Nigerian Journal of Mathematics and Applications
Volume **24**, (2015), 261 − 267
 http : //www.kwsman.com

A NEW NUMERICAL METHOD FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

[1]Abolarin O. E. and [2]Akingbade S. W.

Abstract

A method of representing the exact solution to the initial value problem by an interpolating function is considered to generate a numerical method which enjoys the higher-order advantage over all the existing methods. The algorithm developed is coded using MATLAB programming application language towards its applicability. The effectiveness of the proposed method is tested and therefore found to compete favourably with the exact solution and the existing method.

1. INTRODUCTION

Differential equations are used to model problems in science and engineering that involve the change of some variables with respect to another. Most of these problems require the solution of an initial-value problem, that is, the solution to a differential equation that satisfies a given initial condition. In common real-life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to modify the problem by simplifying the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original problem. The other approach, which will be examined in this work, uses method for approximating the solution of the

Received November 14, 2014. * Corresponding author.
2010 *Mathematics Subject Classification.* 49Mxx & 35E15.
Key words and phrases. Initial Value Problem, Interpolating function, Exact Solution, Algorithm, Differential equations.
[1]Department of Mathematics, Federal University Oye-Ekiti, Ekiti State, Nigeria. Email: olusola.abolarin@fuoye.edu.ng

original problem. This is the approach that is most commonly taken because the approximation method give more accurate results and realistic error information [2]. Initial-value problems obtained by observing physical phenomena generally only approximate the true situation, so we need to know whether small changes in the statement of the problem introduce correspondingly small changes in the solution. This is also important because of the introduction of round-off error when numerical methods are used. To determine whether a particular problem has the property that small changes, or perturbations, in the statement of the problem introduce correspondingly small changes in the solution, the problem must be a well-posed problem. Numerical methods will always be concerned with solving a perturbed problem because any round-off error introduced in the representation perturbs the original problem. Unless the original problem is well-posed, there is little reason to expect that the numerical solution to a perturbed problem will accurately approximate the solution to the original problem [11]. The numerical solution of first order initial value problems have caught much attention recently where most of these integrators (algorithms/methods) were developed by representing the actual solution to initial value problem by an interpolating function . This paper considers the numerical solution to initial value problems of the form

$$y' = f(x,y) \colon y(x_0) = y_0 \tag{1}$$

The scheme was developed using an interpolant which is in the form of a quartic polynomial and trigonometric functions. We were motivated by the work of [9] where they proposed a numerical integrator capable of solving problems of the form (1); an integrator developed by combining polynomials, exponentials and cyclometric functions. The newly proposed method compete favourably when compared with the work of [9] and the exact solution. Many researchers have contributed immensely in the implementation of different numerical methods that solves initial value problem (1). In [1], they worked on the method for the numerical solution of painleve equations i.e. equations having singularities at points at where the solution takes certain finite values. A new numerical scheme was proposed in [8] for the solution of initial value problems in ordinary differential equations which is based on a nonlinear polynomial interpolant. Lambert [13] also work on numerical methods for solving initial value problems in ordinary differential systems. References [3][4][5][6][10][12][14][15] are worth mentioning also.

2. DERIVATION OF THE SCHEME

Let the exact solution $y(x_n)$ evaluated at $x = x_n$ to the first order ordinary differential equation be represented by the interpolant

$$y(x_n) = p_0 + p_1 x_n + p_2 x_n^2 + p_3 x_n^3 + p_4 x_n^4 + q \sin x_n \tag{2}$$

where p_0, p_1, p_2, p_3 and q are unknown parameters. These parameters are determined by following some steps.
We make equation(2) concur with the exact solution at $x = x_{n+1}$,

$$y(x) = y(x_{n+1}) = p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + p_3 x_{n+1}^3 + p_4 x_{n+1}^4 + q \sin x_{n+1} \tag{3}$$

We assume $y(x_n) = y_n$ and $y(x_{n+1}) = y_{n+1}$

$$y(x_n) = y_n = p_0 + p_1 x_n + p_2 x_n^2 + p_3 x_n^3 + p_4 x_n^4 + q \sin x_n \tag{4}$$

and

$$y(x_{n+1}) = y_{n+1} = p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + p_3 x_{n+1}^3 + p_4 x_{n+1}^4 + q \sin x_{n+1} \tag{5}$$

Defining

$$f_n = f(x_n, y_n) \tag{6}$$

such that $y'(x_n) = f_n$ and $y''(x_n) = f'_n$ and so on

$$x_{n+1} = x_0 + (n+1)h, n = 0, 1, 2, 3, \cdots \tag{7}$$

Differentiating (4) with respect to x_n yields,

$$f_n = p_1 + 2p_2 x_n + 3p_3 x_n^2 + 4p_4 x_n^3 + q \cos x_n \tag{8}$$

We now differentiate (8) with respect to x_n

$$f'_n = 2p_2 + 6p_3 x_n + 12 p_4 x_n^2 - q \sin x_n \tag{9}$$

Also,

$$f''_n = 6p_3 + 24 p_4 x_n - q \cos x_n \tag{10}$$

And for f'''_n

$$f'''_n = 24 p_4 + q \sin x_n \tag{11}$$

Lastly,

$$f_n^{(4)} = q \cos x_n \tag{12}$$

$$q = \frac{f_n^{(4)}}{\cos x_n} \tag{13}$$

from (1), we can deduce that

$$p_4 = \frac{1}{4!}(f'''_n - f_n^{(4)} \tan x_n) \tag{14}$$

$$\begin{aligned} f''_n &= 6p_3 + 24 p_4 x_n - q \cos x_n \\ 6p_3 &= f''_n - (f'''_n - f_n^{(4)} \tan x_n) + f_n^{(4)} \\ p_3 &= \frac{1}{3!}[f''_n - (f'''_n - f_n^{(4)} \tan x_n) + f_n^{(4)}] \end{aligned} \tag{15}$$

from (9),

(16)

$$2p_2 = f'_n - [f''_n - (f'''_n - f^{(4)}_n \tan x_n)x_n + f^{(4)}_n]x_n - \frac{1}{2}(f'''_n - f^{(4)}_n \tan x_n)x_n^2 + f^{(4)}_n \tan x_n$$

$$p_2 = \frac{1}{2!}[f'_n - [f''_n - (f'''_n - f^{(4)}_n \tan x_n)x_n + f^{(4)}_n]x_n - \frac{1}{2}(f'''_n - f^{(4)}_n \tan x_n)x_n^2 + f^{(4)}_n \tan x_n$$

rearranging (8)

$$p_1 = f_n - 2p_2x_n - 3p_3x_n^2 - 4p_4x_n^3 - q\cos x_n \tag{17}$$

Hence,

(18)

$$\begin{aligned} p_1 = f_n - [f'_n - [f''_n - (f'''_n - f^{(4)}_n \tan x_n)x_n + f^{(4)}_n]x_n - \frac{1}{2}(f'''_n - f^{(4)}_n \tan x_n)x_n^2 \\ + f^{(4)}_n \tan x_n]x_n - \frac{1}{2!}[f''_n - (f'''_n - f^{(4)}_n \tan x_n)x_n \\ + f^{(4)}_n]x_n^2 - \frac{1}{3!}(f'''_n - f^{(4)}_n \tan x_n)x_n^3 - f^4_n \end{aligned}$$

Since all the parameters are known, we now subtract (4) from (5)

$$\begin{aligned} y_{n+1} - y_n = p_1(x_{n+1} - x_n) + p_2(x_{n+2}^2 - x_n^2) + p_3(x_{n+1}^3 - x_n^3) + \\ p_4(x_{n+1}^4 - x_n^4) + q(\sin x_{n+1} - \sin x_n) \end{aligned} \tag{19}$$

from (7),

$$\begin{aligned} x_{n+1} = x_n + h \\ x_{n+1} - x_n = h \end{aligned} \tag{20}$$

$$x_{n+1}^2 - x_n^2 = h^2 + 2x_nh \tag{21}$$

$$x_{n+1}^3 - x_n^3 = h^3 + 3hx_{n+1}x_n \tag{22}$$

$$x_{n+1}^4 - x_n^4 = h^4 - 2x_n^4 + 4x_{n+1}^3x_n - 6x_{n+1}^2x_n^2 + 4x_{n+1}x_n^3 \tag{23}$$

$$\begin{aligned} y_{n+1} = y_n + p_1(h) + p_2(h^2 + 2x_nh) + p_3(h^3 + 3hx_{n+1}x_n) + \\ p_4(h^4 - 2x_n^4 + 4x_{n+1}^3x_n - 6x_{n+1}^2x_n^2 + 4x_{n+1}x_n^3 \\ + q(\sin x_n \cosh + \cos x_n \sinh - \sin x_n) \end{aligned} \tag{24}$$

we now introduce (13), (14), (15),(16) and (17) in (23) to give the Numerical Scheme

(25)

$y_{n+1} = y_n + f_n - [f'_n - [f''_n - (f'''_n - f_n^{(4)} \tan x_n)x_n + f_n^{(4)}]x_n -$
$\frac{1}{2}(f'''_n - f_n^{(4)} \tan x_n)x_n^2 + f_n^{(4)} \tan x_n]x_n - \frac{1}{2!}[f''_n - (f'''_n - f_n^{(4)} \tan x_n)x_n + f_n^{(4)}]x_n^2 -$
$\frac{1}{3!}(f'''_n - f_n^{(4)} \tan x_n)x_n^3 - f_n^{(4)}]h + \frac{1}{2!}[f'_n - [f''_n - (f'''_n - f_n^{(4)} \tan x_n)x_n + f_n^{(4)}]x_n -$
$\frac{1}{2}(f'''_n - f_n^{(4)} \tan x_n)x^2 + f_n^{(4)} \tan x_n](h^2 + 2x_n h) + \frac{1}{3!}[f''_n - (f'''_n - f_n^{(4)} \tan x_n) +$
$f_n^{(4)}] + (h^3 + 3hx_{n+1}x_n) + \frac{1}{4!}[f'''_n - f_n^{(4)} \tan x_n](h^4 - 2x_n^4 + 4x_{n+1}^3 x_n - 6x_{n+1}^2 x_n^2 +$
$4x_{n+1}x_n^3) + \frac{f_n^{(4)}}{\cos x_n}(\sin x_n \cosh + \cos x_n \sinh - \sin x_n)$

3. NUMERICAL RESULTS AND ANALYSIS

In order to make a valid comparison between the new method (24) and the previous method [9]. We shall make use of some problems in [9]which are the initial value problems (a) and (b) below. Throughout the examples, we shall denote the previous method in [9] as J's method while the new proposed method will remain as 'new method'.

1. One of the most basic examples of differential equations in Malthusian law of population growth $\frac{dp}{dt} = rp$ shows how the population *(p)* changes with respect to time t. The constant r will changes depending on the species. Malthus used this law to predict how a species would grow over time[7].

let us take for example the IVP with respect to the new scheme:

$$y' = 0.2y; y(0) = 1000 \ (a)$$

$0 \leq x \leq 1$ and $h = 0.1$

whose exact solution is given as $y(x) = 1000e^{0.2x}$

Table 1

x	New method	J 's method	Exact Solution	Error in new	Error in J's
0.0	1000.00000000	1000.00000000	1000.00000000	0.00000000	0.000000
0.1	1020.20134003	1020.20135498	1020.20134003	0.00000000	0.000014
0.2	1040.81077419	1040.81079102	104081077419	0.00000000	0.000017
0.3	1061.83654654	1061.83654785	1061.83654654	0.00000000	0.000001
0.4	1083.28706767	1083.28710938	1083.28706767	0.00000000	0.000042
0.5	1105.17091807	1105.17102051	1105.17091807	0.00000000	0.000102
0.6	1127.49685158	1127.49694824	1127.49685158	0.00000000	0.000097
0.7	1150.27379886	1150.27392578	1150.27379886	0.00000000	0.000127
0.8	1173.51087099	1173.51098633	1173.51087099	0.00000000	0.000116
0.9	1197.21736312	1197.21752930	1197.21736312	0.00000000	0.000166
1.0	1221.40275816	1221.40295410	1221.40275816	0.00000000	0.000196

2. We consider the logistics growth modeled by the differential equation $\frac{dp}{dt} = KP(1 - \frac{p}{m})$ for some positive constants k and m. We now take the IVP

$$y' = y(1 - y); y(0) = 0.5 \ (b)$$

with the exact solution

$$y(x) = \frac{0.5}{0.5 + 0.5e^{-x}}$$

Table 2

x	New method	J's method	Exact Solution	Error in new	Error in J's
0.0	0.500000000	0.50000000	0.500000000	0.00000000	0.00000000
0.1	0.524979151	0.52497977	0.524979187	0.00000004	0.00000058
0.2	0.549833927	0.54982515	0.549833997	0.00000007	0.00000115
0.3	0.574442416	0.57444417	0.574442516	0.00000010	0.00000165
0.4	0.598687535	0.59868979	0.598687660	0.00000013	0.00000213
0.5	0.622459187	0.62246192	0.622459331	0.00000014	0.00000259
0.6	0.645656149	0.64565927	0.645656306	0.00000016	0.00000296
0.7	0.668187608	0.66819102	0.668187772	0.00000016	0.00000325
0.8	0.689974316	0.68997794	0.689974481	0.00000017	0.00000346
0.9	0.710949343	0.71095312	0.710949502	0.00000016	0.00000362
1.0	0.731058429	0.73106223	0.731058578	0.00000015	0.00000365

4. CONCLUSION

We have implemented the new scheme (24) which has an advantage over the existing method [9]. The problems under investigation showed that the new scheme is consistent and stable, thereby showing a measure of convergence towards the exact solution. Also, it is more accurate and reliable for solving Initial

Value Problems of first order Ordinary Differential Equations and therefore, recommended.

References

[1] .A Abramov, L.F Yukhno(2013),A method for the numerical solution of the Painleve Equations," *Journal of Computational Mathematics and Mathematical Physics*, vol. 53, no 5, pp. 702-726. 2013

[2] ichard L. Burden and J. Douglas Faires,"Numerical Analysis" Ninth Edition . Brooks/Cole, Cengage learning.

[3] .H Hsiao ,Numerical Solution of stiffs differential equation via Harr wavelets," *International Journal of Computer Mathematics*, vol. 82, issue 9.2005

[4] .W Gear,Numerical Initial Value Problems in ODEs, Prentice Hall, Englewood Cliffs, NJ. 1971

[5] atunla S.O A new algorithm for the numerical solution of ODEs.*Computer and Mathematics with Applications* 2:247-253. 1976

[6] atunla , S.O Numerical Methods for Initial Value Problems in ODEs *Academic Press* New York, NY. 1988

[7] B Maths Resources from British International School Phuket(2014),"Differential Equations in Real life". *http://ibmathsresources.com/2014/02/28/differential equations-in-real-life/*

[8] bijola E.A and Ogunrinde R.B, On a New Numerical Scheme for the solution of IVPs in ODEs, *Australian Journal of Basic and Applied Sciences*.4(10):5277-5282. 2010

[9] . Sunday and M.R Odekunle A New Numerical Integrator for the solution of Initial Value Problems in ODEs. Pacific Journal of Science and Technology. 13(1): 221-227. 2012

[10] .J More, M.Y. Cosnard,"Numerical Solution of Nonlinear Equations," ACM *Transactions on Mathematical Software*, Vol. 5, No. 1, pp.64-85. 1979

[11] .Stoer and R. Bulirsch,Introduction to Numerical Analysis,*Mathematics of computation*, vol46, No Pp 119-133. 1986

[12] ama, P and Ibijola E.A On a new one step method for Numerical Solution of ODEs. *International Journal of Computer Mathematics*. 78(4).2000

[13] ambert, J.D Numerical methods for Ordinary Differential Systems," New York: Wiley, ISBN 978-0-471-92990-1. 1992

[14] .F. Shampine Numerical Solution of ODEs," Chapman and Hall, New York. 1994

[15] . Wambecq, "Non linear methods in solving ODEs," *Journal of Computational and Applied Mathematics* 2(1), 27-33. Retrieved December 10, 2012