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Gradient Analysis in pipe networking for Domestic water supply system in transient flow

Y. M. Aiyesimi[1*], Effiom Uket Effiom[2*] and Gbolahan Bolarin[3*]

Abstract

Transient flow in water distribution networks is studied using the Gradient Method. In an attempt to obtain solutions rapidly, a computer program has been developed in FORTRAN 77. The computer program developed herein is general, easy to use, and allows a dynamic analysis of complex looped and open water distribution networks. Piezometric heads at nodes and flow rates in pipes are computed for water distribution networks. Results obtained from the Gradient Method are compared with an existing dynamic model. The benefits of using the Gradient Algorithm appear to be flexibility in coding, versatility in operational conditions, and its application to complicated water distribution networks.

1. Introduction

If the flow in a pipe is changed from one steady state condition to another, the intermediate temporary unsteady flow is defined as transient flow. Of the many challenges that face water utilities, one critical but too-often-forgotten issue is protecting the system from excessive transient or water hammer conditions. Surge analysis is essential to estimate the worst-case sceneries in distribution systems (Boulos et al, 2005). In essence, transients occur whenever flow conditions

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Department of Mathematics and Statistics, Federal University of Technology, Niger State, Nigeria

are altered, for they are the physics of change, bringing news of any adjustment throughout the network. A large number of papers have been published on numerical methods for determining the solution of unsteady flow problems in water distribution networks. But, among these publications only a handful investigates the problem of gradually varied transient flow in hydraulic networks. Due to the fact that rapid pressure changes are actually local problems in water distribution networks, a literature review on gradually varied unsteady flow analysis is of utmost importance.

Holloway (1985) performed surge analysis to simulate gradually varied flow in water distribution networks. The surge analysis differs from the water-hammer analysis in that, the flow change is felt throughout the pipe instantaneously. Holloway obtained the solution by solving one non-linear ordinary differential equation per pipe line instead of solving numerous non-linear partial differential equations per pipeline. Due to slow rates of flow variation in a water distribution network during its normal operating cycle, surge analysis gives satisfactory results with far less computational effort than that required by water-hammer analysis.

Most of the problems considering unsteady pure liquid flow in pipes are solved using a set of partial differential equations (Wylie and Streeter, 1993). These equations are valid only when the pressure is greater than the vapor pressure of liquid, and are solved numerically using the method of characteristics which was introduced by Streeter and Wylie (1967). But in many flow regimes, small amount of free gas is present in a liquid when local pressure during transient drops below saturated pressure, the liquid releases free gas, if the pressure drops to vapor pressure, cavities are formed (Jeppson, R. W. 1976).

Onizuka (1986) presented a method of analysis based on rigid water column theory for slow transients in pipe networks. Onizuka formulated a system of first order ordinary differential equations that describes the dynamic response of the network. The time integration is performed directly by using, for example, the Runge-Kutta method without involving any iterative procedure. A network example from Onizuka's paper has been presented in this study with minor modifications. Todini and Pilati (1987) proposed a gradient algorithm that operates on the field of piezometric heads and flows simultaneously. Salgado (1987) extended the original method to incorporate pumps into the system, and showed some of the advantages of the method when compared with the methods recommended by Wood (1981). Todini provided a unified mathematical framework for the most widely used existing gradient based methods. This study extends the application of the Gradient Method for simultaneous solution of piezometric heads and flows in distribution networks under the slow-transient conditions.

(Martin 1976) developed a one-dimensional homogeneous bubbly model using a two step Lax-Wendroff scheme. Pressure wave propagation and interactions are

handled well by Lax-Wendroff scheme by introducing a pseudo-viscosity term. The results produced by this model compare favorably with experiment than by using fixed grid method of characteristics. An analytical model was developed by Wiggert and Sundquist (1979) to investigate gaseous cavitation using the method of characteristics. Gas release is assumed mainly due to difference in local unsteady pressure and saturation pressure. Increase in void fraction due to latent heat flow is not considered. Wylie (1984) investigated both gaseous and vapor cavitation using a discrete free gas model. Free gas is lumped at discrete computing locations and pure liquid is assumed in between these locations. Small void fraction and isothermal behavior of fluid are some of the assumptions made. Gaseous cavitation is simulated and it gave close results when compared with other methods. Pezzinga (1999) developed a 2D model, which computes frictional losses in pipes and pipe networks using instantaneous velocity profiles. The extreme values for pressure heads and pressure wave oscillations were well reproduced by this model. Pezzinga (2004) adopted second viscosity to better explain energy dissipation during transient gaseous cavitation. Constant mass of free gas is assumed at constant temperature. Second viscosity or bulk viscosity coefficient accounts for other forms (other than frictional losses) of energy dissipation such as gas release and heat exchange between gas bubbles and surrounding liquid. Cannizzaro and Pezzinga (2005) considered the effects of thermic exchange between gas bubbles and surrounding liquid and gas release and solution process separately to study energy dissipation during gaseous cavitation. Separate 2D models were considered. The results of numerical runs shows that 2d model with gas release allows for a good simulation of the experimental data.
This study extends the application of the Gradient Method for simultaneous solution of piezometric heads and flows in distribution networks under the Slow-transient conditions. Reference to further literature study is embedded throughout this work in order to appreciate the efficiency and robustness of the Gradient Method for solving water distribution network problems.

2. Gradient analysis

Case 1: steady state

A gradient algorithm is derived for flow analysis of pipe networks in steady state. The Newton- Raphson technique is applied both in terms of nodal heads (h) and pipe flows (Q) to obtain a simultaneous solution of the system of equations expressing mass and energy balance. The problem is then solved through an iterative solution of a system of linear equations, the size of which equals the number of unknown heads.
A pipe network is defined by its topology, pipe characteristics and system constraints. System constraints include water demands (q) or fixed heads (h_0)for each node and head loss laws for each pipe (K_iQ_i). The problem is to determine

all of the flow rates in the pipes and all of the unknown heads (h) at the nodes under the steady state assumption.
In matrix form the problem can be formulated as follows Gradient analysis

$$A_{12}h + K(Q) = -A_{10}h_0 \qquad (1)$$
$$A_{21}Q = q$$

$A_{21} = A_{12}^{\tau}$ (ne,nn) unknown head nodes incidence matrix
$A_{10} = A_{01}^{\tau}$ (ne,no) fixed head nodes incidence matrix
$Q^{\tau}\,(Q_1\ Q_2...Q_{ne})$ (1, ne) flow rates in each pipe
$q^{\tau}\,(q_1\ q_2...Q_{ne})$ (1, nn) nodal demands
$h^{\tau}\,(h_1\ q_2...h_{nn})$ unknown nodal heads
$h_0^{\tau}\,(h_{01}\ q_{02}...h_{0n})$ (1, n0) fixed nodal heads
$K^{\tau}(Q) = (K_1 K_2.....K_{ne})$ (1,ne)laws expressing head losses in pipes
where,
nn=number of nodes with unknown head.
ne=number of pipes with unknown flow.
1 if flow of pipe i enters node j.
$A_{12}(ij) = 0$ if pipe i and node j are not connected.
-1 if flow of pipe i leaves node j.
and with A_{10} defined similarly to A_{12} for fixed head nodes.
The system represented by equation (1) may have more than one solution depending upon the form of $K_i(Q_i)$. If all $K_i(Q_i)$ are monotonically increasing functions, it can be proved that the solution of the system exists and is unique (Teruya M. 2001). The first set of Equations (1) are head loss equations for each pipe while the second are the set of nodal balance relationship.

Derivation

The necessary conditions for the steady-state flow are simply the simultaneous fulfillment of nodal balance and the head loss-flow relationship. Both conditions can be expressed in the following compact system of equations:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \begin{pmatrix} Q \\ h \end{pmatrix} = \begin{pmatrix} -A_{10}h_0 \\ q \end{pmatrix} \qquad (2)$$

where
A_{11} is matrix (nn, ne) dependent on the particular head loss-flow relationship.
$A_{12}$is topological (incidence) matrix (ne,nn).
A_{21} is $(A_{12})^{\tau}$
$A_{10}h_0$is known vector (ne, 1)

The upper part of the system represents the head loss-flow relationship and the lower part corresponds to the nodal flow (mass) balance. Since A_{11} clearly depends on the flow, the upper part of the above system is a set of non-linear equations in Q. A generalized relation between flow and head may be written as:

$$h_i = K_i Q_i^m \ (1, 2, 3, ...ne) \tag{3}$$

Using the link to node topological matrix (A_{12}), one can express the head loss (or head gain in the presence of pumps) of each link connecting two different nodes as:

$$A_{11}Q + A_{12}h = A_{10}h_0 \tag{4}$$

Where

$$A_{11} \begin{pmatrix} K_1|Q_1^m| & & 0 \\ & K_2|Q_2^m| & \\ 0 & & K_{\gamma E}|Q_{\gamma E}^m| \end{pmatrix} \tag{5}$$

Because the system represented by equation (2) is non-linear, a direct solution is not possible, but an iterative gradient approach is indeed feasible. Differentiating the system of equation (2) gives:

$$\begin{pmatrix} NA_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \begin{pmatrix} dQ \\ dh \end{pmatrix} = \begin{pmatrix} dE \\ dq \end{pmatrix} \tag{6}$$

Where,
N is diagonal matrix of the exponents of the head loss-flow relationship. At iteration k', where convergence has not yet been achieved, a $(ne, 1)$ vector dE and a $(nn, 1)$ vector dq are defined as:

$$dE = A_{11}Q_i^{K'} + A_{12}h_i^{K'} + A_{10}h_0 \tag{7}$$

and

$$dq = A_{21}Q_1^{K'} - q \tag{8}$$

Equation (7) represents the energy imbalance at each link while (8) represents the flow imbalance at each node. The solution of (8) is:

$$\begin{pmatrix} dQ \\ dh \end{pmatrix} = \begin{pmatrix} NA_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}^{-1} \begin{pmatrix} dE \\ dq \end{pmatrix} \tag{9}$$

Following a similar approach to Todini and Pilati (1987), the inverse of the block triangular matrix in (9) can be computed as another block matrix:

$$\begin{pmatrix} NA_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}^{-1} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{10}$$

Assuming

$$NA_{11}D^{-1} \tag{11}$$

(and therefore $DA_{11} = N^{-1}, N, A_{11}$ and D being diagonal), and computing the blocks of the inverse in equation (10), one gets:

$$\left.\begin{array}{c} B_{11} = D - DA_{12}(A_{12}DA_{11})^{-1}A_{21}D \\ B_{12} = DA_{12}(A_{21}DA_{12})^{-1} \\ B_{21} = (A_{21}DA_{12})^{-1}A_{12}D \\ B_{22} = -(A_{21}DA_{12})^{-1} \end{array}\right\} \tag{12}$$

The solution of equation (2) is thus obtained from the fact that

$$\left.\begin{array}{c} dQ = B_{11}dE + B_{12}dq \\ dh = B_{21}dE + B_{22}dq \end{array}\right\} \tag{13}$$

By substituting set of equations (12) into equations (13), the following can be obtained:

(14)

$$df = (A_{21}DA_{12})^{-1} A_{12}D \left(A_{11}Q^{K'} + A_{12}h_i^{K'} A_{10}h_0\right) - (A_{21}DA_{12})^{-1} \left(A_{11}Q^{K'} - q\right)$$
$$= h^{K'} + (A_{21}DA_{12})^{-1} \left(A_{12}D \left(A_{11}Q^{K'} + A_{10}h_0\right) - \left(q - A_{11}Q^{K'}\right)\right)$$

$$\begin{array}{l} dQ = D - DA_{12}\left(A_{21}DA_{12}\right)^{-1} A_{12}DA_{12}D \left(A_{11}Q^{K'} + A_{12}h_i^{K'} + A_{10}h_0\right) \\ + DA_{12}\left(A_{21}DA_{12}\right)^{-1} \left(A_{11}Q^{K'} - q\right) \end{array} \tag{15}$$

$$\begin{array}{l} dQ = D \left(A_{11}Q^{K'} + A_{10}h_0\right) - DA_{12}\left(A_{21}DA_{12}\right)^{-1} \\ \left\{A_{12}D \left(A_{11}Q^{K'} + A_{10}h_0\right) - \left(q - A_{11}Q^{K'}\right)\right\} \end{array} \tag{16}$$

Using the definition of D as presented earlier

$$\left.\begin{array}{c} dQ = Q^{k'} - Q^{k'+1} \\ dh = h^{k'} - h^{k'+1} \end{array}\right\} \tag{17}$$

A recursive Newton-Raphson algorithm is obtained as follows:

(18)

$$h^{K'+1} = -\left(A_{12}N^{-1}A_{11}^{-1}A_{21}\right) \left(A_{12}N^{-1} \left(A_{11}Q^{K'} + A_{10}h_0\right) - \left(q - A_{11}Q^{K'}\right)\right)$$

$$h^{K'+1} = \left(1 - N^{-1}\right) Q^{K'} - N^{-1} A_{11}^{-1} \left(A_{12} h^{K'+1} + A_{10} h_0\right) \tag{19}$$

Where A_{11} is computed using $Q^{K'}, k'$ and $(k'+1)$ are consecutive iteration steps.
Case 2: Unsteady state
Governing equations for unsteady-State Flow Conditions in a Pipe Network

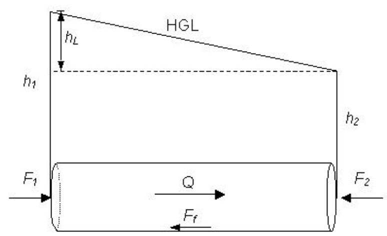


Figure 1. Free Body Diagram with Notation for Dynamic Equations

A free-body diagram of a conduit cross-sectional area is shown in Figure1. The forces acting on the fluid element are:

$$F_1 = \gamma A h_1 \tag{20}$$

$$F_2 = \gamma A h_2 \tag{21}$$

$$F_3 = \gamma A h_3 \tag{22}$$

where
A is cross-sectional area of pipe.
γ is specific weight of fluid.
h_1 is piezometric head at node 1.
h_2 is piezometric head at node 2.
h_3 is friction loss along pipe.
If the downstream flow direction is considered positive, then the net force acting on the fluid element in the positive direction is:

$$\sum F = F_1 - F_2 - F_3 = \frac{d(mv)}{dt} = ma \tag{23}$$

where F_1 and F_2 are the forces at the end of the element due to the total head of the fluid and F_3 is the force due to friction. This balances with the rate of change of momentum of the fluid in the element that has mass,m, and velocity, V. This rate of change is equal to the mass times the acceleration, a, of the fluid mass. Thus, (23) is Newtons Second Law ($F = ma$) Therefore, it follows from equations (19) to (22) that,

$$\sum F = \gamma A(h_1 - h_2 - h_3) \tag{24}$$

The mass of the fluid element is:

$$\frac{\gamma AL}{g} \tag{25}$$

where
L is length of conduit section $\left(\frac{L}{gA}\right)\frac{dQ}{dt} = h_1 - h_2 - KQ|Q^m|$
g is gravitational acceleration
If Q is the conduit flow and t is the time, then the rate of change of momentum of the element can be written as,

$$\left(\frac{\gamma L}{g}\right)\frac{dQ}{dt} \tag{26}$$

According to Newton's second law of motion, the rate of change of momentum is equal to the net applied force. Therefore,

$$\left(\frac{\gamma L}{g}\right)\frac{dQ}{dt} = \gamma A(h_1 - h_2 - h_3) \tag{27}$$

By defining $h_3 = KQ|Q^m|$ in which K and m are constants, depending on the friction formula chosen, equation (27) can be written as,

$$\left(\frac{L}{gA}\right)\frac{dQ}{dt} = h_1 - h_2 - KQ|Q^m| \tag{28}$$

Equation (28) is valid for horizontal and sloping pipes.
The continuity equation for each node can be written as:

$$\sum_{i=1}^{p} Q_i + q = 0 \tag{29}$$

where
q is external flow in or out of the node
p is number of pipes connected to the node
summing up equation (28) around a closed loop in the network yields:

$$\left[\sum_{p}^{i=1} Q_i \left(\frac{L_i}{gA_i}\right)\frac{dQ}{dt}\right] = \sum_{i=1}^{p}(h_{1i} - h_{2i}) - \sum_{i=1}^{p} h_{fi} \tag{30}$$

where
p is number of pipes in the loop

Equation (29) can be integrated as follows:

(31)

$$\int_{Q^t}^{Q^{t}-\Delta}\left\{\left(\sum_{i=1}^{p} Q_i\left(\frac{L_i}{gA_i}\right)dQ\right)\right\}=\int_{t}^{t+\Delta}\left\{\sum_{i=1}^{p}\left(h_{1i}-h_{2i}\right)dt\right\}-\int_{t}^{t+\Delta}\left\{\sum_{i=1}^{p}h_{fi}\right\}dt$$

At any instant in time, Kirchhoff's 2" Law requires the difference in head around a loop to equal zero.
Therefore,

$$\sum_{i=1}^{p}\left(h_{1i}-h_{2i}\right)=0 \tag{32}$$

$$\sum_{i=1}^{p}\left\{\left[\frac{L_i}{gA_i}\right]\left(Q_i^{t-\Delta t}-Q_i^{t}\right)\right\}=\sum_{i=1}^{p}\left[\int_{t}^{t+\Delta}h_{fi}dt\right] \tag{33}$$

Separation of the known Q (at time t) from the unknown Q(at time $t+?t$) in equation (33) yields:

$$\sum_{i=1}^{p}\left\{\left[\frac{L_i}{gA_i}\right]\left(Q_i^{t-\Delta t}-Q_i^{t}\right)\right\}=\sum_{i=1}^{p}\left\{\left[\frac{L_i}{gA_i}\right]\left(Q_i^{t}\right)\right\}+\sum_{i=1}^{p}\left[\int_{t}^{t+\Delta}h_{fi}dt\right] \tag{34}$$

The friction loss term can be rewritten as:

$$\sum_{i=1}^{p}\left[\int_{t}^{t+\Delta}h_{fi}dt\right]=\sum_{i=1}^{p}\left[K_i+\int_{t}^{t+\Delta}Q_i|Q_i^{m}|dt\right] \tag{35}$$

where K and m are constants depending on the friction loss formula chosen (i.e., Darcy- Weisbach, Hazen-Williams) and the units. The head loss term is non-linear in Q, and the variation of Q during the time increment Δt is unknown.

In transient analysis of rapid flow changes analyzed by the method of characteristics, several approximations have been used for the non-linear friction-loss term for water-hammer analysis, such as:

$$\int_{t}^{t+\Delta}KQ_i|Q_i^{m}|dt=KQ_i^{t+\Delta}|Q^{t}| \tag{36}$$

$$\int_{t}^{t+\Delta}KQ_i|Q_i^{m}|dt=K\left(\frac{Q^{t}+Q^{t+\Delta t}+|\left(Q^{t}+Q^{t+\Delta t}\right)^{m}|}{2^{m-1}}\right)\Delta t \tag{37}$$

or

$$\int_{t}^{t+\Delta}KQ_i|Q_i^{m}|dt=K\left(\frac{Q^{t}|(Q^{t})^{m}|+Q^{t+\Delta t}+|(Q^{t+\Delta t})^{m}|}{2}\right)\Delta t \tag{38}$$

where time t represents known values of Q and time $(t+?t)$ the unknown values. The separate ordinary differential equations used to describe water-hammer by the method of characteristics and pipeline surges are integrated in a similar manner. The suggested approximation is as follows:

$$\int_{t}^{t+\Delta} KQ_i|Q_i^m|dt = KQ^{t+\Delta t}|(Q^{t+\Delta t})^m|\Delta t \tag{39}$$

(40)

$$\sum_{i=1}^{p}\left\{\left[\frac{L_i}{gA_i}\right]\left(Q_i^{t-\Delta t}-Q_i^t\right)\right\}=\sum_{i=1}^{p}\left\{\left[\frac{L_i}{gA_i}\right]\left(Q_i^t\right)\right\}-\sum_{i=1}^{p}\left[KQ^{t+\Delta t}|(Q^{t+\Delta t})^m|\Delta t\right]$$

Thus, the numerical representation of the dynamic equation written around a loop in the network is linear in $Q^{t+?t}$.

3. Dynamic Equations

The dynamic head-flow equation for a single pipe in the system is derived in reference to Figure 2 below. The figure represents a simple water distribution network with 5 pipes, 2 fixed grade nodes (FGN), and appropriate nodal consumption. The flow directions are obtained from a steady-state run of a computer program based on the gradient algorithm. The following equations are derived in conjunction with these flow directions.

Pipe 1: $k_1Q_1^m$

$$\textbf{Pipe 1:}\quad k_1Q_1^m + h_1 - h_{01} = \left(\frac{L_1}{gA_1}\right)\frac{dQ_1}{dt} \tag{41}$$

where
K_1 = constant depending on the friction formula (H-W or D-W) used.
m is exponent depending on the friction formula (H-W or D-W) used.
L is length of the pipe.
h is piezometric head at node 1.
h_o is fixed grade head connected to pipe 1.
Q is flow rate in pipe 1.
t is time.
Similarly, head-flow equations for the other pipe elements in the network are,

$$\textbf{Pipe 2:}\quad k_2Q_2^m + h_2 - h_{02} = \left(\frac{L_2}{gA_2}\right)\frac{dQ_2}{dt} \tag{42}$$

(43) $$\textbf{Pipe 3:}\quad k_3Q_3^m + h_3 - h_{03} = \left(\frac{L_3}{gA_3}\right)\frac{dQ_3}{dt}$$

(44) $$\textbf{Pipe 4:}\quad k_4Q_4^m + h_4 - h_{04} = \left(\frac{L_4}{gA_4}\right)\frac{dQ_4}{dt}$$

(45) $$\textbf{Pipe 5:}\quad k_5Q_5^m + h_5 - h_{05} = \left(\frac{L_5}{gA_5}\right)\frac{dQ_5}{dt}$$

Where, h_{02} = fixed grade head connected to pipe 5

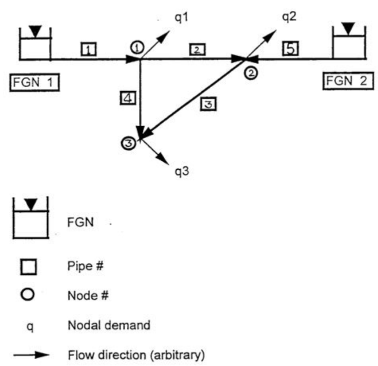


Figure 2. Typical Setup of Water Distribution Network

The following derivation is valid for pipe element 1 (Figure 2). Equation (40) can be integrated to the form:

(46) $$\int_t^{t+\Delta} k_1Q_1^m dt + \int_t^{t+\Delta}(h_1 - h_{01})dt = \frac{L_1}{gA_1}\int_{Q_1'}^{Q_1^{t+\Delta}} Q_1$$

The above equation is non-linear and therefore, following valid approximations are made:

(47) $$\int_t^{t+\Delta} k_1Q_1^m dt = k_1Q_1^{t+\Delta}|Q_1^{m-1}|\Delta t$$

(48) $$\int_t^{t+\Delta} h_1 dt = \left[\frac{h_1^t - h_1^{t+\Delta}}{2}\right]\Delta$$

(49) $$\int_t^{t+\Delta} h_{01} dt = \left[\frac{h_{01}^t - h_{01}^{t+\Delta}}{2}\right]\Delta t$$

where
Δt is time increment
Q^t is known flow rate at time t
$Q^{t+\Delta}$ is unknown flow rate at time $t+?t$
h^t is known piezometric head at time t
$h^{t+\Delta t}$ is unknown piezometric head at time $t+\Delta t$
Equation (47) is the only major numerical simplification. Equations (48) and (49) are arithmetic mean (linear) that could be substituted for geometric mean (non-linear).
The head loss formula due to friction $h_f = KK|Q^{m-1}|$ accounts for reverse flow anywhere in the system.
Substitution of the above approximations and further simplification of equation (46) lead to the following form:

(50)
$$K_1 Q_1^{t+\Delta t}|Q^{m-1}|\Delta t + \left(\frac{(h_1^t - h_1^{t+\Delta t})(h_{01}^t - h_{01}^{t+\Delta t})\Delta t}{2}\right) = \left(\frac{L_1}{gA_1}\right) Q_1^{t+\Delta t} - Q_1^t$$

(51)
$$\left[K_1 \left|Q_1^{t(m-1)}\right| \Delta t\right] - \left(\frac{L_1}{gA_1}\right)\left(Q_1^{t+\Delta t} + \left[\frac{(h_1^t - h_1^{t+\Delta t})(h_{01}^t - h_{01}^{t+\Delta t})\Delta t}{2}\right]\right) = \left(\frac{L_1}{gA_1}\right) Q_1^t$$

Upon placing the known terms to the right hand side of the equation the final form is obtained:

(52)
$$\left[K_1 \left|Q_1^{t(m-1)}\right| \Delta t\right] - \left(\frac{L_1}{gA_1}\right)\left(Q_1^{t+\Delta t} + \left[\frac{(h_1^{t+\Delta t})\Delta t}{2}\right]\right) = -\left(\frac{L_1}{gA_1}\right) Q_1^t + \left[\frac{(h_{01}^t - h_{01}^{t+\Delta t})\Delta t}{2}\right]$$

where the fixed grade head term h_{01} remains a known boundary at all times.
A general equation of the following form can be written for the pipes with two node
(53)
$$\left[K_1 \left|Q_1^{t(m-1)}\right| \Delta t\right] - \left(\frac{L_1}{gA_1}\right) Q_1^{t+\Delta t}\left(\frac{(h_j^{t+\Delta t} - h_i^{t+\Delta t})}{2}\right)\Delta t = -\left(\frac{L_1}{gA_1}\right) Q_1^t + \left[\frac{(h_i^t - h_j^{t+\Delta t})\Delta t}{2}\right]$$

where
i is starting node for a pipe.
j is ending node for a pipe.
The above set of dynamic equations is used to set up the Transient gradient Algorithm below.

3.1. **Algorithm for unsteady state.** Recast (from equation (1-4) derived earlier)

$$A_{11}Q + A_{12}h = A_{10}h_0 \tag{54}$$

$$A_{11} = \begin{pmatrix} K_1|Q_1^m|\Delta - \frac{L_1}{gA_1} & & 0 \\ & K_2|Q_2^m|\Delta - \frac{L_2}{gA_2} & \\ 0 & & K_5|Q_5^{t(m-1)}|\Delta - \frac{L_5}{gA_5} \end{pmatrix} \tag{55}$$

$$A_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{56}$$

Combining all the terms at the known time step, matrix $A_10h_0$is writen in the following compact form:

$$A_{10}h_0 = -\begin{pmatrix} \frac{L_1}{gA_1}Q_1^t \\ \frac{L_2}{gA_2}Q_2^t \\ . \\ . \\ \frac{L_5}{gA_5}Q_5^t \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_{01}^t + h_{01}^{t+\Delta} \\ h_1^t \\ h_2^t \\ h_3^t \\ h_{02}^t + h_{02}^{t+\Delta} \end{pmatrix} \frac{\Delta t}{2} \tag{57}$$

which, upon algebraic manipulation, is a $(ne, 1)$ matrix for the system.
Because the system represented by equation (2) is non-linear, a direct solution is not possible, but an iterative gradient approach has been introduced in the program that solves the problem.
The resulting recursive Newton-Raphson algorithm is:

$$h^{k'+1} = \left(A_{21}N^{-1}A_{11}A_2\right)^{-1}\left\{A_{21}N^{-1}(Q^{K'} + A_{11}^{-1}A_{10}h_0) + \left(q - A_{12}Q^{K'}\right)\right\} \tag{58}$$

$$Q^{k'+1} = (1 - N^{-1})Q^{K'} - N^{-1}A_{11}^{-1}\left(A_{12}h^{k'+1} + A_{10}h_0\right) \tag{59}$$

4. Results and Discussions

The networks below was analyzed by the Gradient Method for a period of 3 hours in figure 3, to compare with results from Holloway's Dynamic Model. The results of head and flow profile are plotted in Figure 4 and 5 respectively.

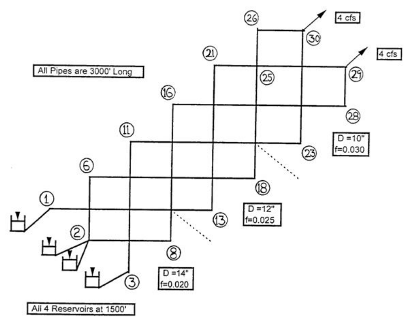


Figure 3.

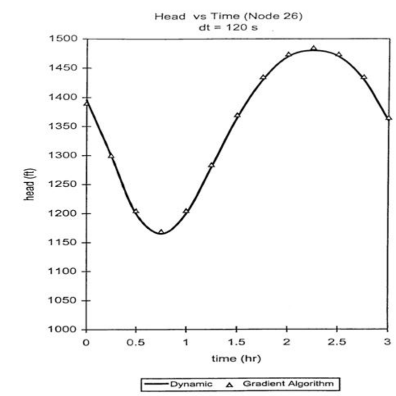


Figure 4. Comparison of Calculated Heads

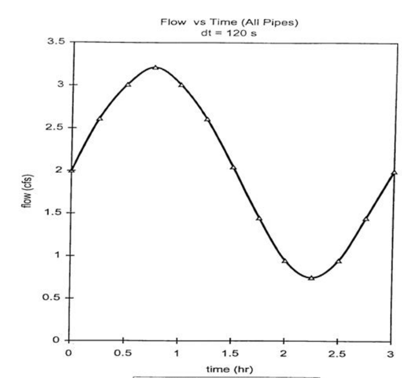
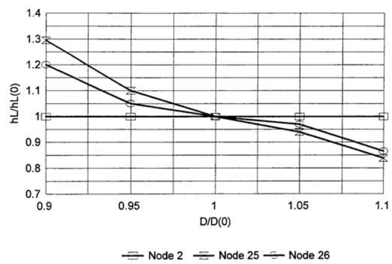


Figure 5. Comparison of Calculated Flows



4.1. **Discussions.** Figure 3, Illustrates the head profile at node 26 of the network. The model was run at a time increment of ?t(dt)=120s to get a near perfect agreement between the two models. With the fluctuating nodal demand, as demand is increased by the appropriate factor, hydraulic head decreased. In time, head started to increase as nodal demand decreased. Figure 4, show the flow profile for all pipes (due to topological symmetry) in the network. Once again, a perfect match is observed. The pattern of the profile corresponds to the head profile in a way that as head decreased, flow rate in all pipes increased and vice versa

4.2. **Conclusions.** Successful completion of Model Verification and Sensitivity leads to the conclusion that the Gradient Algorithm developed herein provides stable hydraulic results. Any size water distribution network can be simulated under slow-transient conditions. The results have been presented to support different operating conditions. Each condition provided distinct patterns for head and flow profiles. This was obtained without any major computational difficulties. The linearized friction term approximation and the linearized minor loss term yields satisfactory results. The task of analyzing gradually varied flows in water distribution networks can thus be accomplished efficiently.

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