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THE COUNTERPART OF A RESULT OF WEBB FOR THE BOUNDED DUAL Eb

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Abstract

Denote by E', E^+ and E^b the continuous dual, the sequential dual and the bounded dual, respectively, of the $lcs(E, \tau)$. Well-known is that, [1, 22.2, p.87][2. Proposition 2.5.7, p.107], $f' \in E \iff kerf$ is closed.
John Webb added that, [5, Proposition 1.9., p.345], $f \in E^+ \iff kerf$ is sequentially closed.
I, hereby, add that $f \in E^b \iff kerf$ is locally closed.
The result substantially improves

[4, Proposition 6.2.15, p.177] A sequentially closed hyperplane of a bornological space is closed. }

to a locally closed hyperplane of a semibornological space is closed.

1. Introduction

Our notation and language shall be pretty standard as found for example in $[1], [6], [2] and [3]$. By a $lcs(E, \tau)$ shall be meant a separated locally convex space. All topologies are assumed separated unless otherwise stated or implied. The field of scalars is $K = R$ or C, the real field or the complex field.
For the $lcs(E, \tau)$ we refer to NOTATON and LANGUAGE and the first paragraph of NOTE 2 in [3] for the notations and language disc, locally converge, local limit point, local null, locally closed set, the algebraic dual $E^\#$, the bounded dual E^b, and the continuous dual E'. By 0 we denote the zero of K and by $kerf$, for

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linear functional $f : (E, \tau) \longrightarrow K$, is meant the kernel, $f^{-1}(0)$, of f. If p is a seminorm on the vector space $E, \sigma p$ denotes the pesudometric topology of p on E. σp is locally convex.
signifies the end or absence of a proof.

LEMMA A:
$[[1], (17.2), p.68]$ Let E be a vector space over K and suppose $\phi \neq A \subseteq E$ a balanced subset of E. Then,$\lambda, \mu \in K$ and $|\lambda| \leq |\mu| \longrightarrow \lambda A \subseteq \mu A$.

LEMMA B :
[2, Proof of **Proposition** 3.5.6, p.208 − 209] For a disc B of $lcs(E, \tau)$ the gauge q_B of B in E_B is a norm, and, the family $\{\lambda_B : \lambda > 0\}$ is a base of neighborhoods of zero of the normed space (E_B, q_B).
LEMMA C :
Let (E, τ) be a lcs and $\{x_n\}_{n=1}^{\infty}$ a local null sequence. If $\beta > 0$, then $\{\beta x_n\}_{n=1}^{\infty}$ is also local null. If $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of scalars, then $\{\alpha_n x_n\}_{n=1}^{\infty}$ is also local null.
Proof:
Let B be a disc of (E, τ) and suppose the sequence $\{x_n\}_{n=1}^{\infty}$ null in $(E_B, \sigma q_B)$. Let $\epsilon > 0$ and consider $\frac{x}{\beta}$. By **Lemma B**, therefore, for some positive integer $N(\epsilon)$, $x_n \in \frac{x}{\beta} B$, for all $n \geq N(\epsilon)$.
Hence $\beta x_n \in \beta \frac{x}{\beta} B$, for all $n \geq N(\epsilon)$.
By **Lemma B**, again, since ϵ was arbitrary, $\{x_n\}_{n=1}^{\infty}$ is local null in (E, τ). This concludes the proof of the first part.
For the second part, suppose $\{x_n\}_{n=1}^{\infty}$ is a local null sequence. Let $\epsilon > 0$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of scalars and > 0 such that

$$|x_n| < \beta \text{ for all } n \tag{4}$$

By **Lemma B**

$$x_n \in \frac{x}{\beta} B, \textbf{ for all } \; n \geq N'(\epsilon)$$

for some positive integer $N'(\epsilon)$. Hence,

$$\alpha_n x_n \in \alpha_n \left(\frac{x}{\beta}\right) B, \text{ for all} n \geq N'(\epsilon)$$

from which follows by (4) and **Lemma A** that

$$\alpha_n x_n \in \beta \left(\frac{x}{\beta}\right) B, \text{ for all } n \geq N'(\epsilon)$$

. That is,

$$\alpha_n x_n \in B, \text{ for all} n \geq N'(\epsilon)$$

Since ϵ was arbitrary, again by **Lemma B**, we have proved the second part.

LEMMA D:
Let (E, τ) be a *lcs*. (i) [4, **Proposition** 3.2.2, p. 82] Let be a disc of (E, τ). The topology, σq_B , of the norm q_B , is finer than the restriction $\tau | E_B$. i.e., $\tau | E_B \leq \sigma q_B$. And so a sequence that locally converges in (E, τ) to $x \in E$, say, also ordinarily $\tau-$converges to x.
(ii) [4, Proposition 5.1.3(i), p. 151]. The sequence $\{x_n\}_{n=1}^{\infty} \tau$-locally converges to $x \in E \Longleftrightarrow \{x_n - x\}_{n=1}^{\infty}$ is local null in (E, τ). (iii) [4 Lemma 5.1.2, p.151] If (E, τ) is metrizable, then a τ-null sequence is local null.

We deduce the following well-known result from **Lemma D**.

THEOREM A :
Convergence is Mackey in a metrizable *lcs*. That is, if $lcs(E, \tau)$ is metrizable, then the sequence $\{x_n\}_{n=1}^{\infty}$ in E ordinarily converges to $x \in E \{x_n\}_{n=1}^{\infty}$ local converge to x.

Proof$\Longrightarrow$**:** Assume $\{x_n\}_{n=1}^{\infty}$ in E locally converges to $x \in E$. then by (ii) of **Lemma D**,

$\{x_n - x\}_{n=1}^{\infty}$ is local null. And so by (i) of **Lemma D**, $\{(x_n - x)\}_{n=1}^{\infty}$ is ordinarily null. By [6, Problem 4.1.1, p.39], therefore, $\{(x_n - x)\}_{n=1}^{\infty}$ ordinarily converges to x.
$\Longleftarrow$: This is the second claim of **Lemma D**(i).
COROLLARY A:
In the scalar space $(K, \sigma|.|)$ convergence is Mackey.

THEOREM B:
Let (E, τ) be a *lcs* and E^b its bounded dual (= the collection of all the bounded linear functionals $f : E \longrightarrow K$ on E). Let $f \in E^{\#}$. Then, $f \in E^b \Longleftrightarrow kerf$ is locally closed.

Proof:
The proof is a modification of the proof of the theorem [5, **Proposition** 1.9, p. 345] whose counterpart for E^b is being established. For clarity we simply have to give it in detail (since different theorems have to be invoked).
$\Longrightarrow$: So, let $f \in E^b$ and a sequence in $kerf$ locally converging to $x \in E$. By **Lemma D**(ii), $\{x_n - x\}_{n=1}^{\infty}$ is local null. And so by **Theorem A** of **Note** 1 of[3], $\{f(x_n - x)\}_{n=1}^{\infty}$ is local null in $(K, \sigma|.|)$. By **Theorem A**, therefore,

$\{f(x_n - x)\}_{n=1}^{\infty}$ is ordinarily null in the scalar space $(K, \longrightarrow |.|)$. Hence, using the linearity of f and [6, **Problem** $4.1.1, p.39$] converges to $f(x)$ ordinarily. But $f(x_n) = 0$ for all n, and so $\{f(x_n - x)\}_{n=1}^{\infty} = (0, 0, ...)$ which converges to 0. By uniquences of limit in $(K, \sigma|.|), f(x) = 0$. Hence, $x \in kerf$. This concludes the proof of the forward implication $\Longrightarrow$.

$\Longleftarrow$: Let $f \in E^{\#}$ and suppose $kerf$ is locally closed. If f is the zero functional we have noting to show. So suppose f is not the zero functional, and so there exist $x_0 \in E$ such that $f(x_0) = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a local null sequence in (E, τ). For each $n, x_n = \alpha_n x_0 + z_n$ where α_n is a scalar and $z_n \in kerf$, since $kerf$ is a hyperplane to which x_0 does not belong. (Hyperplane are the kernels of non-zero linear functionals) [2, Second and third paragraphs of p.41][1, **Theorem** 21.3(a), (b), (e), p. 80]. We show that converges to 0 in $(K, \tau|.|)$ to conclude from **Corollary A** and [3, **Theorem 1** of **Note** 1] that $f \in E^b$. Suppose the opposite that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to 0 in $(K, \sigma|.|)$. And so, since $f(x_n) = \alpha_n$, for all n, the sequence of scalars $\{\alpha_n - x\}_{n=1}^{\infty}$ is not a null sequence. Extracting a subsequence, if necessary, the sequence $\{\frac{1}{\alpha_n}\}_{n=1}^{\infty}$ is a bounded sequence, and so by **LEMMA 3**, $\{\frac{1}{\alpha_n} x_n\}_{n=1}^{\infty}$ is local null. Thus,

$$\left\{\frac{1}{\alpha_n} Z_n - (-x_o))\right\}_{n=1}^{\infty} = \left\{\frac{1}{\alpha_n} Z_n + x_o\right\}_{n=1}^{\infty} = \{\frac{1}{\alpha_n} x_n\}_{n=1}^{\infty}$$

is local null, and so by **Lemma D** (ii), $\{\frac{1}{\alpha_n} x_n\}_{n=1}^{\infty}$ locally converges to $x_0$in (E, τ). But $\{\frac{1}{\alpha_n} x_n\}_{n=1}^{\infty} keff$, for all n, and $kerf$ by hypothesis is locally closed. And so, $x_0 \in keff$, from which follows that $x_0 \in keff$. But this is incompatible with fact that $f(x_0) = 1$.

COROLLARY B: For $lcs(E, \tau)$,
$E' = E^b \Longleftrightarrow$ every locally closed hyperplane in (E, τ) is closed.

Proof : Immediate from (1) of the abstract.

COROLLARY C : A locally closed hyperplane of a semibornological space is closed.

Proof A $lcs(E, \tau)$ for which $E' = E^b$ is called semibornological.
By **Lemma D** (i), a sequentially closed set is locally closed. A bornological space is also semibornological. So **Corollary C** substantially improves the result (4) of the abstract.

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