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ON SPECIAL CASES OF OPIAL'S AND HARDY'S INEQUALITY

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Abstract

In this paper, we establish the relationship between Opial-type and Hardy-type integral inequalities which extend Anthonio and Rauf Opial-type inequalities for convex function.

1. Introduction

The following interesting classical integral inequalities were stated and proved by Opial and Hardy respectively:

Theorem 1.1. *Let $x(t) \in C'[0,b]$ be such $x(0) = x(b) = 0$ and $x(t) > 0$ in (0, b), then*

$$\int_a^b |x(t)x'(t)|dt \leq \frac{b}{4}\int_a^b (x'(t))^2 dt \tag{1}$$

where $\frac{b}{4}$ in the best possible constant. (See [9])

Theorem 1.2. *For $f(x) \geq 0$ and $p > 1$,*

$$\int_0^\infty \left[\frac{1}{x}f(t)dt\right]^p dx \leq q^p \int_0^\infty f^p(t)dt \tag{2}$$

where $q = \frac{p}{p-1}$ is the best possible constant. (see [5])

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In view of the usefulness of these inequalities in analysis and its applications generally, many authors have established the necessary and sufficient conditions on p, q, v, w for the Hardy-type inequality:

$$\left[\int_a^b |u(x)|^q w(x)dx\right]^{\frac{1}{q}} \leq C\left[\int_a^b |u'(x)|^p v(x)dx\right]^{\frac{1}{p}} \tag{3}$$

to hold, where C is a constant depending on p and q. See [7] and the reference therein.

Theorem 1.3. *Let g be continuous and non-decreasing on $[a,b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then*

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}} \tag{4}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{q\delta+p}\right)^{\frac{p}{q}} g(b)^{\frac{q\delta+p}{p}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q(p-1)}{p}} > 0 \tag{5}$$

(See [2])

It was pointed out by Oguntuase (2009) that the constant $C(a,b,p,q,\delta)$ at the right hand side of (4) is wrong and stated the following:

Theorem 1.4. *Let g be a continuous and nondecreasing function on $[a,b], 0 \leq a < b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and let $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then,*

$$\left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f^p(x) dg(x)\right]^{\frac{1}{p}} \tag{6}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{p}{q\delta+p}\right)^{\frac{1}{q}} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{p-1}{p}} \left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}}\right)^{\frac{1}{q}}$$

(See [8])

Some special cases of the result were obtained. The purpose of the work is to extend the work of Anthonio and Rauf (2015) with a view to obtain the relationship between Opial-type and Hardy-type classical inequalities.

2. Main Results

Throughout this paper, we shall define: $h(x,t) = g(x)^{\zeta} f(t)^p g(t)^{p(1+\zeta)} d\mu(t)$ and $d\mu(t) = g(t)^{-(1+\zeta)} dg(t)$

The statement of the main results are as follows:

Lemma 2.1. *Let $h(x,t)$ be non negative, $x \geq 0, t \geq 0$ and $\mu \geq 0$ be non decreasing.*
Let$-\infty \leq 0 \leq x < \infty$, then the following holds:

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(x)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{7}$$

Proof:
Let $f(t)$ and $g(x)$ are absolutely continuous functions.

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = \left[\int_a^x g(x)^{\zeta} f(t)^p g(t)^{(p-1)} g(t)^{(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{8}$$

$$\left[\int_a^x h(x,t)d\mu(t)\right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \tag{9}$$

which complete the proof of the Lemma.

Lemma 2.2. *Let $h(x,t)$ be non negative, $t \geq 0, x \geq 0$, $p \geq 1$ and $\mu \geq 0$ be non decreasing.*
Let$-\infty \leq 0 \leq x < \infty$, then the following holds:

$$\left[\int_a^x d\mu(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{-\zeta}}{-\zeta}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}} = [g(x) - g(a)]^{\frac{(p-1)\zeta\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}} \tag{10}$$

Proof:

$$\left[\int_a^x d\mu(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\int_a^x g(t)^{-(1+\zeta)} dg(t)\right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{(-1-\zeta+1)}}{-1-\zeta+1}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}}$$

$$= \left[\frac{g(t)^{-\zeta}}{-\zeta}\Bigg|_a^x\right]^{\frac{(1-p)\nu}{p}} = \left[g(x)^{\zeta} - g(a)^{\zeta}\right]^{\frac{(p-1)\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}} \tag{11}$$

This completes the proof of the Lemma.

Lemma 2.3. *Suppose all the conditions of Lemma 2.2 hold, then we have:*

$$\left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)dg(t)\right]^\nu \tag{12}$$

Proof:

$$\left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu = \left[\int_a^x \left(g(x)^\zeta f(t)^p g(t)^{p(1+\zeta)}\right)^{\frac{1}{p}} g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x \left(g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)}\right) g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)} g(x)^{-\zeta+1} dg(x)\right]^\nu$$

$$= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) dg(x)\right]^\nu$$

$$\Rightarrow \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(t)\right]^\nu \geq g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)dg(t)\right]^\nu \tag{13}$$

The proof is completed.
Using the well known Jensen's inequality of the form: (see [3] and [4])

$$\int_a^x h(x,t)d\mu(s) \geq \left[\int_a^x d\mu(s)\right]^{1-p} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(s)\right]^p \tag{14}$$

Raising both sides of inequality (15) to power $\frac{\nu}{p}$ yields

$$\left[\int_a^x h(x,t)d\mu(s)\right]^{\frac{\nu}{p}} \geq \left[\int_a^x d\mu(s)\right]^{\frac{(1-p)\nu}{p}} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\mu(s)\right]^\nu \tag{15}$$

Theorem 2.4. *Let $f(t)$ and $g(t)$ be a absolutely continuous function which is non-decreasing on $[a,b], 0 \leq a \leq b < \infty$. Suppose that $p \geq \nu \geq 1, \zeta > 0$ and $f(x)$ is Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Then,*

(16)

$$\left[\int_a^x f(t)dg(t)\right]^\nu \leq (-\zeta)^{\frac{(1-p)\nu}{p}} [g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}}$$

Proof :

Multiply both side of (16) with $g(x)^{\frac{\zeta\nu}{p}}$ and by combining the results of Lemma 2.1, 2.2 and 2.3 in inequality (17), we get

(17)

$$g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t)\right]^{\frac{\nu}{p}} \geq (-\zeta)^{\frac{(p-1)\nu}{p}} [g(x)-g(a)]^{\frac{(p-1)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^b f(t)dg(t)\right]^\nu$$

that is
(18)
$$(-\zeta)^{\frac{(p-1)\nu}{p}}[g(x)-g(a)]^{\frac{(p-1)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu} \le g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}$$
which implies
(19)
$$g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu} \le (-\zeta)^{\frac{(1-p)\nu}{p}}[g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}$$
Integrating both sides of (20) with respect to $g(x)$ on $[a,b]$ and then raising both sides to power $\frac{p}{\nu}$ to obtain the following inequality:

$$\left[\int_a^b g(x)^{\frac{\zeta\nu}{p}}\left[\int_a^x f(t)dg(t)\right]^{\nu}dg(x)\right]^{\frac{p}{\nu}} \le (-\zeta)^{\frac{(1-p)\nu}{p}}\left[\int_a^b [g(x)-g(a)]^{\frac{(1-p)\zeta\nu}{p}}g(x)^{\frac{\zeta\nu}{p}}\right.$$

$$\left.\times\left[\int_a^x f(t)^p g(x)^{(p-1)(1+\zeta)}dg(t)\right]^{\frac{\nu}{p}}dg(x)\right]^{\frac{p}{\nu}} \tag{20}$$

which is the type of the result in [2] and [8] generalization.
This has successfully suggested that Hardy-type and Opial-type of the two classical inequalities can be found in Jensen's inequality for convex function.

References

[1] Adeagbo-Sheikh, A. G. and Fabelurin. O. O., (2011). On a Bessack's Inequality related to Opial's and Hardy's. *Krag. J. Math.* **35** (1), 145-150.

[2] Adeagbo-Sheikh, A. G. and Imoru, C. O., (2006). An Integral Inequality of the Hardy's -type. *Krag. J. Math.* **29**, 57-61.

[3] Anthonio Yisa Oluwatoyin and Rauf Kamilu, (2015). On New Variations of Opial-type Integral Inequalities, *Global J. of Math.* **3**(1), 226-231.

[4] Anthonio Y. O., Salawu S. O. and Sogunro S. O., (2014). Dual Results of Opial's inequality, *IOSR J. Math.* **10**, 01-04.

[5] Hardy, G. H. (1925). G.H. Hardy, Notes on some points in the integral calculus, LX. An inequality between integrals, Messenger of Math. 54, 150?56.

[6] Imoru, C. O. and Adeagbo-Sheikh, A. G., (2013). On an Integral Inequality of the Hardy's-type, Austral J. of Math. Ana. and App. **5**(4), 56-64.

[7] Kufner, A., Maligranda, L. and Persson,L-E. (2007). The Hardy Inequality. About its History and some Related Results, Vydavatelsky Servis Publishing House, Pilsen.

[8] Oguntuase, James Adebayo (2009). Remark on an Integral Inequality of the Hardy type, *Krag. J. Math.* **32**, 133-138.

[9] Opial, Z., (1960). Sur une in*tégalité*, *Ann., Polon. Math.* **8**, 29-32.