Nigerian Journal of Mathematics and Applications
Volume **32**, (2022), 163 − 173. *Printed by Unilorin press*

http://www.njmaman.com

Estimates for some Classes of Analytic Functions Associated with Pascal Distribution Series, Error Function, Bell numbers and q-Differential Operator

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Abstract

In this paper, we considered two classes of analytic functions. The classes are associated with Pascal distribution series, error function, Bell numbers and q-derivative operator. Some early coefficient estimates for the classes were established and some interesting nexus between our estimates and that of some earlier known classes were presented.

1. Introduction

In this investigation, let the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1}$$

be denoted by $\mathcal{A}$. Also, let $\mathcal{S}$, a subclass of $\mathcal{A}$, be the class of analytic functions that are also univalent in Δ. Two of the subclasses of class $\mathcal{S}$ that are of interest in this work are the classes of starlike functions and convex functions. A function f is said to be starlike in Δ if $\mathcal{R}e(zf'/f) > 0$ and it is said to be a convex function

Received: 02/02/2022, Accepted: 19/03/2022, Revised: 05/04/2022. * Corresponding author.
2015 *Mathematics Subject Classification.* 30C45 & 30C50.
Keywords and phrases. Analytic function, Pascal distribution series, error function, Bell numbers, q-differentiation, coefficient estimate, subordination, convolution
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if $\mathcal{R}e(1+z(f''/f'))>0$. Let these two classes be respectively denoted by $\mathcal{S}^\star$ and $\mathcal{K}$. Let the class of Schwarz functions $w(z)$ analytic in Δ be represented as

$$\mathcal{W} := \Big\{w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots : w(0)=0,\ |w(z)|<1,\ z\in\Delta\Big\}. \tag{2}$$

In the sequel, let

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{A} \quad (z\in\Delta). \tag{3}$$

Then we say that function f is subordinate to F, notationally represented as $f \prec F$ if there exist a Schwarz function $w(z)$ such that

$$f(z) = F(w(z)) \quad (z\in\Delta).$$

Suppose $F\in\mathcal{S}$, then

$$f \prec F if and only if f(0)=F(0) \quad \text{and} \quad f(\Delta)\subset F(\Delta).$$

From (1) and (3), the *convolution* (or *Hadamard product*) of f and F notationally represented by $(f\star F)(z)$ is defined by

$$(f\star F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n \quad (z\in\Delta). \tag{4}$$

A special function which occurs in probability, statistics, material science and partial differential equation is the *error function*

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}z^{n+1}}{(2n+1)!} \quad (z\in\Delta) \tag{5}$$

found in [1]. For more properties of function $erf(z)$ see the works of Alzer [4], Coman [10], Elbert [11] and Altinkaya and Olatunji [3]. In 2018, Ramachandran et al. [31] modified (5) by introducing the modified error function

$$\mathcal{E}f(z) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}z^n \quad (z\in\Delta). \tag{6}$$

Next we present an analytic function $\mathcal{P}$ whose coefficients are the probabilities of the Pascal distribution, thus we have

$$\mathcal{P}(z) = z + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}p^{(n-1)}(1-p)^m z^n \tag{7}$$
$$(m\geqslant 1,\ 0<p\leqslant 1,\ z\in\Delta).$$

The function (7) was introduced and studied by Porwal [29], see also [26]. Now applying the principle of convolution and in view of the functions in (6) and (7) we hereby define the function

$$\begin{aligned}\mathcal{G}(z) &= (\mathcal{E}f \star \mathcal{P})(z)\\ &= z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} p^{(n-1)}(1-p)^m \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n \end{aligned} \tag{8}$$

for $m \geqslant 1$, $0 < p \leqslant 1$ and $z \in \Delta$.
Another function of interest in this work is the function $\mathcal{Q}(z)$ studied by Kumar et al. [16] and defined by

$$\mathcal{Q}(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \quad (z \in \Delta) \tag{9}$$

where the coefficients B_n $(n \in \mathbb{N} \cup \{0\})$ are called Bell numbers $(B_0 = B_1 = 1)$ while some few early Bell numbers are $1, 1, 2, 5, 15, 52, 203, 877, 4140, \cdots$. Bell [7, 8] presented these numbers as a count of the possible partitions of a set. Meanwhile, Kumar et al. [16] proved that function $\mathcal{Q}(z)$ is starlike with respect to 1. It is this starlikeness property that attracted our attention to further study this function. For more details, see [9, 16, 23, 24, 30].

For $q \in (0, 1)$, the Jackson's q-differentiation of a function $f \in \mathcal{A}$ of the form (1) is defined by

$$\left.\begin{aligned} &\mathcal{D}_q f(0) = f'(0) = 1 \quad (z = 0) \quad \text{if it exists,}\\ &\mathcal{D}_q f(z) = \tfrac{f(z) - f(qz)}{z(1-q)} = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (z \neq 0),\\ &\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2}. \end{aligned}\right\} \tag{10}$$

and note that

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1} \quad \text{so that } \lim_{q \to 1^-} [n]_q = n. \tag{11}$$

For some historical details, properties, applications, and some results on some subclasses of analytic functions involving q-differentiation see the works in [2, 5, 6, 13, 14, 15, 17, 18, 19, 20, 32].

Definition 1.1. Using the concept of q-difference operator in (10) and in view of the respective functions $\mathcal{G}(z)$ and $\mathcal{Q}(z)$ in (8) and (9) we hereby define two new classes $\mathcal{S}^{\star}_{q,b}(\mathcal{G}, \mathcal{Q})$ and $\mathcal{K}_{q,b}(\mathcal{G}, \mathcal{Q})$ as follows. A function $\mathcal{G} \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\star}_{q,b}(\mathcal{G}, \mathcal{Q})$ if, and only if, $\mathcal{G}$ satisfies the condition

$$1 + \frac{1}{b}\left\{\frac{z\mathcal{D}_q \mathcal{G}(z)}{\mathcal{G}(z)} - 1\right\} \prec \mathcal{Q}(z) \quad (z \in \Delta). \tag{12}$$

and $\mathcal{G} \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{q,b}(\mathcal{G}, \mathcal{Q})$ if, and only if, $\mathcal{G}$ satisfies the condition

$$1 + \frac{1}{b}\left\{\frac{z\mathcal{D}_q(\mathcal{D}_q\mathcal{G}(z))}{\mathcal{D}_q\mathcal{G}(z)}\right\} \prec \mathcal{Q}(z) \quad (z \in \Delta) \tag{13}$$

for $q \in (0,1)$, $m \geqslant 1$, $0 < p \leqslant 1$ and $b \in \mathbb{C} - \{0\}$.

2. Applicable Lemmas

Let $w \in \mathcal{W}$ defined in (2). The following lemmas shall be applied in the course of proving our results.

Lemma 2.1 ([12]). *Let $w(z) \in \mathcal{W}$, then $|w_n| \leqslant 1$ $(n \in \mathbb{N})$. Equality occurs for functions $w(z) = e^{i\vartheta} z^k$ $(\vartheta \in [0, 2\pi))$.*

Lemma 2.2 ([12]). *Let $w \in \mathcal{W}$, then for $t \in \mathbb{C}$, $|w_2 + tw_1^2| \leqslant \max(1, |t|)$. The inequality is sharp for the functions $w(z) = z^2$.*

Motivated by the works of the aforementioned authors in the Introductory section and more importantly the works of Altinkaya and Olatunji [3], Oladipo [21], Olukoya and Oyekan [22], Oyekan and Awolere [25] Oyekan and Opoola [27], Oyekan et al. [28] and Porwal [29]. Henceforth, we shall assume that $q \in (0,1)$, $m \geqslant 1$, $0 < p \leqslant 1$, $b \in \mathbb{C} - \{0\}$ and that $\mathcal{G} \in \mathcal{A}$ unless otherwise stated.

3. Main Results

The following are the established results.

Theorem 3.1. *Let $\mathcal{G}(z) \in \mathcal{S}^{\star}_{q,b}(\mathcal{G}, \mathcal{Q})$, then*

$$|mp(1-p)^m| \leqslant \frac{3|b|}{q} \tag{14}$$

$$|m(m+1)p^2(1-p)^m| \leqslant \frac{12|b|}{q(1+q)} \max\left\{1, \left|\frac{(q+b)}{q}\right|\right\} \tag{15}$$

$$\begin{aligned} &|m(m+1)p^2(1-p)^m - \mu(mp(1-p)^m)^2| \\ &\qquad \leqslant \frac{12|b|}{q(1+q)} \max\left\{1, \left|\frac{12(q+b)^2 + 12qb + 9\mu q(1+q)b}{12q^2}\right|\right\} \end{aligned} \tag{16}$$

$$\begin{aligned} &|m(m^2+m+2)p^3(1-p)^m| \\ &\qquad \leqslant \frac{126|b|}{q(1+q+q^2)} \max\left\{1, \left|\sigma\left[\frac{t}{\sigma} + \frac{q+b}{q} + \left(1 + \frac{2}{\sigma}\right)\right]\right|\right\} \end{aligned} \tag{17}$$

where

$$\sigma = \frac{[(1-[3]_q)w_1]b}{(1-[2]_q)(1-[3]_q)} \quad \textit{and} \quad t = \frac{5}{6} - \frac{b^2}{(1-[2]_q)^2}. \tag{18}$$

Proof. Suppose $\mathcal{G}(z) \in \mathcal{S}^{\star}_{q,b}(\mathcal{G}, \mathcal{Q})$, then applying the subordination principle implies that (12) can be expressed as

$$1 + \frac{1}{b}\left\{\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right\} = \mathcal{Q}(w(z)) \quad (z \in \varDelta).$$

or

$$[z\mathcal{D}_q\mathcal{G}(z) - \mathcal{G}(z)][\mathcal{G}(z)]^{-1} = [\mathcal{Q}(w(z)) - 1]b \tag{19}$$

Now using (10) in (7) we get

$$\mathcal{D}_q\mathcal{G}(z) = 1 + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}p^{(n-1)}(1-p)^m[n]_q z^{n-1} \tag{20}$$

$$(m \geqslant 1,\ 0 < p \leqslant 1,\ z \in \varDelta)$$

and by expansion we get

$$\mathcal{D}_q\mathcal{G}(z) = 1 + C^m_{m-1}(1-p)^m p^{-1/3} z^2 + C^{m+1}_{m-1}(1-p)^m p^2 \frac{1}{10} z^3 + C^{m+2}_{m-1}(1-p)^m p^3 - \frac{1}{42} z^4 + C^{m+3}_{m-1}(1-p)^m p^4 \frac{1}{216} z^5 + \cdots \tag{21}$$

or

$$z\mathcal{D}_q\mathcal{G}(z) = z - \frac{1}{3}mp(1-p)^m z^2 + \frac{1}{20}m(m+1)p^2(1-p)^m z^3 + \frac{1}{225}m(m^2+m+2)p^3(1-p)^m z^4 - \frac{1}{5184}m(4m^2+5m+6)p^4(1-p)^m z^5 + \cdots \tag{22}$$

Applying binomial expansion shows that

$$[\mathcal{G}(z)]^{-1} = z^{-1} + \frac{m}{3}p(1-p)^m + \frac{m^2}{9}p^2(1-p)^{2m}z - \frac{m^3}{27}p^3(1-p)^{3m}z^2 + \frac{m^4}{81}p^4(1-p)^{4m}z^3 + \cdots \tag{23}$$

Putting (22) (23) into the LHS of (19) and simplifying completely gives

$$
\begin{aligned}
(24)\quad [z\mathcal{D}_q\mathcal{G}(z)-\mathcal{G}(z)][\mathcal{G}(z)]^{-1} &= \left\{\frac{m}{3}(1-[2]_q)p(1-p)^m\right\}z \\
&+\left\{\frac{m}{9}(1-[2]_q)p^2(1-p)^{2m}-\frac{m(m+1)}{20}(1-[3]_q)p^2(1-p)^m\right\}z^2 \\
&+\left\{\frac{m^3}{27}(1-[2]_q)p^3(1-p)^{3m}-\frac{m^2(m+1)}{60}(1-[3]_q)p^3(1-p)^{2m}\right. \\
&\left.+\frac{m(m^2+m+2)}{225}(1-[4]_q)p^3(1-p)^m\right\}z^3 \\
&+\left\{\frac{m^4}{81}(1-[2]_q)p^4(1-p)^{4m}-\frac{m^3(m+1)}{180}(1-[3]_q)p^4(1-p)^{3m}\right. \\
&+\frac{m^2(m^2+m+2)}{675}(1-[4]_q)p^4(1-p)^{2m} \\
&\left.-\frac{m(4m^2+5m+6)}{5184}(1-[5]_q)p^4(1-p)^m\right\}z^4+\cdots
\end{aligned}
$$

Careful expansion of the RHS of (12) shows that

$$
\begin{aligned}
(25)\quad b[Q(w(z))-1] &= bw_1z+b\{w_2+w_1^2\}z^2+b\{w_3+2w_1w_1+\frac{5}{6}w_1^3\}z^3 \\
&+b\{w_4+w_2^2+2w_1w_3+\frac{5}{2}w_1^2w_2+\frac{5}{8}w_1^4\}z^4+\cdots
\end{aligned}
$$

and comparing of the coefficients in (24) and (25) gives

$$
(26)\qquad \frac{m}{3}(1-[2]_q)p(1-p)^m = bw_1
$$

$$
(27)\qquad \frac{m}{9}(1-[2]_q)p^2(1-p)^{2m}-\frac{m(m+1)}{20}(1-[3]_q)p^2(1-p)^m = b\{w_2+w_1^2\}
$$

$$
\begin{aligned}
(28)\quad &\frac{m^3}{27}(1-[2]_q)p^3(1-p)^{3m}-\frac{m^2(m+1)}{60}(1-[3]_q)p^3(1-p)^{2m} \\
&+\frac{m(m^2+m+2)}{225}(1-[4]_q)p^3(1-p)^m = b\{w_3+2w_1w_1+\frac{5}{6}w_1^3\}
\end{aligned}
$$

$$(29)\quad \frac{m^4}{81}(1-[2]_q)p^4(1-p)^{4m} - \frac{m^3(m+1)}{180}(1-[3]_q)p^4(1-p)^{3m} + \frac{m^2(m^2+m+2)}{675}(1-[4]_q)p^4(1-p)^{2m} - \frac{m(4m^2+5m+6)}{5184}(1-[5]_q)p^4(1-p)^m = b\{w_4 + w_2^2 + 2w_1w_3 + \frac{5}{2}w_1^2w_2 + \frac{5}{8}w_1^4\}.$$

Now from (26) we get

$$(30)\quad mp(1-p)^m = \frac{3bw_1}{(1-[2]_q)}$$

and applying triangle inequality we have

$$(31)\quad |mp(1-p)^m| = \left|\frac{3bw_1}{(1-[2]_q)}\right| = \frac{3|b||w_1|}{(1-[2]_q)}$$

and applying (11) and Lemma 2.1 gives (14). Also by putting (30) into (27) and completely simplifying we get

$$(32)\quad m(m+1)p^2(1-p)^m = \frac{-12b}{1-[3]_q}\left\{w_2 + \left(1-\frac{b}{1-[2]_q}\right)w_1^2\right\}$$

so that by triangle inequality we get

$$(33)\quad |m(m+1)p^2(1-p)^m| = \left|\frac{-12b}{1-[3]_q}\right|\left|w_2 + \left(1-\frac{b}{1-[2]_q}\right)w_1^2\right|$$

and applying Lemma 2.2 we get (15). Next by putting (30) and (32) into (28) and completely simplifying we get

(34)

$$m(m^2+m+2)p^3(1-p)^m = \frac{126b}{((1-[4]_q)}\left[w_3 + \left(\frac{5}{6}-\frac{b^2}{(1-[2]_q)^2}\right)w_1^3 + 2w_1w_2 - \frac{[(1-[3]_q)w_1]b}{((1-[2]_q)(1-[3]_q))}\left\{w_2 + \left(1-\frac{b}{(1-[2]_q)}\right)w_1^2\right\}\right]$$

so that by triangle inequality we get

(35)

$$|m(m^2+m+2)p^3(1-p)^m| = \left|\frac{126b}{((1-[4]_q)}\right|\left[\left|w_3 + \left(\frac{5}{6}-\frac{b^2}{(1-[2]_q)^2}\right)w_1^3\right| + |2w_1||w_2| + \frac{[(1-[3]_q)|w_1|]|b|}{((1-[2]_q)(1-[3]_q))}\left|w_2 + \left(1-\frac{b}{(1-[2]_q)}\right)w_1^2\right|\right]$$

and applying (11) and Lemmas 2.1 and 2.2 we get (16). Putting (30), (32) and (34) into (29) and completely simplifying gives

$$\begin{aligned}(36)\quad &\frac{m^4}{81}(1-[2]_q)p^4(1-p)^{4m}-\frac{m^3(m+1)}{180}(1-[3]_q)p^4(1-p)^{3m}\\ &+\frac{m^2(m^2+m+2)}{675}(1-[4]_q)p^4(1-p)^{2m}\\ &-\frac{m(4m^2+5m+6)}{5184}(1-[5]_q)p^4(1-p)^{m}\\ &=b\{w_4+w_2^2+2w_1w_3+\frac{5}{2}w_1^2w_2+\frac{5}{8}w_1^4\}\end{aligned}$$

or by using (30), (32) and (34) into (29) and simplifying completely we get

$$\begin{aligned}&m(m^2+m+2)p^3(1-p)^m\\ &=\frac{126b}{(1-[4]_q)}[w_3+\sigma\left(\frac{t}{\sigma}w_1^3+\left(\frac{1-[2]_q-b}{(1-[4]_q)}\right)\right)w_1^3+\left(\frac{1+2}{\sigma}\right)2w_1w_2]\end{aligned}$$

so that by applying triangle inequality, (18), (11) and Lemmas 2.1 and 2.2 we get (17). □

Theorem 3.2. *Let $\mathcal{G}(z)\in\mathcal{K}_{q,b}(\mathcal{G},\mathcal{Q})$, then*

$$|mp(1-p)^m|\leqslant\frac{3|b|}{1+q}$$

$$\begin{aligned}&|m(m+1)p^2(1-p)^m-\mu(mp(1-p)^m)^2|\\ &\leqslant\frac{6|b|}{q(1+q)}\max\left\{1,\left|\frac{6[2]_q(1+b)-9\mu b[3]_q}{6[2]_q}\right|\right\}\end{aligned}$$

and

$$\begin{aligned}&|m(m^2+m+2)p^3(1-p)^m|\\ &\leqslant\frac{42|b|}{[4]_q}\max\left\{1,\left|\sigma\left[\frac{t}{\sigma}+(1+b)+\left(1+\frac{2}{\sigma}\right)\right]\right|\right\}\end{aligned}$$

where

$$\sigma=\frac{b[(1-[2]_q)+(1-[3]_q)]}{(1-[2]_q)(1-[3]_q)}\quad\text{and}\quad t=\frac{5}{6}-\frac{b^2}{(1-[2]_q)^2}.$$

Proof. The proof is omitted since it is akin to the technique used in proving Theorem 3.1. □

4. Nexus between Bound Results for The Pascal Distribution Series for Analytic Function Classes and The Generalized Distribution for Analytic Function Classes both of which are Associated with Error Function and Bell Numbers

The Pascal Distribution Series for Analytic Function Class $\mathcal{S}^{\star}_{q,b}(\mathcal{G},\mathcal{Q})$ Associated with Error Function and Bell Numbers	The Generalised Distribution for Analytic Function Class $\phi S^{q}_{b}(Q)$ Associated with Error Function and Bell Numbers defined in [3]	Remarks
$\lvert mp(1-p)^m\rvert \leqslant \frac{3\lvert b\rvert}{q}$	$\lvert\frac{a_1}{S}\rvert \leqslant \frac{3\lvert b\rvert}{q}$	$\lvert mp(1-p)^m\rvert \leq \lvert\frac{a_1}{S}\rvert = \frac{3\lvert b\rvert}{q}$
$\lvert m(m+1)p^2(1-p)^m\rvert \leqslant \frac{12\lvert b\rvert}{q(1+q)}\max\{1,\lvert\frac{(q+b)}{q}\rvert\}$	$\lvert\frac{a_2}{S}\rvert \leqslant \frac{10\lvert b\rvert}{q(1+q)}\max\{1,\lvert\frac{(q+b)}{q}\}$	$\lvert m(m+1)p^2(1-p)^m\rvert \leq \frac{6}{5}\lvert\frac{a_2}{S}\rvert$
$\lvert m(m^2+m+2)p^3(1-p)^m\rvert \leqslant \frac{126\lvert b\rvert}{q(1+q+q^2)}\max\{1,\lvert\sigma[\frac{t}{\sigma}+\frac{q+b}{q}+(1+\frac{2}{\sigma})]\rvert\}$	$\lvert\frac{a_3}{S}\rvert \leqslant \frac{42\lvert b\rvert}{q(1+q+q^2)}\max\{1,\lvert\sigma[\frac{t}{\sigma}+\frac{(q+b)}{q}+(1+\frac{2}{\sigma})]\rvert\}$	$\lvert m(m^2+m+2)p^3(1-p)^m \leq 3\lvert\frac{a_3}{S}\rvert$
$\lvert m(m+1)p^2(1-p)^m-\mu(mp(1-p)^m)^2\rvert \leqslant \frac{12\lvert b\rvert}{q(1+q)}\max\{1,\lvert\frac{(12(q+b)^2+12qb+9q(1+q)b)}{12q^2}\rvert\}$	$\lvert\frac{a_2}{S}-\mu\frac{a_1}{S^2}\rvert \leqslant \frac{10\lvert b\rvert}{q(1+q)}\max\{1,\lvert\frac{(10(q+b)^2+10qb+9q(1+q)b)}{(10q^2)}\rvert\}$	$\lvert m(m+1)p^2(1-p)^m-\mu(mp(1-p)^m)^2\rvert \leq \frac{6}{5}\lvert\frac{a_2}{S}-\mu\frac{a_1^2}{S^2}\rvert$

Conclusions: Two new classes of generalized analytic functions defined by some known special functions such as Pascal distribution series, error function and Bell numbers and the q-derivative operator were introduced and investigated in this work. Some results obtained were the early coefficient estimates for functions in each of the classes. However, if we vary some underlaying parameters in each of the new classes, then the results of Theorems 3.1 and 3.2 will reduce to some known results. Some of these results are featured in Section 4.

Acknowledgement. The authors thank the referee(s) for their individual and collective valuable suggestions that helped to produce this quality paper.

Competing interests: The manuscript was read and approved by all the authors. They therefore declare that there is no conflicts of interest.

Funding: The authors received no financial support for the research, authorship, and/or publication of this article.

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