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METRICIZATION ON FULL TRANSFORMATION SEMIGROUP

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Abstract

Representation of positional transformation semigroup is discussed using Hamming distance function. It is shown that transformation semigroup is metricizable.

1. Introduction

Let $X_n = \{1, 2, 3...n\}$ be a natural ordering of numbers and
$\alpha : Dom(\alpha) \subseteq X_n \longmapsto Im(\alpha) \subset X_n$.

The full transformation T_n is defined as the set of all functions $\alpha : X_n \longrightarrow X_n$ on a non - empty set $X_n = \{1, 2, \ldots n\}$. T_n is a semigroup under composition of functions.

A metric space is defined by Kreyzig, (1978) as a pair (X_n, d) where X_n is a set and d is a metric on X_n(or distance function on X_n) that is a function defined on $X_n \times X_n$ such that for all $x, y, z \in X_n$ we have:
(i) d is real - valued, finite and nonnegative i.e. $d(x, y) > 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$ (Coincidence axiom);
(iii) $d(x, y) = d(y, x)$ (Symmetric);
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ Triangular Inequality.

Property (ii) is referred to as "'coincidence axiom"' by Niemytzki in (1927). David, (1979) defined a metric operation ,d satisfying $d(x, x) = 0, d(x, y) = 0 \Rightarrow x = y$ and postulate (iv) above as being necessary and sufficient for a finite lattice, L to be modular, distributive and Boolean.

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Tripathi and Kumar, (2012) defined a cone metric space (X, d) as a space that satisfies the four axioms above on a Banach space, E and it's subset.

Frink in one of his papers in (1937), stated that the metrization problem is concerned with conditions under which a topological space is metrizable, that is, is homeomorphic to a metric space. He defined a space to be metric if to every two points a and b, a non - negative real number ab is assigned satisfying the well known conditions:

I. $ab = 0$ if and only if $a = b$;

II. $ab = ba$, (symmetry);

III. $ac \leq ab + bc$, (triangle property).

He opined that a metrization theorem is usually proved by introducing such a distance function into the space. Also, that it is easier to introduce first into a topological space a distance function satisfying the following conditions IV or V instead of III:

IV. If $ab < \epsilon$ and $cb < \epsilon$, then $ac < 2\epsilon$ (generalized triangle property);

V. For every $\epsilon > 0$, there exists $\phi(\epsilon) > 0$ and that if $ab < \phi(\epsilon)$ and $cb < \phi(\epsilon)$ then $ac < \epsilon$ (uniformly regular). Condition V reduces to IV if $\phi(\epsilon) = \epsilon/2$.

Let X be the set of all ordered triples of zeros and ones. A metric d on X is defined by $d(x, y) =$ number of places where x and y have different entries. $d(x, y)$ is called the Hamming distance between x and y. Hamming, (1950) claimed that this space and similar spaces of n - tuples play a role in switching and automata theory and coding. The distance function of full transformation semigroup T_n is denoted by $d(x, y)$ for any two elements $x, y \in T_n$. $d(x, y)$ denotes the number of positions in which x and y have different values as images, for all $x, y \in T_n$ for the purpose of this study.

Standard definition of terms on metric space and theory of Semigroup are contained respectively in Kreyzig, (1978) and Ganyushkin and Mazorchuk, (2009).

Thus the distance function is the sum total of all the positional differences between elements of the full transformation semigroup, T_n. For example, let

x = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ and y = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, $d(x, y) = 2$.

There are two different positional differences in x and y. Let $\xi_i \in \text{Im }(x)$ and $\eta_i \in \text{Im } y$, where Im stands for image. One line notation is preferred in this work as 211 means $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$.

Young and Chung, (1999) defined a selfmap f on a metric space (X, d) to be $\alpha-$contractive if it satisfies

$d(f(x), f(y)) \leq \alpha(d(x, y)), x, y \in X_n$.

Howie, (2006) defined a contraction as a transformation for which

$|x\alpha - y\alpha| \leq |x - y|$ for $x, y \in X_n$.

Catarino and Higgin, (1999) defined a set $C \subseteq X_n$ to be convex if C has the form $i, i+t$ for some $i \in X_n$ and $0 \leq t \leq n-1$.

2. RESULTS

The metric space $\Re^2$, called the Euclidean plane is obtained if we take the set of ordered pairs of real numbers written $X = (\xi_1, \xi_2), Y = (\eta_1, \eta_2)$, etc and the Euclidean metric defined by

. $D(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$.

. In this work, metric space induced transformation is also expressed as above with a difference in inequality. That is, take a set of elements of a semigroup, $S = \{x, y, z, \ldots\}$ such that $\text{Im}x = (\xi_1, \xi_2, \xi_3, \ldots)$, $\text{Im}y = (\eta_1, \eta_2, \eta_3 \ldots)$ and $\text{Im}z = (\delta_1, \delta_2, \delta_3 \ldots)$. The Euclidean metric on transformation semigroup is defined by

. $D(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 \ldots}$.

. The metric defined on the semigroup is Hamming distance function counting the differences in the order of images of elements of the semigroup of transformations as defined in section 1. $d(x,y) = 2$ in the example used in section 1 while the corresponding value for the Euclidean representation is

. $\sqrt{(2-1)^2 + (1-1)^2 + (1-2)^2} = \sqrt{2}$.

. The metric defined in this work also satisfies the first three properties stated by Frink, (1937) but the fourth property is satisfied by introducing equality as it can be seen in Theorem 2.1.

. Theorem 2.1: Let $x_1, x_2, x_3 \in T_n$ and $\epsilon > 0$ for $\epsilon \in X_n$. If $d(x_1, x_2) < \epsilon$ and $d(x_2, x_3) < \epsilon$ then $d(x_1, x_3) < 2\epsilon$(generalized triangle property for full transformation semigroup).

Proof: Choosing any $\epsilon \in X_n$ obviously implies that $\epsilon > 0$. The metric

$d(x_1, x_2) > 0 \forall x_1, x_2 \in X_n$ if $x_1 \neq x_2$

and also satisfies

$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$.

Hence the proof of the theorem.

. **Theorem 2.2:** $D(x,y) \leq d(x,y)$ when convexity is preserved between x and y, especially x.

Proof:

Convexity is as defined by Catarino and Higgin (1999). One of the connections between Euclidean and Hamming distances is that, given

$i, i+t$ for some $i \in X_n$ and $0 \leq t \leq n-1$, then $D(x,y) \leq d(x,y)$.

. **Theorem 2.3:** (X, d) is a metric space for

$D(x,y) = \{^{0,}_{n \in N,} \quad ^{x=y}_{x \neq y}, x, y \in S$.

Proof: The distance between a point, x and itself is zero. One of the properties of a metric space is that, the metric $d(x,y) = 0$ if and only if $x = y$. This means reflexivity since $x - y = 0$ and $x = y$ depict that x relates to itself or y relates to itself, which is true for the distance function $D(x,y)$.

If $x - y \neq 0$, then $x \neq y$ implying that $0 < D(x, y) \leq n$ for all $x, y \in S$.

. **Theorem 2.4:** $\sum d(x, y) = n^{2n}$.

Proof: The metric on $d(x, y)$ for $x = y$ is zero and the cardinality of full transformation semigroup is n^n. The length of image, $|Imx|$ was used to partition positional behaviour of transformation semigroup with respect to metric space. There are n^n summed up partitions for each n and each positional difference. Hence the total metric positional differences is $(n^n)(n^n) = n^{2n}$.

. **Theorem** 2.5 : The distance function is symmetrical with 0's as the diagonal line of symmetry.

Proof: Let $x_1, x_2, x_3, \dots x_n \in T_n$, and $d(x, y) = d(y, x)$ as one of the properties of metric space. The distance between x_1 and x_2 is the same as the distance between x_n and $x_n - 1$ and all the elements follow this pattern by inspection. That is,

$d(x_1, x_2) = d(x_n, x_{n-1})$,

$d(x_1, x_3) = d(x_n, x_{n-2})$,

$\vdots$

$d(x_n, x_n) = d(x_1, x_1)$.

The diagonal elements are

$d(x_1, x_1), d(x_2, x_2), \dots d(x_n, x_n)$

with each equals zero.

Illustratively, the metric distance defined on transformation semigroup form a square table with mirrored equal sides. The figures generated have the form

$d(x_1, x_1), d(x_1, x_2), \dots d(x_1, x_n)$

$= d(x_n, x_1), d(x_n, x_2), \dots d(x_n, x_n)$

and

$d(x_n, x_n), d(x_n, x_{n-1}), \dots d(x_n, x_1), d(x_n, x_1)$

$= d(x_{n-1}, x_1), d(x_{n-2}, x_1), d(x_{n-3}, x_1) \dots d(x_1, x_1)$

in reverse order.

Let $x, y \subseteq T_n$. A linear transformation is a function

$\alpha : X_n \longmapsto X_n, \xi_i \in Im(x) \subseteq N, \eta_j \in Im(y) \subseteq N$,

$i = j = 1, 2, \dots n - 1, i + j = j + i$ and $i + j \leq n$

with the following properties:

1. For any $\xi_1, \xi_2 \in x$ we have $\alpha(\xi_1 + \xi_2) = \alpha(\xi_1) + \alpha(\xi_2)$.
2. For any $i, \xi_i \in x, r \in X_n$ we have $\alpha(r\xi_i) = r\alpha(\xi_i)$.

Note that $r\xi_i \leq n$.

In other words, a linear transformation is a function between vector spaces which is compatible with addition and scalar multiplication.

Linearity is defined on full transformation semigroup in this study as above with the following conditions:

A function α on a linear space is said to be a norm if

$(1)\alpha(\lambda x) = |\lambda|\alpha(x)$ for $\lambda \in R$ (homogeneity),

(2)$\alpha(x) \geq 0$ and $\alpha(x) = 0 \Leftrightarrow x = 0$ (positivity),
(3)$\alpha(x + y) \leq \alpha(x) + \alpha(y)$ (convexity).
A linear space with a norm is said to be a normed linear space.

A norm defined on transformation semigroup S, over a set of nonnegative integers, $Z^{nonnegative}$, is a function
$\|.\| : S \to Z^{nonnegative}$
such that for all $x_1, x_2 \in S$,

- $|x_1\| \geq 0$; and $\|x_1\| = 0$ if and only if $x_1 = 0$
- $\|ax_1\| = |a| \, \|x_1\|$ for every $a \in Z^{nonneg}$(homogeneous).
- $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$(triangle inequality).

The pair $(S, \|.\|)$ is called a normed transformation semigroup. A relationship is established between $d(x_1, x_2)$ and $D^2(x_1, x_2) = \|x_1 - x_2\|$ with the following theorem:

Theorem 2.6 : If $\|.\|$ is a norm on S then $d(x_1, x_2) \leq \|x_1 - x_2\|$ is a metric on S where $\|x_1 - x_2\| = (\xi_1 - \eta_1) + (\xi_2 - \eta_2) + (\xi_3 - \eta_3) \ldots (\xi_n - \eta_n)$.

Proof: The properties of the metric and norm defined on S are satisfied using the Hamming distance function as:

- $d(x_1, x_2) \geq 0 \Rightarrow \|x_1 - x_2\| \geq 0$
- $d(x_1, x_2) = 0$ iff $x_1 = x_2 \Rightarrow \|x_1 - x_1\| = 0$
- $d(x_1, x_2) = d(x_2, x_1) \Rightarrow \|x_1 - x_2\| = \|x_2 - x_1\|$
- $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \Rightarrow \|(x_1 - x_2) + (x_2 - x_3)\|$
  $\leq \|(x_1 - x_2)\| + \|(x_2 - x_3)\|$
  $\leq d(x_1, x_2) + d(x_2, x_3)$.

. **Theorem 2.7:** The identity map is a linear transformation in semigroup of transformations.

Proof: Let $x = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_n \\ \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_n \end{pmatrix}$.

$\Rightarrow$ Taking $\xi_1 = 1, \xi_2 = 2, \xi_3 = 3, \ldots \xi_n = n$ and $\alpha(\xi_1) = 1, \alpha(\xi_2) = 2, \ldots \alpha(\xi_n) = n$.

$\Rightarrow \alpha(\xi_1 + \xi_2) = \alpha(\xi_1) + \alpha(\xi_2)$ and
$\alpha(r\xi_i) = r\alpha(\xi_i)$.

. **Theorem 2.8:** Let S be a semigroup of transformation, $\alpha(a\xi_i) = a\alpha(\xi_i)$ if and only if $a = 1$.

Proof: Let $\xi_i \in X_n$, for each $i = 1, 2, \ldots n$ and α is the transformation defined as

$\alpha : X_n \longmapsto X_n$

where $\alpha(xi_i)$ denote the image of the transformation. Define any element x in $T_n, t_{i=1,2,\ldots n}$ as:

$x = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_n \\ t_1 & t_2 & t_3 & \ldots & t_n \end{pmatrix}$.

Take $a > 1$, $\alpha(a\xi_i) = \alpha(K)$. If $K \leq \xi_n$ then $\alpha(K)$ exists in x,$(K = a\xi_i)$, otherwise $\alpha(K)$ does not exist.

$\alpha(K)$ exists for $K \leq \xi_n$ implies $a = \frac{K}{\xi_i}$,

$\Rightarrow a\alpha(\xi_i) = \frac{K}{\xi_i}\alpha(\xi_i) \neq \alpha(\frac{K}{\xi_i}.\xi_i)$,

$\Rightarrow \frac{K}{\xi_i}\alpha(\xi_i) \neq \alpha(K)$.

If $a = 1$ and $K = a\xi_i, \Rightarrow K = \xi_i$,

$\Rightarrow \alpha(a\xi_i) = a\alpha(\xi_i)$.

3. CONCLUSION

Transformation semigroup is metricizable using the technique of Hamming distance function.

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