Nigerian Journal of Mathematics and Applications
Volume **24**, (2015), 100 − 111

http : //www.kwsman.com

A REVIEW OF THE P^{th} MOMENT EXPONENTIAL STABILITY AND ALMOST SURE EXPONENTIAL STABILITY OF NON-LINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS WITH CONSTANT TIME LAG.

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Abstract

The paper tries to connect the pth- moment exponential stability and the almost sure exponential stability of non-linear stochastic delay differential equations (SDDEs) with small time lag. It is generally true that the pth- moment exponential stability of SDDEs and the almost sure exponential stability do not imply each other. By applying the Lyapunov function and an idea of generalized moment inequality as well as Borel - Cantelli lemma, it is established that under certain conditions on the drift and diffusion coefficients, the Pth moment exponential stability implies the almost sure exponential stability. To illustrate the effectiveness of the main result an example is presented.

1. Introduction

Presently, there exists a collection of monographs and research articles that indicate areas of application of deterministic ordinary differential equations (ODEs) and delay differential equations (DDEs). Moreover, the fact that many situations frequently modelled by deterministic ordinary differential equations and delay differential equations can better be modelled by stochastic delay differential equations (SDDEs) has not escaped the attention of qualitative and numerical analysis

Received June 12, 2015. * Corresponding author.
2010 *Mathematics Subject Classification.* 34Dxx & 34K50.
Key words and phrases. P^{th} moment exponential stability, almost sure exponential stability, stochastic delay differential equations, constant time lag, non-linear.
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community. For articles on application areas of DDEs and SDDEs we refer to the important works of Tlusty (2000), Elabbasy et al (2000), Chen (2002), Atay (2003), etc. As far back as 1892, A. M. Lyapunov developed a method for obtaining the stability of deterministic dynamic systems, especially those described by ordinary differential equations. Over the recent years, the concept of stability of differential equations has attracted the research interest of many mathematicians and a lot of research work done. See for instance, the articles of Rodkina and Mao (2001), Caraballo et al (2003), Sadek (2003), Rodkina and Schurz (2004) and Tunc (2006) and some of the references therein. Zhu et al (2015) studied the stability properties of a stochastic differential equation with Markovian switching of the form:

$$dx(t) = f\left(x(t), x(t - r(t), t, r(t)\right)dt + g\left(x(t), x(t - r(t), t, r(t)))\right) dw(t) \tag{1}$$

where $t \geq$ with initial condition $r(0) = t_0 \in \mathbb{N}$ and $x(s) = \epsilon(s) \in C^B_{F_0}([-r, 0], \mathbb{R})$ for $s \in [-r, 0], f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{N} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{N} \longrightarrow \mathbb{R}^{n\times m}$ are measurable mappings and $W(t)$ is an n-dimensional Wiener process. Using the Lyapunov function, the generalized Hanalay inequality and stochastic techniques, the authors derived some sufficient conditions to ensure the p^th moment exponential stability of (1). Yuan and Mao (2003) studied the asymptotic stability in distribution of nonlinear stochastic differential equations with Markovian switching of the form:

$$dX(t) = f\left(X(t), r(t)\right)dt + g(X(t), r(t)))\, dB(t) \tag{2}$$

where $t \geq$ with initial condition $X(0) = x \in \mathbb{R}^n$ where $f : \mathbb{R}^n \times \mathbb{N} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{N} \longrightarrow \mathbb{R}^{n\times m}$, $\mathbb{N} = \{1, 2, 3, ..., , n\}$. The authors assumed the following conditions:
P_1 : for any $(x, i) \in \mathbb{R}^n \times \mathbb{N}$ for any $\epsilon > 0$, there exists a constant $R > 0$ such that $p\left\{\left|x^{X,i}(t)\right| \geq R\right\} < \epsilon \;\forall\; t \geq 0$
P_2 : for any $\epsilon > 0$ and constant K of $\mathbb{R}$ there exists a $T = T(\epsilon, K) > 0$ such that $p\left\{\left|X^{X,i}(t) - X^{Y,i}(t)\right| \geq R\right\} < \epsilon \;\forall\; t \geq T$ whenever $(x, y, i) \in K \times \times T \times N$
The authors then obtained a sufficient criterion for the asymptotic stability of (2) as below:
Theorem 1: [Yuan and Mao (2003)]:
Assume that (2) satisfies the local Lipschitz and the linear growth conditions. That is, if the functions f and g are such that for each $K = 1, 2, 3, ..., n$ there exists an $h_k > 0$ such $|f(x, i) - f(y, i)| + |g(x, i) - g(y, i)| \leq h_k|x - y| \;\forall i \in \mathbb{N},\; x, y \in \mathbb{R}$ with $|x| \vee |y| \leq K$ and there is moreover an $h > 0$ such that $|f(x, i) + g(y, i)| \leq h(1 + |x|)\forall\; x \in \mathbb{R}$ and $i \in \mathbb{N}$. If in addition, (2) has the properties P_1 and $P2$, then (2) is asymptotically stable in distribution.
Yu and Liu (2011) investigated the almost sure asymptotic stability of Euler type methods for neutral stochastic delay differential equations (NSDDEs).

$$(3) \qquad d\left[x(t) - N(t - r(t)\right] = f\left(t, x(t), x(t - r)\right)dt + g(t, x(t), x(t - r)))\, dB(t)$$

on $t > 0$ with initial data $\{x(0) : -r \leq \theta \leq 0\} = \epsilon \in C^b_{F_0}([-r.0], \mathbb{R}^n)$ where $N : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Using the discrete semi martingale convergence theorem, it was shown that the Euler method and the backward Euler method can reduce the almost sure asymptotistability of exact solutions to NSDDEs under additional conditions.

Most of the articles on stability are mainly concerned with an attempt to obtain necessary and sufficient conditions which ensure the stability in probability, stochastic asymptotic stability, almost sure exponential stability, p^{th} moment exponential stability, etc. To the best of the authors knowledge, there is scarcity in the articles devoted to the study of the connection between p^{th} moment exponential stability and the almost sure exponential stability of stochastic delay differential equations with small but constant time lag.

Motivated by the publication of the above interesting results, the aim of the present article is to attempt to connect the pth moment exponential stability and the almost sure exponential stability of stochastic delay differential equations with constant but small time lag driven by an Ito type noise. This type of SDDE is presented in the next section. The article of Driver et. al. (1973) indicated that when the delay r is small, then certain similarities exist between the solutions of deterministic SDDEs and DDEs and those of an equation without delay.

2. Notations and Preliminaries

Unless otherwise stated, throughout this article, the space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ shall denote a filtered probability space with filtration $\{F_t\}_{t\geq 0}$ which is assumed to be right continuous and contains all P-null sets. In this article, the nonlinear stochastic delay differential equation considered is of the form:

$$(4) \qquad dX(t) = f\left(t, x(t), x(t - r)\right)dt + g(t, x(t), x(t - r)))\, dB(t),\ t \in [0, T]$$

where, r is a small constant time lag, $f : \mathbb{R}\times \longrightarrow \mathbb{R}$ and $g : \mathbb{R}\times \longrightarrow \mathbb{R}$ are continuous functions, the initial function $x(t_0) = x_0 \in C^b_{F_0}([-r.0], \mathbb{R}^n)$, the family of all bounded F_0- measurable $C([-r.0], \mathbb{R}^n)$ valued random variables $\xi = \{\xi(s) : -r \leq s \leq 0\}$ such that $Sup_{-r\leq s\leq 0} E|\xi(s)| \leq \infty$, where E denotes the expectation operator corresponding to the probability measure P. is a 1-dimensional Brownian motion defined on the probability space (Ω, F, P) With filtration $\{F_t\}_{t\geq 0}$.

Definition 1

A real valued stochastic process $\left\{x(t)_{t\geq 0}\right\}$ defined on the probability space (Ω, F, P) is called a strong solution of (4) if it is a measurable sample path

continuous process such that $\frac{x}{[0,T]}$ is $\{F_t\}_{t\in[0,T]}$ - adapted, f and g are continuous functions and $x(t)$ satisfies (4) and its initial function.
Assume that there exists an equivalent version or solution $\left\{\bar{x(t)}\right\}_{t\geq 0}$ of (4) with continuous sample paths which is indistinguishable from $\{x(t)\}_{t\geq 0}$, that is if
$P\left\{x(t) = \bar{x(t)}, \text{for all} t \in [-r, T]\right\} = 1$, then the solution $\{x(t)\}_{t\geq 0}$ is called a unique solution. That is to say that $\left\{\bar{x(t)}\right\}_{t\geq 0}$ has almost surely the same sample paths as $\{x(t)\}_{t\geq 0}$ in the sense that $P\left\{\lim_{0\leq t\leq T}\left|\bar{x(t)} - x(t)\right| > 0\right\} = 0$, for any $T > 0$.
Assume that the initial function $x(t) = x_0 \in \mathbb{R}^\delta$ and (4) has the global or strong solution $x(t :, 0, x_0)$. Moreover, suppose that $f(0,t) = 0, g(0,t) = 0,\ \forall\ t \geq 0$ then (4) has a solution $x(t) = 0$ which corresponds to the initial value $x(0) = x_0$ called the trivial solution or equilibrium position of (4).

Definition 2

By stability of the trivial solution of the SDDE (4), one means that for all $\epsilon > 0$ there exists some $\delta = \delta(\epsilon, t_0) > 0,\ t \geq 0$ such that $|x, (t, t_0, x_0)| < \epsilon$ whenever $|x_0| > 0$. If the equilibrium position is not stable, it is said to be unstable.

Definition 3

Let $x, (t, t_0, x_0)$ denote the trivial solution. Then $x, (t, t_0, x_0)$ is said to be almost surely exponentially stable if the Lyapunov exponent is negative, that is, if $\lim Sup \frac{1}{t}|x, (t, t_0, x_0)| < 0\ \forall\ x_0 \in \mathbb{R}^\delta$.

Definition 4

The trivial solution or equilibrium position $x, (t, t_0, x_0)$ of (4) is said to be p^{th} moment exponentially stable if there exist positive constants K such that for all sufficiently small $c, |X, (t, t_0, x_0)|^p \leq K|c|^p e.^{-\beta(t-t_0)}\ \forall t \geq t_0$.
Let $x(t)$ be an Ito process, then $d(x)t = U(t) + V(t)dB(t)$. Assume that $g(t) \in C^2(\mathbb{R})$ which is a twice continuously differentiable function. If $g(x(t)) = L^2$, then $y(t) = g(x(t))$ is an Ito process

$$d(y(t)) = \frac{\partial g}{\partial x}(x(t))dx(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x(t))(dx(t))^2$$

and

$$d(y(t)) = \left(\left[\frac{\partial g}{\partial x}x(t)\right]U(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}x(t)V(t)^2\right)dt + \frac{\partial g}{\partial x}x(t)(V(t))dB(t).$$

This can be re-written as below and called Ito formula.
Define the positive constant $h \in S_h = (0, \infty)$ and denote by $C^{1,2}(S_h, \mathbb{R}^+, \mathbb{R}^+)$, the family of all nonnegative functions $V(x,t)$ on $S_h \times \mathbb{R}^+$ which are continuously once differentiable in t and twice differentiable in x. Note that the function $V(x,t)$ called Lyapunov function. Also for every $C^{1,2}(S_h, \mathbb{R}^+, \mathbb{R}^+)$, define the operator

$LV : S_h, \times\mathbb{R}, \longrightarrow \mathbb{R}$ by

$$L = V_t + \sum_{i=1}^{d} f_t(x,t)V_x(x,t) + \frac{1}{2}\sum_{ij=1}^{d}\left[g^T(x,t)g(x,t)\right]_{ij} V_{x_i x_i}(x,t)$$

Assume that L operates on $V \in C^{1,2}\left(S_h, \mathbb{R}^+, \mathbb{R}^+\right)$ one gets

$$LV(x,t) = V_t(x,t) + V_{xi}(x,t)f(x,t)\sum_{i=1}^{d} + \frac{1}{2}\text{trace}\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right]$$

Let $x(t) \in S_h$ one gets by Ito formula that

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t)$$

Assumptions

The following assumptions are important:

H_1: f and g are continuous functions.

H_2: There exist some constants $K_1 > 0, K_2 > 0, K_3 > 0$ and K_4 given some $\lambda_1, \lambda_2, \beta_1, \beta_2 \in \mathbb{R} f$ and g such that

$$|f(\lambda_1,\beta_1) - f(\lambda_2,\beta_2)| \leq K_1|\lambda_1 - \lambda_2| - K_2|\beta_1 - \beta_2|$$

and

$$|g(\lambda_1,\beta_1) - g(\lambda_2,\beta_2)| \leq K_3|\lambda_1 - \lambda_2| - K_4|\beta_1 - \beta_2|$$

That is, the functions f and g satisfy the Lipschitz condition (Mao (1977)). H_3: There exist $M_1 > 0, M_2 > 0$ such that for all $\lambda_1, \lambda_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$|f(\lambda_1,\lambda_2)| \leq M_1\left(1 + |\lambda_1|^2 + |\lambda_2|^2\right)$$

and

$$|g(\beta_1,\beta_2)| \leq M_2\left(1 + |\beta_1|^2 + |\beta_2|^2\right)$$

That is, the functions f and g satisfy the linear growth condition. The conditions of assumptions H_2 and H_3 are imposed to guarantee the existence and uniqueness of the solution of the SDDE (4).

The following important result which is helpful to our main result is called Borel Cantelli Lemma. It can be found in Arnold (1974) and Mao (1977).

Lemma 1

Assume that $\{A_n\}$ is an arbitrary sequence of events in a probability space (Ω, F, P) . Also let $A = \{w : w \in A_n \text{for infinitely many n}\}$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P\left(\lim_{n\longrightarrow\infty} Sup(A_n)\right) = 0$. If $A_n \subset F$ and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P\left(\lim_{n\longrightarrow\infty} Sup(A_n)\right) = 1$.

Presented below is an important example of moment inequality for stochastic integrals which is easily verified using Ito formula. It is called the Burkholder-Davis-Gundy inequality. See Mao (1977) Theorem 1.7.3.

Theorem 2
Let $g : L^2(\mathbb{R}^+, \mathbb{R}^{d\times m})$. Define for $t \geq 0$, $x(t) = \int_0^t g(s)dB(s)$ and $A(t) = \int_0^t |g(s)|^2 d(s)$. Then for every $p > 0$, there exist positive constants c_p, C_p which depend entirely on p such that

$$c_p E|A(t)|^{\frac{1}{2}} \leq E\left(Sup_{0\leq s\leq t}|x(t)|^p\right) \leq C_p E|A(t)|^{\frac{1}{2}} \text{ for } t \geq 0$$

. In particular one may take

$$\begin{aligned} &c_p = (\frac{p}{2})^p,\ C_p = (\frac{32}{p})^{\frac{2}{p}} \text{ for } 0 \leq p \leq 2 \\ &c_p = 1,\ C_p = 4 \text{ for } p = 2 \\ &c_p = (2p)^{-\frac{p}{2}},\ C_p = \left(\frac{P^{p+1}}{2(p-1)}\right)^{\frac{2}{p}} \text{ for } p \geq 2 \end{aligned}$$

3. Main Results

In this section, a connection will be established between the P^{th} moment exponential stability and the almost sure exponential stability of stochastic delay differential equations.
Consider the impulsive stochastic delay differential system:

$$\begin{aligned} &dx(t) = f\left((t, x(t), x(t-r)) + g(t, x(t), x(t-r))\right) dB(t) \\ &\Delta x(t_k) = x(t_k) - x(t_k^*),\ K \in N \\ &x(t_0) = \zeta(t_0 + \theta), \theta \in [-r, 0] \end{aligned} \tag{5}$$

where $\{t_k : k \in N\}$ is a strictly non-decreasing sequence so that $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and $x(t^*) = \lim_{s\uparrow t} x(s)x(t)$ is defined by $x_t(t_\vartheta) = \zeta(t+\theta), \theta \in [-r, 0]$, where as, f and g are Borel measurable functions. In addition, we suppose that $f(t, 0) = 0, g(t, 0) = 0$ and $I(t, 0) = 0$ for all $t \geq t_0$. Then the delay differential system (5) admits a trivial solution or equilibrium position.
The following useful result states that under certain conditions the effect of the noise perturbation does not destroy the Pth moment exponential stability of a given system of stochastic delay differential equations. It is a special case of the result found in Wu et al (2012)**Theorem 3.1**.
Proposition 1
Assume that there exist positive constants k_1, k_2, P, λ such that
$H_4 :\ k_1|x|^p \leq V(t, x) \leq k_2|x|^p$
$H_5 :\ $ for all$[t_{k-1}, t_k]\ K \in N$

$$ELV(t, \vartheta(\theta)) \leq \lambda EV(t, \vartheta(\theta)) \tag{6}$$

provided that $\varphi \in L^p_{f_t}([-r,0],\mathbb{R}^\delta)$ satisfying

$$EV(t+\varphi,\varphi(\theta)) \leq qEV(t,\varphi(\theta)), \theta \in [-r,0]$$

$H_6:$ There exists a positive constant $\mu > 1$ such that

$$EV(t_k, x+I(t_k,x)) \leq \mu EV(t_k,x) \tag{7}$$

$H_7:$ $e^{\lambda t} \leq q, T_\alpha > In\frac{\mu}{\lambda}$

Then the equilibrium position of (5) is P^{th} moment potentially stable.

Proof

Following from H_4, one sees that

$$EV(t,x(t)) \leq Me^{-\lambda(t-t_0)}\ t \in [t_0-r,t_0] \tag{8}$$

Where $M = k_2E\|\zeta\|^p$ and such

$$EV(t,x(t)) \leq Me^{-\lambda(t-t_0)}\ t \in [t_0,t] \tag{9}$$

Assume that (9) is not true, then exist some $t \in (t_0,t)$ such that $EV(t,x(t)) > Me^{-\lambda(t-t_0)}$ Choose $t^{-1} = inf\left\{t \in (t_0-r,t_0) : EV(t,x(t)) > Me^{-\lambda(t-t_0)}\right\}$. It follows that $t^- \in [t_0,t]$ and $EV(t^-,x(t^-)) = Me^{-\lambda(t-t_0)}$. Also there exists a sequence $\{S_m\}_{m\geq 1}$ and $\{S_m\} \uparrow t^-$ such that

$$EV(S_m,x(S_m)) > Me^{-\lambda(t-t_0)}\ S_m \in (t^-,t) \tag{10}$$

$$EV(t^-+\theta,\varphi(\theta)) \leq e^{-\lambda\theta}EV(t^-,x(t^-)) \leq qEV(t^-,x(t^-)) \tag{11}$$

This implies that

$$EV(t^{-1},\varphi(\theta)) \leq \lambda EV(t^-,\varphi(0)) \tag{12}$$

while one sees from the solution that the functionals V, EV are continuous on (t^{-1},t) and as one gets:

$$EV(T,\varphi(0)) \leq -\lambda EV(t,\varphi(0))\ t \in [t^{-1},t] \tag{13}$$

for sufficiently small $h > 0$. Applying Itos formula, one obtains

$$EV(t^-+h) = EV(t^-) + \int_{t^-}^{t^-+h} ELV(s,\varphi(\theta))ds \tag{14}$$

which results in

$$EV(t^-+h) \leq Me^{-\lambda(t^-+t_0)}e^{-\lambda h} \tag{15}$$

This contradicts (10) and as such (9) holds. One can now assume that

$$EV(t,X(t)) \leq M\mu^{k-1}e^{\lambda(t,t_0)},\ t \in [t_k,t_{k+1}) \tag{16}$$

which similarly holds for

$$EV(t,x(t)) \le M\mu^k e^{\lambda(t,t_0)},\ t \in [t_k, t_{k+1}) \tag{17}$$

Using (16) andH_6, one sees that (16) holds for $t = t_k$. Assume that (17) is not true, then there exists some (t_k, t_{k+1}) such that
$EV(t,x(t)) > M\mu^k e^{-\lambda(t,t_0)}$
Setting $t^- = inf\left\{t \in (t_k, t_{k+1}) : EV(t,x(t)) \ge M\mu^k e^{-\lambda(t,t_0)}\right\}$ one gets $t^- \in [t_k, t_{k+1})$ and $EV(t^-, x(t^-)) = M\mu^k e^{-\lambda(t^-,t_0^-)}$. As such there exists a sequence $\{S_m\}\ m \ge 0$ and $S_m \uparrow t^-$ such that

$$EV(S_m, x(S_m)) \le M\mu^k e^{-\lambda(S_m,t_0)},\ S_m \in [t^-, t_{k-1}) \tag{18}$$

For $\theta \in [-r, 0]$, there exists an integer $j \in [0, k]$ such that $t^- + \theta \in [t_j, t_{j+1})$ and as such,

$$EV(t^- + \theta)) \le M\mu^j e^{-\lambda(t^-+\theta-t_0)} \le M\mu^k e^{-\lambda(t^-+\theta-t_0)} \le qEV(t^- + \theta) \tag{19}$$

It thus follows from (H_5) that

$$ELV(t^-, \varphi(\theta)) \le \lambda EV(t^-, \varphi(0)) \tag{20}$$

Similarly, this can lead to a contradiction which implies that (17) holds. It follows from definition of an average impulsive interval t_0, t of the impulsive sequence $\{t_k\}_{K \in N}$ that there exist positive numbers T_α ans N_0 such that

$$\frac{t-t_0}{T_\alpha} - N_0 \le t - t_0 N_0 \le \frac{t-t_0}{T_\alpha} + N_0,\ t \ge t_0 \tag{21}$$

where $N(t - t_0)$ denotes the number of impulsive sequence in the interval $t - t_0$. Consequently,
(22)

$$EV(t,x(t))) \le M\mu^{N(t-t_0)}e^{-\lambda(t-t_0)} \le M\mu^{N_0}e^{\frac{-\lambda(t-t_0)In\mu}{T_\alpha}}e^{-\lambda(t-t_0)} \le M\mu^{N_0}e^{-\delta(t-t_0)}$$

where $\delta = \frac{\lambda - In\mu}{T_\alpha}$ and hence the equilibrium position (5) is P^{th} moment exponentially stable.
The following proposition specifies that the equilibrium position of the stochastic delay differential system (5) is almost sure exponentially stable under some additional conditions. It is a similar to the result found in Wu et al (2012) **Theorem 3.7**.

Proposition 2
Assume that $p \ge 1, \beta = inf\{t_k - t_{k-1}\}$ and there exists a positive integer u such that $(u-1)\beta < r < u\beta$. Assume that the conditions in Proposition (1) hold. If in addition, there exists a constant $U > 0$ such that

$$E\left(|f(t,x(t),x(t-r),\varphi)|^p + |g(t,x(t),x(t-r),\varphi)|^p\right) < USup_{\theta\in[-r,0]}E|\varphi(\theta)|^p$$

Then the equilibrium position of (5) is almost surely exponentially stable.
In the remaining part of the work, the main result which establishes a connection between the P^{th} moment exponential stability and the almost sure exponential stability of the nonlinear stochastic delay differential equation (4) is presented.

Theorem 3
Assume that the functions f and g are uniformly Lipschitz continuous and there exists that there is a positive constant C such that

$$x^T f(x,t) \leq |g(x,t)|^2 \leq |x|^2 \tag{23}$$

for all $(x,t) \in \mathbb{R}^d \times [t_0, \infty)$. Then the p^{th} moment exponential stability of the trivial solution of (4) implies the almost sure exponential stability.
Proof: Recall that the trivial solution of the SDDE (4), that is, $x(t) = x(t : t_0, x_0)$ is almost surely exponentially stable if $\lim Sup_{t \longrightarrow \infty} \frac{1}{t} |x(t : t_0, x_0)| < 0$. Choose a fix value $x_0 \in \mathbb{R}^\delta$ such that $x_0 \neq 0$. By definition of the p^{th} moment exponential stability of the trivial solution of the SDDE (5), there exists a pair of constants such that

$$E\,|X(t)|^p \leq K\,|x_0|^p\, e^{-\beta(t-t_0)} \text{ on } t \geq t_0 \tag{24}$$

Applying Ito formula and the condition in (4) for $n = 1, 2, 3, ...$ with $t_0 + n - 1 \leq t \leq t_0 + n$ one gets

$$|x(t)|^P = |x(t_0+n-p)|^p + \int_{t_0+n-p}^{t} p|x(s)|^{p-2} x^T f(x, x(s), x(s-r)) ds$$
$$\frac{1}{2}\int_{t_0+n-p}^{t} \left[p|x(s)|^{p-2}|g(x, x(s), x(s-r))|^2 + p(p-1)|x|^{p-4}|x^T g(x, x(s), x(s-r))|^2\right] ds$$
$$+ \int_{t_0+n-p}^{t} p(p-1)|x|^{p-4}|x^T g(x, x(s), x(s-r))|^2 ds(B)$$

$$\leq |(t_0+n-p)|^p + k_1 \int_{t_0+n-p}^{t} |x(s)|^p ds + \int_{t_0+n-p}^{t} p|x(s)|^{p-2} x^T g(x, x(s), x(s-r)) ds(B)$$

where $c_1 = \frac{pC + p(P+(1+|p-1|))C}{2}$ and as such,

$$\begin{aligned} &E\left(Sup_{t_0+n-1 \leq t_0+n}|x(t)|^p\right) \leq E\,|x(t_0+n-1)|^p + k_1 \int_{t_0+n-1}^{t} E|x(s)|^p ds \\ &+ E\left(Sup_{t_0+n-1 \leq t_0+n} \int_{t_0+n-1}^{t} p|x(s)|^{p-2} x^T(s) g(x, x(s), x(s-r)) ds(B)\right) \end{aligned} \tag{25}$$

Applying Burkholder- Davis Gundy inequality, one has that

$$E\left(Sup_{t_0+n-1\leq t_0+n}\int_{t_0+n-1}^{t_0+n} p|x(s)|^{p-2}x^T(s)g(x,x(s),x(s-r))ds(B)\right)$$

$$\leq 4\sqrt{2}E\left(Sup_{t_0+n-1\leq t_0+n}\int_{t_0+n-1}^{t_0+n} p^2|x(s)|^{2(p-2)}|x^T(s)g(x,x(s),x(s-r))|2ds(B)\right)^{\frac{1}{2}}$$

$$\leq 4\sqrt{2}E\left(Sup_{t_0+n-1\leq t_0+n}|x(s)|^p\int_{t_0+n-1}^{t} p^2C|x(s)|^p|\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}E\left(Sup_{t_0+n-1\leq t_0+n}|x(s)|^p\right)+16P^2C\int_{t_0+n-1}^{t_0+n} E|x(s)|^p|ds$$

which follows from the inequality $\sqrt{mn}\leq\frac{(m+n)}{2}$. One now uses (25) and obtains

$$E\left(Sup_{t_0+n-1\leq t_0+n}|x(t)|^p\right)\leq 2E|x(t_0+n-1)|^p+k_2\int_{t_0+n-1}^{t_0+n} E|x(s)|^p ds$$

where $K_2=2K_1+32P^2C$

$$E\left(Sup_{t_0+n-1\leq t_0+n}|x(t)|^p\right)\leq k_3e^{-\beta(n-1)} \tag{26}$$

where $K_3=K|x_0|^p(2+k_2)$ Choose an arbitrary element $\epsilon\in(0,\beta)$. One gets from (26) that

$$P\left\{Sup_{t_0+n-1\leq t_0+n}|x(t)|^p>k_3e^{-(\beta-\delta)(n-1)}\right\}$$
$$\leq e^{-(\beta-\delta)(n-1)}E\left(Sup_{t_0+n-1\leq t_0+n}|x(t)|^p\right)\leq k_3e^{-\delta(n-1)}$$

Following from the Borel Cantelli lemma, one sees that for almost all $w\in\Omega$

$$Sup_{t_0+n-1\leq t_0+n}|x(t)|^p\leq e^{-(\beta-)(n-1)} \tag{27}$$

$$\frac{1}{t}\log|x(t)|\leq\frac{1}{pt}\log\left(|x(t)|^p\right)$$

$$\frac{-(\beta-)(n-1)}{p(t_0+n-1)},\ t_0+n-1\leq t\leq t_0+n,\ n\geq n_0$$

Therefore,

$$\lim_{t\longrightarrow\infty} Sup\frac{1}{t}\log|x(t)|\leq\frac{-(\beta-\epsilon)}{p},\text{a.s}$$

Since $\epsilon>0$ is arbitrary, we holds for all but finitely many n. As such, there exists an $n_0=n_0(w)$ for all $w\in\Omega$ excluding a $p-$null set for which (27) holds whenever $n\geq n_0$. Therefore, for all $w\in\Omega$ must have $\lim_{t\longrightarrow\infty} Sup\frac{1}{t}\log|x(t)|\leq\frac{-(\beta-\epsilon)}{p}$, a.s. It follows from definition that the trivial solution of (4) is almost surely exponentially stable as required.

4. Numerical Example

To justify the effectiveness of the main result, we present the following numerical example which follows from the result in **Proposition 1** Consider the impulsive stochastic delay differential system

$$\begin{aligned} & d(t) = [-x(t) + 0.125x(t-0.5)]\,dt + 0.5x(t-0.5)dB(t), t \neq t_k, t \geq t_0 \\ & \Delta x(t) = 0.2x(t^-), k \in N \end{aligned} \tag{28}$$

Take $p = 2, V(t,x) = x^2, k_1 = k_2 = 1, q = \frac{4}{3}$ and apply **Proposition 1**. One gets that

$$ELV(t,x) = -2E|x|^2 + 0.25E|x|^2 + 0.25Ex(t)(t-0.5) + 0.25E\,|x(t)(t-0.5)|^2$$

$$\leq 1.5E|x(t)|^2 + \frac{3q}{4}E|x(t)|^2 = 0.5E|x(t)|^2$$

By allowing $\lambda = 0.49$ one sees that $ELV(t,x) < -\lambda LV(t,x)$. It follows that $\mu = 1.44\ e^{-\lambda r} = 1.278 < q = \frac{4}{3},\ T_\alpha \geq In\frac{\mu}{\lambda} = 0.744$ For all$\epsilon > 0$ one can now set $t_{(2k-1)} - t_{2(k-1)} = 1.488,\ t_{2k} - t_{2k-1} = \epsilon, k \in N$ and as such it follows from **Proposition 1** that the trivial solution of system (28) is p^{th} moment exponentially stable. Again setting $u = 10$ in Proposition 2, one sees that the trivial solution of the system (28) is also almost surely exponentially stable.

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