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# CONJUGACY OF A SUBGROUP OF SYMMETRIC GROUP

Adeniji, A. O.

Abstract

This paper studied conjugacy on a subgroup that is not invariant generated by plain hunting and other related properties.

## 1. Introduction

A subgroup $H$ of a group $G$ is defined in [3] as a normal(or invariant) subgroup of $G$ if $gHg^{-1} = H$ for all $g \in G$, that is, if it's left and right cosets coincide. It has been pointed out before [6] that permutations were first studied in the 1600's in the context of the ringing of bells in a certain order and not (as usually stated) in the 1770's by Lagranges.

Gary, in his article [2], further provided a brief explanation of how permutation arise in the ringing of bells. The subgroup obtained by plain hunt method of ringing of bells in [2] is used in this paper for $n \geq 3$. The permutations A and B generate a subgroup $H_n$ of $S_n$ of order $2n$ and is obtained by referring to n bells listed in a particular order as row. For example, rounds is the row 123456 where $n = 6$. The permutation is done in such a way that the last element obtained is an identity element. He further stated that rows are compiled, subject to the following rules:

- The first and last rows must be rounds;
- No row may be repeated(apart from round which appear twice);
- Each bell may only change place by one position when moving from one row to the subsequent row;
- No bell occupies the same position for more than two successive rows.

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Department of Mathematics, Faculty of Science, University of Abuja, Abuja, Nigeria
*adeniji4love@yahoo.com

In his article, Peter Wenham[8] explained plain hunting of bells from rounds for four bells as:

| | | | | |
|---|---|---|---|---|
| Rounds | 1 | 2 | 3 | 4 |
| Change two pairs | 2 | 1 | 4 | 3 |
| Change the middle pair | 2 | 4 | 1 | 3 |
| Change the outside pairs | 4 | 2 | 3 | 1 |
| Change the middle pair | 4 | 3 | 2 | 1 |
| Change the outside pairs | 3 | 4 | 1 | 2 |
| Change the middle pair | 3 | 1 | 4 | 2 |
| Change the outside pairs | 1 | 3 | 2 | 4 |
| Change the middle pair | 1 | 2 | 3 | 4 |

It can be seen that all the numbers(bells) follow similar pattern in plain hunting path. The same subgroup is generated following Gary's method in [2] of plain hunting, but the elements do not occupy same positions.

Change Ringing is a team sport, a highly coordinated musical performance, an antique art, and a demanding exercise that involves a group of people ringing rhythmically a set of tuned bells through a series of changing sequences that are determined by mathematical principles and executed according to learned patterns [9]. Church bells have been rung in England for almost 1000 years to call parishioners to church. Originally churches had just one bell, probably rung by the priest who conducted the service. In time more bells were bought and rung. About 400 years ago the idea of changing the order of the bells was suggested, with a conductor calling the changes to tell the ringers which bells to swap. Later, methods were developed where the order of the bells changes according to a pattern memorised by the ringers. Call Changes are often the first thing a beginner learns after rounds; they are still rung in most church towers from time to time. Bells are normally tuned to a major scale; some sequences sound more pleasing than others.

Bells are heavy, often very heavy; a ringer can only make a small alteration to the swing of a bell when it is in motion. However, because the bell can be held stationary at the balance, it is possible for the bells to change their striking order but each bell can only move by one position in the order. The convention for Call Changes is that the instruction to change is called by one of the ringers, often known as the Caller, near the beginning of a handstroke; during that stroke and the next backstroke the other ringers prepare for the new order which happens when the next handstroke is rung [10].

In a group $G$, two elements x and y are conjugate when
$x = hyh^{-1}$ for some $h \in G$.

Conjugacy is studied on the subgroup $H_n$ in this work. The conjugacy class of an element x in a group G is the set of elements conjugate to it and it is written as:
$C_x = \{hxh^{-1} : h \in G\}$.
That is, the collection of all conjugate elements together in a group is called a conjugate class. John and Mark [4] established a general result about the distribution of the conjugacy classes of a finite group $G$ between the cosets of a normal subgroup $H$ such that $G/H$ is cyclic. Xiaolei,L., Yanming, W. and Huaguan, W.[7] investigated influences of lengths of conjugacy classes of finite groups on the structure of finite groups. Certain conditions were imposed on the lengths of the conjugacy classes of G and the group structure of $G$ was described under these conditions[1]. Patrick[5] also studied the number of conjugacy classes in a finite group. The action of any group on itself by conjugation and the corresponding conjugacy relation play an important role in group theory.

## 2. MAIN RESULTS

Plain hunt on n bells uses two permutations denoted by A and B in the cycle forms.
For odd $n \geq 3, i = 1$ here,
$A = (i \quad i+1)(i+2 \quad i+3)\ldots(i+n-3 \quad i+n-2)$
$B = (i+1 \quad i+2)(i+3 \quad i+4)(i+5 \quad i+6)\ldots(i+n-2 \quad i+n-1)$
For even $n \geq 3$,
$A = (i \quad i+1)(i+2 \quad i+3)(i+4 \quad i+5)\ldots(i+n-2 \quad i+n-1)$
$B = (i+1 \quad i+2)(i+3 \quad i+4)(i+5 \quad i+6)\ldots(i+n-3 \quad i+n-2)$
Plain Hunt on n bells uses two permutations applied alternately to rounds until rounds comes back again by Gary [2].
2.1 **Examples:**

1. Let $n = 3$(odd)
$A = (12), B = (23)$; the elements in the subgroup, $H_3$ can be generated as:
1 2 3

$$\text{A} = (12)$$

2 1 3

$$\text{AB} = (132)$$

3 1 2

$$\text{ABA} = (13)$$

3 2 1

$$\text{ABAB} = (123)$$

2 3 1

ABABA = (23)

1 3 2

ABABAB = identity

1 2 3

2. Let $n = 8$(even); $A = (12)(34)(56)(78)$; $B = (23)(45)(67)$. The elements of the subgroup $H_8$ are thus generated:

1 2 3 4 5 6 7 8

A = (12)(34)(56)(78)

2 1 4 3 6 5 8 7

AB = (13578642)

3 1 5 2 7 4 8 6

ABA = (14)(36)(58)

4 2 6 1 8 3 7 5

ABAB = (1584)(2376)

5 3 7 1 8 2 6 4

ABABA = (16)(24)(38)(57)

6 4 8 2 7 1 5 3

ABABAB = (17438256)

7 5 8 3 6 1 4 2

ABABABA = (18)(26)(37)

8 6 7 4 5 2 3 1

ABABABAB = (18)(27)(36)(45)

8 7 6 5 4 3 2 1

ABABABABA = (17)(28)(35)(46)

7 8 5 6 3 4 1 2

ABABABABAB = (16528347)

6 8 4 7 2 5 1 3

ABABABABABA = (15)(27)(48)

5 7 3 8 1 6 2 4

ABABABABABAB = (1485)(2673)

4 6 2 8 1 7 3 5

ABABABABABABA = (13)(25)(47)(68)

3 5 1 7 2 8 4 6

ABABABABABABAB = (12468753)

2 4 1 6 3 8 5 7

ABABABABABABABA = (23)(45)(67)

1 3 2 5 4 7 6 8

ABABABABABABABAB = (identity)

1 2 3 4 5 6 7 8

The permutations A and B generate a subgroup $H_n$ of $S_n$ of order $2n$ which [2] called the hunting subgroup.

. **Theorem 2.1:** Let n be the number of bells. $H_n$ is not a normal subgroup.

**Proof:** Let $a, b, c \in H_n$, where $n \in N$. $S_n$ is the collection of all permutations of n and is a group under permutation multiplication. Permutations are functions and $H_n$ is a set of functions.

To show that the operation performed to generate $H_n$ is closed, then for $a, b, c \in H_n$

$(a.b) = c \in H_n$.

Also, $(a.b).c = a.(b.c) \in H_n$, then the function composition is associative. The permutation $i \in H_n$ such that $i.a = a.i = a$, for all $a \in H_n$ implies existence of identity in $H_n$.

Let $a^{-1}$ be another permutation in $H_n$ for each a, such that $aa^{-1} = a^{-1}a = i(identity)$. This established the existence of inverse for each element $a \in H_n$.

Let $s \in S_n$ and $h \in H_n, s.h = a$ and $h.s = b$, then $s.h \neq h.s \ldots (1)$

Multiply both sides of (1) by $s^{-1}$ from the right,

$shs^{-1} \neq hss^{-1}$

$shs^{-1} \neq hi$, i is the identity element,

$shs^{-1} \neq h$.

Generally for the subgroup $H_n$, $sH_ns^{-1}$ is not contained in $H_n$.

. **Theorem 2.2:** Every element with only product of transpositions(cycles of length two) is the inverse of itself by composition of mappings.

**Proof:** Let the product of transpositions

$(b_1\ \ b_2)(b_3\ \ b_5)(b_4\ \ b_6)(b_7\ \ b_8)\ldots(b_t\ \ b_k)$

be an element obtained from plain hunting, where

$b_1, b_2, b_3, b_4, b_5, b_6, \ldots b_t, b_k \in H_n$.

The transpositions are in order of occurrence and a bell does not appear twice in a row.

If T $= \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \ldots & b_t \\ b_3 & b_4 & b_1 & b_2 & b_5 \ldots & b_x \end{pmatrix}$,

then $T^2 = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_x \ldots & b_t \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_x \ldots & b_t \end{pmatrix}$.

. **Corollary 2.1:** $H_n$ has n -even and n - odd permutations.

**Proof:** The subgroup $H_n$ obtained by plain hunting, the method of changing pairs as seen in section 1 above has 2n elements for $n \geq 3$. $S_n$ has a equal number of both even and odd permutations, since $H_n$ is a subgroup of $S_n$, $H_n$ also has equal number of even and odd permutations.

**Lemma 2.1:** The lengths of cycles of elements in each conjugate class are equal.

**Proof:** This claim is obvious from the set of elements in each conjugacy class.

**Lemma 2.2:** Two permutations in $H_n$ are conjugate if and only if they have the same cycle type but elements with the same cycle type may not necessarily conjugate.

**Proof:** The lengths of cycles of elements in each conjugacy class are equal from Lemma2.1 implies that elements in a conjugacy class have the same cycle type. Choosing any two elements $x$ and $y$ of the same cycle type from $H_n$, is not a criterion for conjugacy.

**Lemma 2.3:** All cycles of the same length in $H_n$ do not belong to the same conjugacy class.

**Proof:** The proof of this follows the reasoning of Lemma 2.2.

**Theorem 2.3:** Let $y \in H_n$ be a disjoint permutation of a particular cycle type. If for all $h \in H_n$, then $hyh^{-1} = x$, also a disjoint permutation of the same cycle type as $y$.

**Proof:** Assuming that

$$y = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix}$$

with the cycle type $y = (b_1b_4)(b_2b_3)$

and

$$x = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix},$$

for each $h \in H_n$, then $hyh^{-1} = x$, a disjoint permutation of the same cycle type. This implies that both $x$ and $y$ belong to the same conjugacy class.

**Lemma 2.4:** Any two elements of a conjugacy class have the same order but the converse is not true.

**Proof:** Let $e$ be the identity element in $H_n$. If $\gamma$ is a conjugacy class such that $x, y \in \gamma$ and $\alpha$ is the order of $x$ and $y$, it implies that $x^\alpha = y^\alpha = e$. If the two elements $x$ and $y$ have the same order in G and $C_x = hxh^{-1}$; $C_y = gyg^{-1}$ for some $h, g \in G$. $x^\alpha = y^\alpha = e$ does not imply that $C_x = C_y$.

**Lemma 2.5:** The order of the subgroup $H_n$ is the product of the order of each element of cycle length n and the number of elements in it's conjugate class.

**Proof:** Let $\alpha$ be the order of each element $x \in H_n$ and $k$ is the number of elements of a conjugate class each of which is of cycle length n. So $x^\alpha = kn$.

**Lemma 2.6:** Let $G$ be a finite symmetric group of order $n!$ and $H_n$ its subgroup. By Lagranges theorem, $H_n \leq G$, then the quotient of $\frac{|G|}{|H_n|}$ is $\frac{(n-1)!}{2}, n \geq 3$.

**Proof:** Given that the order of group $G$ is $n!$ and the order of $H_n$ is $2n$, then the result follows.

**Theorem 2.4:** Any conjugate class whose elements are cycles of order n is abelian.

**Proof:** Let $C_x$ be a conjugate class of $H_n$. Let $x, y \in C_x \subset H_n$ such that
x = $\begin{pmatrix} a & b & c & d & e & f & g \\ c & a & e & b & g & d & f \end{pmatrix}$ and y = $\begin{pmatrix} a & b & c & d & e & f & g \\ b & d & a & f & c & g & e \end{pmatrix}$.
$\Rightarrow x = y^{-1}$
$xy = y^{-1}y$
$xy = 1$(identity element).
Also, $x^{-1} = y$
$x^{-1}x = yx$
$1 = yx$
$\Rightarrow xy = yx = 1.$

**Theorem2.5:** The order of each element in a conjugate class is the length of the cycle.

**Proof:** Let $e$ be the identity element of $H_n$, cycle $(t_1, t_2, \ldots t_m)$ and $C_x$, a conjugate class of $H_n$. Let $x, y \in C_x$ such that $x = (t_1 t_3 t_4 t_2)$ then $x^4 = e$.

**Theorem2.6:** Conjugacy classes of $H_n$ are obtained using the odd and even occurrence of elements as: For odd n,
$2k + 1; k = 0, 1, 2, 3, \ldots n - 1$ with n elements;
$2k, 2n - 2k; k = 1, 2, 3, \ldots n - 2$ with k elements for each n;
and 2n is also a conjugate class.
For even n,
$2k + 1; k = 0, 2, 4, \ldots n - 2$ with $\frac{n}{2}$ elements;
$2k + 1; k = 1, 3, 5, \ldots n - 1$ with $\frac{n}{2}$ elements;
$2k, 2n - 2k; k = 1, 2, 3, \ldots \frac{n-2}{2}$ with k elements for each n;
$n$ is a conjugate class and $2n$ is also a conjugate class.

**Proof:**

Let $C_x$ denote a conjugate class of $H_n$. The status of an element of $H_n$ is the position it occupies. The cardinality of $H_n$ is $2n$. The conjugate classes for odd numbered and even numbered elements are obtained separately for each n.

For odd n,

Let $H_n = \{a, b, c, d, e \ldots 2n\}$ such that the position of a is 1, the position of b is 2, c is 3 …. Elements $a, c, e, \ldots 2n - 1$ are odd - numbered. If a is conjugate to c then

$c = hah^{-1}$, for some $h \in H_n$. A conjugate class is obtained containing odd positioned elements with the expression $2k+1; k = 0, 1, 2, 3, \ldots n-1$ and the class contains n elements. Elements with even positions form k conjugate classes with $\{2k, 2n-2k\}$ elements in each class for $k = 1, 2, 3, \ldots n-2$. The latter conjugate class has 2 elements in each class. The last conjugate class is the last element $2n$, which is the identity element. Conjugate classes for odd n have three different characterization.

For even n,

The elements of $H_{n(even)}$ can be subdivided into conjugate classes by splitting, using odd - even criterion. Elements having status(position) $2k+1$, k is odd number up to $n-1$, constitute a conjugate class with $\frac{n}{2}$ number of elements.

Another conjugate class also emanates by the elements occupying positions $2k, 2n-2k, k = 1, 2, 3, \ldots \frac{n-2}{2}$. This implies that a conjugate class has only two elements with $\frac{n-2}{2}$ occurrences.

The last two conjugate classes for n - even are the identity element of status 2n and element of status n. Each of the two classes has an element each.

**Theorem2.7:** A conjugate class is a generator of other conjugate classes in $S_n$.

**Proof:** Let $x_1, x_2, x_3, \ldots x_n \in S_n$. If $x_1, x_2 \in S_n$ are conjugate elements defined as

$x_1 = x_n x_2 x_n^{-1}$ for some $x_n \in S_n \ldots (*)$.

Assuming that there are other elements in the same conjugate class with $x_1$ and $x_2$. Then, from the right hand side of equation $(*)$, for each $x_n \in S_n$, computing conjugate elements of $x_2$ implies:

$x_1 x_2 x_n^{-1} = x_t$
$x_2 x_2 x_2^{-1} = x_r$
$\vdots$
$x_p x_2 x_p^{-1} = x_1$
$x_q x_2 x_q^{-1} = x_2$.

This implies that $x_t, x_r, x_1$ and $x_2$ belong to the same conjugate class. The set $x_t, x_r, x_1$ and $x_2$ is a generator of another conjugate class which in turn is a generator of another conjugate class until all the conjugate classes are generated. Thus, other conjugate classes are obtained from one conjugate class.

## 3. SUMMARY

The hunting subgroup $H_n$ is not an invariant and that Conjugacy classes of $H_n$ are obtained using the odd and even occurrence of elements as: For odd n,

$2k+1; k = 0, 1, 2, 3, \ldots n-1$ with n elements;
$2k, 2n-2k; k = 1, 2, 3, \ldots n-2$ with k elements for each n;
and 2n is also a conjugate class.

For even n,

$2k+1; k = 0, 2, 4, \ldots n-2$ with $\frac{n}{2}$ elements;
$2k+1; k = 1, 3, 5, \ldots n-1$ with $\frac{n}{2}$ elements;
$2k, 2n-2k; k = 1, 2, 3, \ldots \frac{n-2}{2}$ with k elements for each n;
$n$ is a conjugate class and $2n$ is also a conjugate class.

## References


[1] . David, and H. Marcel, *On the Length of the Conjugacy Classes of Finite Groups*, Journal of Algebra, 131(1), $(1990)110 - 125$. doi: 1016/0021 - 8693(90)90168-N.

[2] . Gary, *Bells, Motels and Permutation Groups*, http://arxiv.org/abs/1203.1835(2012).

[3] . F. John, *A First Course in Abstract Algebra* , Addison - Wesley Longman. Inc.(1998)

[4] . B. John, and W. Mark, *On the Distribution of Conjugacy Classes Between the Cosets of a Finite Group in a Cyclic Extension*, London Mathematical Society, (2008). doi: 10.1112/0000/000000.

[5] . G. Patrick, *The number of Conjugate Classes in a Finite Group*, Mathematische Zeitschrift, 118(3), (1970), 175 - 179.

[6] . T. White and Fabian Stedman, *The First Group Theorist?* Amer. Math. Monthly, 103, (1996), 771 - 778.

[7] . Xiaolei, W. Yanming, and W. Huaguan, *Notes on the Length of Conjugacy Classes of Finite Groups*, Journal of Pure and Applied Algebra, 196(1), (2005) 111 - 117. doi: 10.1016/j.jpaa.2004.08.020.