}{h}$$

Consider a quadratic polynomial function

$$p(x) = a_1 + a_2\frac{x}{h} + a_3\frac{x}{h}^2 \; C_2^0 : [0, \frac{x}{h}, h] \tag{8}$$

Interpolating at the nodal points$(0, \frac{h}{2}, h)$ we obtain
$P_1 = P(0) = a_1$

$P_2 = P(\frac{h}{2})) = a_1 + \frac{1}{2}a_2 + \frac{1}{4}a_3$
$P_3 = P(h) = a_1 + a_2 + a_3$

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{pmatrix}$$

The basis functions are:

$$\phi_i = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{pmatrix}, \ i = 1,2,3 \tag{9}$$

i.e
$\phi_1 = 1 - 3\frac{x}{h} + 2(\frac{x}{h})^2$
$\phi_2 = 4\frac{x}{h} - 4(\frac{x}{h})^2$
$\phi_3 = -\frac{x}{h} + 2(\frac{x}{h})^2$
For a cubic polynomial function

$$P(x) = a_1 + a_2(\frac{x}{h}) + a_3(\frac{x}{h})^2 + a_4(\frac{x}{h})^3 \ C_2^0 : [0, \frac{h}{5}, \frac{2h}{3}, h] \tag{10}$$

$$P_1 = P(0) = a_1$$

$$P_2 = P(\frac{h}{3}) = a_1 + \frac{2}{3}a_2 + \frac{1}{9}a_3 + \frac{1}{27}a_4 \tag{11}$$

$P_3 = P(\frac{h}{3}) = a_1 + \frac{2}{3}a_2 + \frac{4}{9}a_3 + \frac{8}{27}a_4$
$P_4 = P(h) = a_1 + a_2 + a_3 + a_4$
$P_i = A * a_i \quad i = 1,2,3,4$

$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 \\ 1 & 2/3 & 4/9 & 8/27 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -11/2 & 9 & -9/2 & 1 \\ 9 & -45/2 & 18 & -9/2 \\ -9/2 & 27/2 & -27/2 & 9/2 \end{pmatrix}$

$$\phi_i = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 & (\frac{x}{h})^3 \end{pmatrix} A^{-1}, \ i = 1,2,3,4 \tag{12}$$

$\phi_1 = 1 - \frac{11}{2}\frac{x}{h} + 9(\frac{x}{h})^2 - \frac{9}{2}(\frac{x}{h})^3$
$\phi_2 = 9\frac{x}{h} - \frac{45}{2}(\frac{x}{h})^2 + \frac{27}{2}(\frac{x}{h})^3$
$\phi_3 = -\frac{9}{2}\frac{x}{h}\frac{x}{h} + 18(\frac{x}{h})^2 - \frac{27}{2}(\frac{x}{h})^3$
$\phi_4 = \frac{x}{h} - \frac{9}{2}(\frac{x}{h})^2 + \frac{9}{2}(\frac{x}{h})^3$
Consider a polynomial of degree four with five nodal points
$C_4^0 : [0, \dfrac{h}{4}, \frac{h}{2}, \frac{3h}{4}, h]$

$$p_i(x) = a_1 + a_2\frac{x}{h} + a_3(\frac{x}{h})^2 + a_4(\frac{x}{h})^3 + a_5(\frac{x}{h})^4,\ i = 1,2,3,4,5 \tag{13}$$

$P_1 = P(0) = a_1$
$p_2 = P(\frac{h}{4}) = a_1 + a_2\frac{1}{4} + a_3(\frac{1}{4})^2 + a_4(\frac{}{4})^3 + a_5(\frac{1}{4})^4$
$p_3 = P(\frac{h}{2}) = a_1 + a_2\frac{1}{2} + a_3(\frac{1}{2})^2 + a_4(\frac{1}{2})^3 + a_5(\frac{1}{2})^4$
$p_4 = P(\frac{3h}{4}) = a_3 + a_4\frac{3}{4} + a_3(\frac{3}{4})^2 + a_4(\frac{3}{4})^3 + a_5(\frac{3}{4})^4$
$p_5 = P(h) = a_1 + a_2 + a_3 + a_4 + a_5$
$P_i = A * a_i,\ i = 1(1)5$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 \\ 1 & 3/4 & 9/16 & 27/64 & 81/256 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -25/3 & 16 & -12 & 16/3 & -1 \\ 70/3 & -208/3 & 76 & -112/3 & 22/3 \\ -80/3 & 96 & -128 & 224/3 & -16 \\ 32/3 & -128/3 & 64 & -128/3 & 32/3 \end{pmatrix}$$

The basis functions are:

$$\phi_i(x) = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 & (\frac{x}{h})^3 & (\frac{x}{h})^4 \end{pmatrix} A^{-1},\ i = 1(1)5 \tag{14}$$

Proceeding the same way, using MATLAB to evaluate the inverse of A ($i.e\, A^{-1}$) at each step, we were able to obtain the basis functions for $C_i^0$ $(i = 1(1)10)$ whose results could be seen as follows. It deeds be noted that for $i > 10$ matrix A becomes invertible. Thus, recording no basis function.

**To obtain a basic functions in** $C_1^1 : [0, h]$, we consider a polynomial of degree 3.
i.e. $(n + 1)(r + 1) - 1$
where

$$P_i = a_1 + a_2\frac{x}{h} + a_3(\frac{x}{h})^2 + a_4(\frac{x}{h})^3 \tag{15}$$

By differentiating equation (15) and interpolating at the nodal points [0,h], we have
$P_1 = a_1$

$P_2 = \frac{a_2}{h}$
$P_3 = a_1 + a_2 + a_3 + a_4$
$p_4 = \frac{a_2}{h} + 2\frac{a_3}{h} + 3\frac{a_4}{h}$
By generalization,
$P_i = Aa_i \;\; i = 1, 2, 3, 4$
where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/h & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1/h & 2/h & 3/h \end{pmatrix}$$

The basis functions are
$\phi_i = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 & (\frac{x}{h})^3 \end{pmatrix} A^{-1} \; i = 1(1)4$
$\phi_1 = 1 - 3(\frac{x}{h})^2 + 2(\frac{x}{h})^3$
$\phi_2 = h[\frac{x}{h} - 2(\frac{x}{h})^2 + (\frac{x}{h})^3]$
$\phi_3 = 3(\frac{x}{h})^2 - 2(\frac{x}{h})^3$
$\phi_4 = h[-(\frac{x}{h})^2 + (\frac{x}{h})^3$
To obtain basis function in $C_2^1 : [0, \frac{h}{2}, h]$, we consider polynomial of degree 5

$$P = a_1 + a_2(\frac{x}{h}) + a_3(\frac{x}{h})^2 + a_4(\frac{x}{h})^3 + a_5(\frac{x}{h})^4 + a_6(\frac{x}{h})^5 \tag{16}$$

By differentiating equation (16) and interpolating at the nodal points $C_2^1 : [0, \frac{h}{2}, h]$, we have: $P_1 = a_1$
$P_2 = a_2\frac{1}{h}$
$P_3 = a_1 + \frac{1}{2}a_2 + \frac{1}{4}a_3 + \frac{1}{8}a_4 + \frac{1}{16}a_5 + \frac{1}{32}a_6$
$P_4 = \frac{1}{h}a_2 + \frac{1}{h}a_3 + \frac{3}{4h}a_4 + \frac{1}{12h}a_5 + \frac{5}{16h}a_6$
$P_5 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
$P_6 = \frac{1}{h}a_2 + \frac{2}{h}a_3 + \frac{3}{h}a_4 + \frac{4}{h}a_5 + \frac{5}{h}a_6$
$P_i = Aa_i \;\; i = 1, 2, 3, 4, 5, 6$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \\ 0 & \frac{1}{h} & \frac{1}{h} & \frac{3}{4h} & \frac{1}{2h} & \frac{5}{16h} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{h} & \frac{2}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} \end{pmatrix}$$

The basis functions are:
$\phi_i = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 & (\frac{x}{h})^3 & (\frac{x}{h})^4 & (\frac{x}{h})^5 & (\frac{x}{h})^6 \end{pmatrix} A^{-1} \; i = 1(1)6$
*i.e*
$\phi_1 = 24(\frac{x}{h})^5 - 68(\frac{x}{h})^4 + 66(\frac{x}{h})^3 - 23(\frac{x}{h})^2 + 1$
$\phi_2 = 4h(\frac{x}{h})^5 - 12h(\frac{x}{h})^4 + 13h(\frac{x}{h})^3 - 6h(\frac{x}{h})^2 + h(\frac{x}{h})$
$\phi_3 = 16(\frac{x}{h})^4 - 32(\frac{x}{h})^3 + 16(\frac{x}{h})^2$
$\phi_4 = 16h(\frac{x}{h})^5 - 40h(\frac{x}{h})^4 + 32h(\frac{x}{h})^3 - 8h(\frac{x}{h})^2$

$\phi_5 = 24(\frac{x}{h})^5 + 52(\frac{x}{h})^4 - 34(\frac{x}{h})^3 + 7(\frac{x}{h})^2$
$\phi_6 = 4h(\frac{x}{h})^5 - 8h(\frac{x}{h})^4 + 5h(\frac{x}{h})^3 - h(\frac{x}{h})^2$
The process continue to degree 6. But for the purpose of this presentation, we decided to present only the process with degrees 1 and 2 while other basis functions with higher degrees can be found in the main thesis.

**Construction of basis functions with order 2**

For basis functions in $C_1^2 : [0, h]$
Consider a polynomial equation of degree 5:

$$P(x) = a_1 + a_2(\frac{x}{h}) + a_3(\frac{x}{h})^2 + a_4(\frac{x}{h})^3 + a_5(\frac{x}{h})^4 + a_6(\frac{x}{h})^5 \tag{17}$$

By differentiating equation (17)twice and interpolating at nodal points [0,h], we obtain:
$P_1 = a_1$
$P_2 = a_2\frac{1}{h}$
$P_3 = (\frac{1}{h})^2 a_3$
$P_4 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$
$P_5 = \frac{1}{h}a_2 + \frac{2}{h}a_3 + \frac{3}{h}a_4 + \frac{4}{h}a_5 + \frac{5}{h}a_6$
$P_6 = (\frac{1}{h})^2 2a_2 + (\frac{1}{h})^2 6a_4 + (\frac{1}{h})^2 12a_5 + (\frac{1}{h})^2 20a_6$
$P_i = Aa_i \ \ i = 1, 2, 3, 4, 5, 6$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(\frac{1}{h})^2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{h} & \frac{1}{h} & \frac{3}{h} & \frac{4}{h} & \frac{5}{h} \\ 0 & 0 & 2(\frac{1}{h})^2 & (\frac{1}{h})^2 & 12(\frac{1}{h})^2 & 20(\frac{1}{h})^2 \end{pmatrix}$$

The basis functions are:
$\phi_i = \begin{pmatrix} 1 & \frac{x}{h} & (\frac{x}{h})^2 & (\frac{x}{h})^3 & (\frac{x}{h})^4 & (\frac{x}{h})^5 & (\frac{x}{h})^6 \end{pmatrix} A^{-1} \, i = 1(1)6$
$i.e$
$\phi_1 = 6(\frac{x}{h})^5 + 15(\frac{x}{h})^4 - 10(\frac{x}{h})^3 + 1$
$\phi_2 = -3h(\frac{x}{h})^5 + 8h(\frac{x}{h})^4 - 6h(\frac{x}{h})^3 + h(\frac{x}{h})$
$\phi_3 = -\frac{h^2}{2}(\frac{x}{h})^5 + 3\frac{h^2}{2}(\frac{x}{h})^4 + 3\frac{h^2}{2}(\frac{x}{h})^3 + \frac{h^2}{2}(\frac{x}{h})$
$\phi_4 = 6(\frac{x}{h})^5 - 15(\frac{x}{h})^4 + 10(\frac{x}{h})^3$
$\phi_5 = -3h(\frac{x}{h})^5 + 7h(\frac{x}{h})^4 - 4h(\frac{x}{h})^3$
$\phi_6 = \frac{h^2}{2}(\frac{x}{h})^5 - h^2(\frac{x}{h})^4 + \frac{h^2}{2}(\frac{x}{h})^3$
Equally, the basis functions of higher degrees can be found in the main thesis of this research work.

Table 1: **BASIS FUNCTIONS OF SELECTED ORDERS**($C_r^k$)

| Order (k) | Degree (r) | y=x/h |
|---|---|---|
| 0 | 1 | $\phi_1 = 1 - y$ |
| 0 | 1 | $\phi_2 = y$ |
| 0 | 2 | $\phi_1 = 1 - 3y + 2y^2$ |
| 0 | 2 | $\phi_2 = 4y - 4y^2$ |
| 0 | 2 | $\phi_3 = -y + 2y^2$ |
| 0 | 3 | $\phi_1 = 1 - 3y + 2y^2$ |
| 0 | 3 | $\phi_2 = 1 - (11/2)y + 9y^2 - (9/2)y^3$ |
| 0 | 3 | $\phi_3 = -(9/2)y + 18y^2 - (22/2)y^3$ |
| 0 | 3 | $\phi_4 = y - (9/2)y^2 + (9/2)y^3$ |
| 1 | 1 | $\phi_1 = 1 - 3y^2 + 2y^3$ |
| 1 | 1 | $\phi_2 = h[y - 2y^2 + y^3]$ |
| 1 | 1 | $\phi_3 = 3y^2 - 2y^3$ |
| 1 | 1 | $\phi_4 = h[-y^2 + y^3]$ |
| 1 | 2 | $\phi_1 = 24y^5 - 68y^4 + 66y^3 - 23y^2 + 1$ |
| 1 | 2 | $\phi_2 = h[4y^5 - 12y^4 + 13y^3 - 6y^2 + y]$ |
| 1 | 2 | $\phi_3 = 16y^4 - 32y^3 + 16y^2$ |
| 1 | 2 | $\phi_4 = h[16y^5 - 40y^4 + 32y^3 - 8y^2]$ |
| 1 | 2 | $\phi_5 = 24y^5 + 52y^4 - 34y^3 + 7y^2$ |
| 1 | 2 | $\phi_6 = h[4y^5 - 8y^4 + 5y^3 - y^2]$ |
| 2 | 1 | $\phi_1 = -6y^5 + 15^4 - 10y63 + 1$ |
| 2 | 1 | $\phi_2 = h[-3y^5 + 8y^4 - 6y^3 + y]$ |
| 2 | 1 | $\phi_3 = h^2/2[-y^5 + 3y^4 - 3y^3 + y^2]$ |
| 2 | 1 | $\phi_4 = 6y^5 - 15y^4 + 10y63$ |
| 2 | 1 | $\phi_5 = h[-3y^5 + 7y^4 - 4y^3]$ |
| 2 | 1 | $\phi_6 = h^2/2[y^5 - y^4 + y^3]$ |

## 3. Numerical Examples

Illustration 1.
We consider the solution to a growth equation bellow using the constructed basis functions via Galerkin Weighted Residual approach

$$\frac{d^2u(x)}{dx^2} - u(x) = 0,\ 0 < x < 2 \tag{18}$$

with $u(0) = 1\, and\, u(2) = exp(2)$
by employing Galerkin Weighted Residual Method

$(\in, \phi_i) = \int_0^2 (\frac{d^2u(x)}{dx^2} - u(x))\phi_i dx = 0,\ i = 1, 2, 3, ...,$
$\longrightarrow \int_0^2 (\frac{d^2u(x)}{dx^2}\phi_i - u(x)\phi_i)dx = 0$

$\longrightarrow \phi_i \frac{du}{dx} \mid_0^2 - \int_0^2 (\frac{du}{dx}\frac{d\phi_i}{dx} + u\phi_i)dx = 0$
since $\phi_i$ does not include the boundary
$\phi_i \frac{du}{dx} \mid_0^2 = 0$
$\longrightarrow \int_0^2 (\frac{du}{dx}\frac{d\phi_i}{dx} + u\phi_i)dx = 0$
using the relation $\phi_i = \sum_{e=1}^{\in} \phi_N^e \triangle_{N_i}^e$
we have
$\sum_{e=1}^{\in} \int_0^h (\frac{d\phi_N^e}{dx}\frac{d\phi M^e}{dx} + \phi_N^e \phi_M^e) \triangle_{N_i} \triangle_{M_j} u_j dx = 0$
$A_{NM}^e = \int_0^h (\frac{d\phi_N^e}{dx}\frac{d\phi M^e}{dx} + \phi_N^e \phi_M^e)dx$
since $A_{ij} = \sum_{e=1}^{\in} A_{NM}^e \triangle_{N_i}^e \triangle_{M_j}^e$
with
$\phi_1 = 1 - \frac{x}{h} \, and \, \phi_2 = \frac{x}{h}$

$$A_{NM}^e = \begin{pmatrix} h/3 + 1/h & h/6 - 1/h \\ h/6 - 1/h & h/3 + 1/h \end{pmatrix}$$

with the domain of the problem divided into four distinct elements with 5 nodes, we have $h = 0.5$ such that

$$A_{NM}^e = \begin{pmatrix} 2.1667 & -1.9167 \\ -1.9167 & 2.1667 \end{pmatrix}$$

The global finite element equation is
$AU = B$ where

$$A = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 \\ 0 & 4.3333 & -1.9167 & 0 & 0 \\ 0 & -1.9167 & 4.3333 & -1.9167 & 0 \\ 0 & 0 & -1.9167 & 4.3333 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

$$B = \begin{pmatrix} 1.0000 \\ 1.9167 \\ 0 \\ 14.1625 \\ 7.3891 \end{pmatrix}$$

(Note that the boundary conditions have been imposed)
by solving the resulting equation we have

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 1.6348 \\ 2.6961 \\ 4.4608 \\ 7.3891 \end{pmatrix}$$

For 8 equal elements $[h = 0.25]$
the global finite element equation with the boundary conditions imposed reads

$$\begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8.1666 & -3.9583 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.9583 & 8.1666 & -3.9583 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.9583 & 8.1666 & -3.9583 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.9583 & 8.1666 & -3.9583 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.9583 & 8.1666 & -3.9583 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.9583 & 8.1666 & -3.9583 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.9583 & 8.1666 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0000 \\ 3.9583 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 29.2483 \\ 7.3891 \end{pmatrix}$$

which gives

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 1.2822 \\ 1.6453 \\ 2.1124 \\ 2.7129 \\ 3.4847 \\ 4.4766 \\ 5.7512 \\ 7.3891 \end{pmatrix}$$

with the quadratic cases in( $C_2^0$), the results obtained for two quadratic elements reads

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 1.6486 \\ 2.7198 \\ 4.4801 \\ 7.3891 \end{pmatrix}$$

while the results obtained for four quadratic elements reads

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 1.284012 \\ 1.648787 \\ 2.117016 \\ 2.718389 \\ 3.490336 \\ 4.481802 \\ 5.75449 \\ 7.3891 \end{pmatrix}$$

The MATLAB code bellow was used to obtain the solution to the problem above using the basis function in $C_1^1$

$function x = Gauss(A, b)$
$b = [1.0000; -0.1032738; 4.7678571;$
$-0.0980655; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 35.2301729; 0.72461586; 7.3891; 0.7631004];$
$A = [100000000000000000;$
$00.0334821 - 0.09806550.008444900000000000000;$
$0 - 0.09806559.78571420 - 4.76785710.0980655000000000000;$
$0 - 0.008444900.0669642 - 0.0980655 - 0.0084449000000000000;$
$00 - 4.7678571 - 0.09856559.78571420 - 4.76785710.09806550000000000;$
$000.0980655 - 0.008444900.0669642 - 0.0980655 - 0.00844490000000000;$
$0000 - 4.7678571 - 0.09806559.78571420 - 4.76785710.098065500000000;$
$00000.0980655 - 0.008444900.0669642 - 0.0980655 - 0.008444900000000;$
$000000 - 4.7678571 - 0.09806559.78571420 - 4.76785710.0980655000000;$
$0000000.0980655 - 0.008444900.0669642 - 0.0980655 - 0.0084449000000;$
$00000000 - 4.7678571 - 0.09806559.78571420 - 4.76785710.09806550000;$
$000000000.0980655 - 0.008444900.0669642 - 0.0980655 - 0.00844490000;$
$0000000000 - 4.7678571 - 0.09806559.78571420 - 4.76785710.098065500;$
$00000000000.0980655 - 0.008444900.0669642 - 0.0980655 - 0.008444900;$
$000000000000 - 4.7678571 - 0.09806559.7857142000.0980655;$
$0000000000000.0980655 - 0.008444900.06696420 - 0.0084449;$
$000000000000000010; 000000000000000.0980655 - 0.008444900.0334821];$
$[n, n] = size(A);$
$[n, k] = size(b);$
$x = zero(n, k);$
$for i = 1 : n - 1$
$m = -A(i + 1 : n, i)/A(i, i);\ A(i + 1 : n, :) = A(i + 1 : n, :) + M * A(i, :);$
$b(i + 1 : n, :) = b(i + 1 : n, :) + M * b(i, :);$
$end;$
$x(n, :) = b(n, :)/A(n, n);$
$for i = n - 1 : -1 : 1$
$x(i, :) = (b(i, :) - A(i, i + 1 : n) * x(i + 1 : n, :))/A(i, i);$

*end*

Table 2: Summary of the results

| | Order of Continuity | | | | | | |
|---|---|---|---|---|---|---|---|
| | $C_1^0$ | $C_1^0$ | $C_2^0$ | $C_2^0$ | $C_1^1$ | $C_1^1$ | |
| Nodal point | 4 Elements | 8 Elements | 4 Elements | 8 Elements | 4 Elements $U(x)$ | 4 Elements $\frac{du}{dx}$ | Exact |
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.3429 | 1.0000 |
| 1/4 | | 1.2822 | | 1.2840 | 1.2722 | 1.1843 | 1.2840 |
| 1/2 | 1.6348 | 1.6453 | 1.6486 | 1.6488 | 1.6378 | 1.6419 | 1.6487 |
| 3/4 | | 2.1124 | | 2.1170 | 2.1084 | 2.130 | 2.1170 |
| 1 | 2.6961 | 2.7129 | 2.7198 | 2.7184 | 2.7120 | 2.7262 | 2.7183 |
| 5/4 | | 3.4847 | | 3.4903 | 3.4860 | 3.4974 | 3.4903 |
| 3/2 | 4.4608 | 4.4766 | 4.4801 | 4.4818 | 4.4790 | 4.4879 | 4.4817 |
| 7/4 | | 5.7512 | | 5.7545 | 5.7533 | 5.7602 | 5.7546 |
| 2 | 7.3891 | 7.3891 | 7.3891 | 7.3891 | 7.3891 | 7.3934 | 7.3891 |

## 4. Conclusion

In this work, finite order basis functions $\phi_i(x)[i = 1, 2, (r+1)(n+1)]$ which are not only continuous but have in addition, continuous derivatives have been derived, as an invaluable tool for use in the expansion methods. Computational advantages of the generalized basis are illustrated by the numerical results obtained through it for a test problem, demonstrating the versatility of the new approximating tool. Equally, we also observed that the higher the degree of the basis function the more accurate the results. It is indeed an ongoing research; efforts shall be geared towards presenting results on non-homogeneous and non-linear differential equation problems.

## References

[1] Bamigbola, O. M. and Ibiejugba, M. A. and Onumanyi, P. (1988), Higher order chebyshev basis functions for two-point boundary value problems, Intl. J. Numer. Method in Engr.,**26**, 313-327

[2] Bamigbola, O. M. and Ibiejugba, M. A. (1992), A Galerkin weihted residual finite element method for linear differential equations, Jour. of Nig. Math. Society., **2**(3), 149-156

[3] Bamigbola, O. M. (1995), A comparison of the computational efficiency of sone quadratic polynomial basis functions in the finite element method, ABACUS., **25**, 23-42

[4] Zienkiewicz, O. C. (1976), Finite elements the background story in the Mathematics of Finite Elements and Applications, Academic Press, London., **25**, 23-42

[5] Ibiejugba, M. A. and Onumanyi, P. and Bamigbola, O.M. (1987), A Chebyshev Finite element method for solving differential equations, Nig. Jour. Pure Appl.Sc, **2**, 70-85

[6] Le Thi Hoai An and Vaz, A.I.F. and Vicente, L. N. (2012), Optimizing D.C. programming and its use in direct search for global derivatives.free optimization, TOP., **20**, 190-214

[7] Bamigbola, O. M. (1984), Construction of some new polynomial basis function in $C^0$ space, University of Ilorin.

[8] Devies, L. M. (1976), Expansion methods in modern numerical methods for Ordinary differential equations, Calendon Press, Oxford

[9] Devies, L. M.(1980), The Finite Element; A first approach, Calendon Press, Oxford.

[10] Ibiejugba, M.A. (1983), The Ritz Penalty method for solving the control of a diffusion equations, Opt. theory and appl., **3**, 431-449.

[11] Aliu, T. (1995), A Legendre Finite element method for differential equations, University of Ilorin, 431-449.
[12] Mitchel, A. R. and Wait, R. (1977), The finite element method for partial differential equations, John Wiley, Chichestert, 431-449
[13] Robert, M. C. and Stephen, M. W. (2010), Bernstein Bases are Optimal, but, sometimes Lagrange bases are better, Comp. Appl. Math., **29**(1),
[14] Doha, E. H. and Bhrawy, A. H. and Saker, M. A. (2011), On the Derivatives of Bernstein Polynomials: Application of Solution of High Even-Order Differential Equations, BVP, Comp. Appl. Math., Article ID 829543, 16 pp
[15] Brian, D and Hahn and Daniel, T. V. (2007), Essential MATLAB for Engineers and Scientist, Elsevier.