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FIVE SHORT NOTES ON BOUNDED MAPS

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Abstract

This paper contributes five observations on bounded maps: (i) An observation on the definition of a bounded set, (ii) Three characterizations of bounded linear maps that parallel the usual general topology characterizations of continuity, (iii) An analogue of [2, PROPOSITION 3.12.1 p.254][7, **Lemma** 11.1.1, p.164] for bounded maps, (iv) That the Banach-Steinhaus Closure Theorem is also true for bounded maps, and (v) That the associated bornological topology is also Hellinger-Teoplitz.

1. Introduction

A bounded linear map $f : (X, \tau) \longrightarrow (Y, \mu)$ between separated locally convex spaces, (X, τ) and (Y, μ), henceforth simply called a bounded map is one that preserves bounded sets; that is, $f(B)$ is μ-bounded for every τ-bounded subset B of X. **NOTE 1** explains the definition of a bounded set in a locally convex space. [2, Exercise 3.7.7(f), p.225] furnishes some characterizations of bounded maps. We here add three more characterization (i) $\Longleftrightarrow$ (iii), (iv) and (v) of the theorem of **NOTE 2** below. That our characterizations parallel the usual General Topology characterizations of continuity [3, **THEOREM** 1.4, p.59] is the contribution of **NOTE 2**; [4, **LEMMA** 5.1.23, p.156] and [4, Proposition 5.1.24, p.156] fall out as corollaries of our characterizations.
For separated locally convex spaces $(X, \tau)$and (Y, μ) with continuous duals X' and Y', respectively, and associated weak topologies $\sigma(X, X')$ and $\sigma(Y, Y')$, if$u$: $(X, \tau) \longrightarrow (Y, \mu)$ is a linear map, the continuity of the compositions fou,

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for all continuous linear functionals f on (Y, μ), does not force the continuity of u; it only forces the$(\sigma(X, X'), \sigma(Y, Y'))$-continuity of u[2, PROPOSITION 3.12.1, p.254][7, **Lemma** 11.1.1, p.164]. We show in **NOTE 3** below that if the compositions fou are bounded maps for all bounded linear functionals f on (Y, μ), this forces the boundedness of$u$.

The Banach-Steinhaus Closure Theorem [7, **THEOREM** 9.3.7, p.137][2, **COROLLARY** to **PROPOSITION** 3.6.5, p.216] asserts the continuity of the pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous linear maps $f_n : (X, \tau) \longrightarrow (Y, \mu)$, from a barrelled space$(X, \tau)$ into an arbitrary separated locally convex space (Y, μ). In **NOTE 4** below, we show that if the sequence $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded maps [on locally complete (X, μ), its pointwise limit is also bounded, and thus uphold the Banach-Steinhaus Closure Theorem for bounded maps.

In **NOTE 5**, we identify another Hellinger-Toeplitz topology, the associated bornological topology. We also show that the collection of Hellinger-Toeplitz topologies is closed under taking supremum. The numbered results are the results established in this paper.

NOTATION AND LANGUAGE:

We shall follow [2], [4] and [7] for language and notation. All spaces considered are assumed separated. By a $lcs(X, \tau)$ we shall mean a separated locally convex space with continuous dual X', algebraic dual $X^{\#}$, bounded dual X^b, and associated bornological topology τ_b[7, Problem, 4.106, p.50, Theorem 4.4.5, p.48]; τ_b is the finest lcs topology on X having same bounded sets as τ.

For the dual pair $\langle X, Y \rangle$, always separated [7, Paragraph preceding 8.2.2, p.107: The separation conditions (a) and (b) on b...] we denote the weak, the Mackey and the strong topology on X by $\sigma(X, Y'), \tau(X, Y)$ and $\beta(X, Y)$. The topology on X of uniform convergence on the $\beta(Y, X)$-bounded subsets of Y is denoted $\beta^*(X, Y)$. The scalar field of our spaces is $K = R$ or C, the real field or the field of the complex numbers. We denote by θ the zero of the linear space X while 0 denotes the zero of K. We signify with the end or absence of a proof.

NOTE 1 :

AN OBSERVATION ON THE DEFINITION OF A BOUNDED SET IN A LOCALLY CONVEX SPACE

This observation becomes necessary since most books on locally convex space theory, if not all, simply give the definition of a bounded set in a topological vector space without motivating it from the definition of a bounded set in a normed linear space [7, First three lines of p.47][6, Second paragraph under ξ topologies, p.167]; at best a book simply remarks that the definition clearly generalizes the notion of a bounded set in a normed linear space [2, First two lines of the last paragraph of p.108].

OBSERVATION 1:

Let $(E, ||.||)$ be a normed linear space and suppose $B(\alpha, \epsilon)$ denotes the ball of radius ϵ centered on θ. Then;

$$\text{for all } \alpha > 0, \alpha B(\theta, \epsilon) = B(\theta, \alpha\epsilon) \tag{1}$$

OBSERVATION 2

With notation as in the preceding, then

$$B(\theta, \epsilon) \subseteq |\lambda| B(\theta, \alpha\epsilon) \text{for all } |\lambda| \geq \alpha > 0$$

Proof: By (1)

$$\alpha B(\theta, \epsilon) = B(\theta, \alpha\epsilon) \text{and}$$
$$|\lambda| B(\theta, \epsilon) = B(\theta, |\alpha|\epsilon)$$

And clearly, $\alpha\epsilon \leq |\beta|\epsilon$, and so

$$B(\theta, \alpha\epsilon) \subseteq B(\theta, |\lambda|\epsilon)$$

.

Now suppose $(E, \|.\|)$ is a normed space and V is a neighborhood of zero, $theta$. So, there exists $\epsilon > 0$ such that $V \supseteq B(\theta, \epsilon)$. Let $\phi \neq D \subseteq E$ bounded and so there exists $K > 0$ such that $\{||x|| \leq K \text{for all } x \in D\}$.

Hence,

$$\left\| \frac{1}{K} \epsilon x \right\| \leq 1 \text{ for all } \ x \in D$$

And therefore,

$$\left\| \frac{1}{K} \epsilon x \right\| \leq \epsilon \text{ for all } \ x \in D$$

i.e.,

$$\left\| \frac{1}{K} x \right\| \in B(\theta, \epsilon) \text{ for all } \ x \in D$$

from which follows that

$$x \in \frac{\epsilon}{K} B(\theta, \epsilon) \text{ for all } \ x \in D$$

.i.e.

$$x \in \alpha B(\theta, \epsilon) \text{ for all } \ x \in D, \text{ where } \frac{K}{\epsilon}$$
$$\|x\| \leq K \text{ for all } x \in D.$$

Hence,

$$D\alpha \subseteq B(\theta, \epsilon) \tag{2}$$

But by **OBSERVATION 2:**,

$$|\lambda| \geq \alpha \longrightarrow \alpha B(\theta, \epsilon) \subseteq |\lambda| B(\theta, \epsilon) \tag{3}$$

and so by (2) and (3), therefore,

$$D \subseteq |\lambda| B(\theta, \epsilon) \subseteq |\lambda| V \text{ for all } |\lambda| \geq \alpha.$$

So, we have

THEOREM A:

If $(E, \|.\|)$ is a normed space, then, D is a bounded set in $(E, \|.\|) \Longleftrightarrow$ for every neighborhood of zero V of $(E, \|.\|)$, there exists $\alpha > 0$ (depending on V) such that

$$D \subseteq |\lambda| B(\theta, \epsilon) \subseteq |\lambda| V \text{ for all } |\lambda| \geq \alpha.$$

We therefore have

THEOREM B:

If $(E, \|.\|)$ is a normed linear space and $\phi \neq D \subseteq E$, then, D is a bounded set of $(E, \|.\|) \Longleftrightarrow$ For every neighborhood V of zero θ of $(E, \|.\|)$ there exists $\alpha V > 0$ such that

$$D \subseteq |\lambda| B(\theta, \epsilon) \subseteq |\lambda| V \text{ for all } |\lambda| \geq \alpha V$$

Now let V be a neighborhood of zero and V^* a balanced neighborhood of zero contained in the neighborhood V of the normed linear space $(E, \|.\|)$ [At least every ball is absorbing]. Let $\phi \neq D \subseteq E$ be a bounded set of $(E, \|.\|)$. By the preceding theorem, therefore there exists $\alpha V^* > 0$ and $\epsilon^* > 0$ such that

$$D \subseteq |\lambda| B(\theta, \epsilon^*) \subseteq |\lambda| V \text{ for all } |\lambda| \geq \alpha V^* \quad (4)$$

Since V^* is balanced, by [7, Problem 1.5.5, p.9] then $|\lambda| V^* = \alpha V^*$. Therefore, since this argument can be reversed, (1) gives D is a bounded set of $(E, \|.\|) \Longleftrightarrow D \subseteq V^* \subseteq \lambda V$ for all $|\lambda| \geq \alpha V^*$, But by [Definition 2.6.1, p.708] this means D is absorbed by V. So, we have

THEOREM C:

Suppose $(E, \|.\|)$, is a normed linear space and $\phi \neq D \subseteq E$ Then, D is a bounded set of $(E, \|.\|) \Longleftrightarrow D$ is absorbed by every neighborhood of zero.

This explains the adopted definition of a bounded set in a topological vector space.

DEFINITION B: [2, **Definition** 2.6.2, **p.**108] A set D of a topological vector space is bounded if it is absorbed by every neighborhood of zero.

Of course a normed space is a locally convex space, and, a locally convex space is a topological vector space.

NOTE 2:

THREE CHARACTERIZATIONS OF BOUNDED LINEAR MAPS

Let (X, τ) be a *lcs* and B a disc, i.e., an absolutely convex bounded set, in (X, τ). Then,B is an absorbing subset of its linear span X_B in X, and $(X_B, q_B), X_B$ with the gauge q_B of B in X_B, is a normed space [2, **Proposition** 3.5.6(a), p. 207][4, **Proposition** 3.2.2, p. 82]. The sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to locally converge to $x \in X$ in (X, τ) provided converges to x in (X_B, q_B) for some disc B of (X, τ)[2, **Exercise** 3.7.7 p.225][4, **Definition** 5.1.1, p.151]; if $x = \theta$, the zero of X, $\{x_n\}_{n=1}^{\infty}$ is then called a local null sequence. Well known is that $\{x_n\}_{n=1}^{\infty}$ is local null if and only if $\{\lambda_n x_n\}_{n=1}^{\infty}$ is a null sequence for some increasing unbounded real sequence $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n > 0$ for all n [4, **Proposition** 5.1.3 (ii), p.151][2,**Exercise** 3.7.7(b), p.225]. Let $A \subseteq X$. A point $x \in X$is called a local limit point of A if there exists a sequence of elements of A that locally converges to x. A is called a locally closed set of (X, τ) if A is empty or contains all its local limit points, while by the local closure, $\bar{A}^{1c}$, of A is meant the intersection of all the locally closed sets of (X, τ) containing A [4, **Definition** 5.1.18, p. 155]. Clearly, the intersection of an arbitrary collection of locally closed sets is locally closed, and so for $A \subseteq X, \bar{A}^{1c}$ is locally closed. Clearly, also A is locally closed if and only if $A = \bar{A}^{1c}$. Trivially also, a closed set is locally closed, as local convergence implies ordinary convergence [2, **Exercise** 3.7.7(a), (c), p. 225].

Let (X, τ) and (Y, μ) be $lcss$ and $g : (X, \tau) \longrightarrow (Y, \mu)$ a linear map. We shall, in this paper, call g a local sequentially continuous map if the sequence $\{g(x_n)\}_{n=1}^{\infty}$ is a local null sequence in (Y, μ) for every local null sequence $\{x_n\}_{n=1}^{\infty}$ in (X, τ). Clearly, g is local sequentially continuous if $\{x_n\}_{n=1}^{\infty}$ locally converges to xx in (X, τ) implies $\{g(x_n)\}_{n=1}^{\infty}$ locally converges to $g(x)$[4, **Proposition** 5.1.3(i), p. 151][4, **Exercise** 3.7.7 (a), p.225]. We shall also in this paper call the linear map $g : (X, \tau) \longrightarrow (Y, \mu)$ a lc-map if $h^{-1}(A)$ is a locally closed set in (X, τ) for every locally closed set A of (Y, μ). [|Compare the statement of [4, **Lemma** 5.1.23(ii), p.156]|]. This definition is similar to the definition of an sc-map by the author in [5].

We note the following two theorems that fall out as corollaries of our characterizations.

[4, **Lemma** 5.1.23, p.156] Let (X, τ) and (Y, μ) be css, and, $f : (X, \tau) \longrightarrow (Y, \mu)$ be a continuous linear map. Then, (i) f is local sequentially continuous, and
(ii) f is a lc-map.

[4, **Proposition** 5.1.24, p.156] Let $f : (X, \tau) \longrightarrow (Y, \mu)$ be a continuous linear map between the $lcss(X, \tau)$ and (Y, μ) Then, $f(\bar{A}^{1c}) \subseteq \bar{f(A)}^{1c}$.

We now state our new characterizations; these are (i) $\Longleftrightarrow$ (iii), (i)$\Longleftrightarrow$(iv) and (i) $\Longleftrightarrow$ (v) of the following.

THEOREM D

Let (X, τ) and (Y, μ) be $lcss$ and $f : (X, \tau) \longrightarrow (Y, \mu)$ a linear map. The following are equivalent. (i) f is bounded.
(ii) f is local sequentially continuous.
(iii) f is a lc-map.
(iv) $f(\bar{A}^{1c}) \subseteq \bar{f(A)}^{1c}$, for every $A \subseteq X$.
(v) $\bar{f^{-1}(B)}^{1c} \subseteq f^{-1}(\bar{B})^{1c}$, for every $B \subseteq Y$.

Proof We shall show that (i) $\iff$ (ii)$\iff$(iii) $\implies$ (iv) $\iff$ (v) $\implies$ (iii)

(i) $\iff$ (ii) : This is [2, **Exercise** 3.7.7 (f), p. 225]. (ii) $\implies$ (iii) : The empty set is locally closed, and so we have nothing to show if A is empty. So, assume the non-empty set A is locally closed in (Y, μ) and f local sequentially continuous. We proceed as in the proof of [4, **Lemma** 5.1.23, p. 156]. Let $x_n \in f^1(A)$for all n and suppose $\{x_n\}_{n=1}^{\infty}$ locally converge to x. Then, by hypothesis, $\{f(x_n)\}_{n=1}^{\infty}$ converges locally to $f(x)$. Since A is locally closed, $f(x) \in A$ and so $x \inf^1(A)$. Hence, $f^1(A)$ is locally closed.
(ii) $\impliedby$ (iii) : Suppose f is a lc-map and $\{x_n\}_{n=1}^{\infty}$ a local null sequence in (X, τ). Hence, there exists an increasing unbounded sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers such that $\{f(x_n)\}_{n=1}^{\infty}$ is null in (X, τ). Suppose $\{f(x_n)\}_{n=1}^{\infty}$ is not local null in (Y, μ). Then, there exists an open neighborhood of zero, U, of (Y, μ) such that is not eventually in U. So, suppose is a subsequence of $\{f(\lambda_n x_n)\}_{n=1}^{\infty}$ with terms not in$U$, and so it (the subsequence) is a sequence in the complement U U' is closed in (Y, μ) and so locally closed. By hypothesis, $f^{-1}(U')$ is locally closed in (X, τ) and does not contain the zero θ of (X, τ) since U' does not contain the zero of (Y, μ) and f is linear.
Now $\lambda_n x_n \in f^1(U')$ for all k and $\{\lambda_n x_n\}_{n=1}^{\infty}$ is null in (X, τ). Hence, θ is a local limit point of $f^1(U')$. Since $f^1(U')$ is locally closed, $\theta \in f^1(U')$, and we have a contradiction! Hence, the supposition that $\{f(x_n)\}_{n=1}^{\infty}$ is not local null is false, and so is local null. Since $\{f(x_n)\}_{n=1}^{\infty}$ was arbitrary, it follows that f is local sequentially continuous. (iii) $\implies$ (iv): The proof here is the proof of [4, **Proposition** 5.1.24, p.156]. But for clarity, we give the proof in detail. Let f be a lc-map, and $A \subseteq X$. $f(\bar{A})^{1c}$ is locally closed and so by hypothesis, $f^{-1}(f(\bar{A}))^{1c}$ is locally closed in (X, τ). Since $A \subseteq f^{-1}(f(\bar{A}))^{1c}$ it follows that $\bar{A}^{1c} \subseteq f^{-1}(f(\bar{A}))^{1c}$. Hence, $f((\bar{A})^{1c}) \subseteq f(\bar{A}))^{1c}$.

(iv) $\impliedby$ (v) : Let $A \subseteq X$ and put $B = f(A)$ in (v). Then, we have

$$f^{-1}(\bar{f}(A))^{1c} \subseteq f^{-1}\left(f(\bar{A})^{1c}\right) \tag{5}$$

Clearly, $A \subseteq f^{-1}(f(A))$ and so

$$(\bar{A})^{1c} \subseteq f^{-1}(f(A)) \tag{6}$$

By (5) and (6), it now follows that

$$(\bar{A})^{1c} \subseteq f^1(f(A)) \subseteq f^{-1}\left(f(\bar{A})^{1c}\right)$$

$$f(\bar{A}))^{1c} \subseteq f(\bar{A}))^{1c}$$

(iv) $\Longrightarrow$ (v) : The proof here is the proof of [3, $f \Longrightarrow g$ of **THEOREM** 1.4, p.59/60] mutatis mutandis. (v) $\Longrightarrow$ (iii) : Suppose B is a locally closed set. Then $\bar{B}^{1c} = B$, and so by hypothesis

$$f^{-\bar{1}}(B)^{1c} \subseteq f^{-1}(\bar{B})^{1c} = f^{-1}(B) \subseteq f^{-\bar{1}}(B)^{1c}$$

.

Hence,

$$f^{-\bar{1}}(B)^{1c} = f^{-1}(B)$$

Since $f^{-\bar{1}}(B)^{1c}$ is locally closed it follows that $f^{-1}(B)$ is locally closed.

REMARK:

In both normed linear space theory and locally convex space theory, linear map $f : (E, \|.\|) \longrightarrow (F, \|.\|), f : (G, \tau) \longrightarrow (H, \mu)[(E, \|.\|)$ and $(F, \|.\|)$ normed spaces, and (G, τ) and (H, μ) locally convex spaces] is called bounded if it preserves bounded sets. In both theories also every continuous linear map f is bounded; but while every bounded linear map between normed spaces is also continuous, not every bounded linear map f between locally convex spaces is continuous [7, Example 4.4.11, p.49].

NOTE 3:

AN ANALOGUE FOR BOUNDED MAPS :

Let (X, τ) and (Y, μ) be lcss with continuous duals X' and Y', respectively, and u : $(X, \tau) \longrightarrow (X, \mu)$ a linear map. If for all continuous linear functionals f on (X, τ) the compositions fou are continuous, the u is $(\sigma(X, X'), \sigma(Y, Y'))$-continuous and vice vasa [7, **Lemma** 11.1.1, p.164][2, **PROPOSITION** 3.12.1, p.254]. We show in this **NOTE 3** that, however, if four is a bounded linear functional for all bounded linear functionals f on (Y, μ), then u is itself bounded. Thus, the theorem we want to prove in this **NOTE 3** is.

THEOREM E:

Let (X, τ) and (Y, μ) be *lcss* and $u : (X, \tau) \longrightarrow (Y, \mu)$ a linear map. Then, u is bounded if an only if and only if the compositions fou, for all bounded linear functionals f on (Y, μ), are bounded maps.

For the proof we shall need some two lemmas which are of interest in themselves.

LEMMA A:

Let (X, τ) and (Y, μ) be *lcss* and $f : (X, \tau) \longrightarrow (Y, \mu)$ a bounded linear map. Then, f is (τ^b, μ^b)-continuous.

Proof:

Let V be an absolutely convex bornivore of (Y, μ); a bornivore is a set absorbing all bounded sets. We CLAIM that $f^{-1}(V)$ is also an absolutely convex bornivore of (X, τ). The absolute convexity of $f^{-1}(V)$ is easily checked [2, p.80 and 85]. Suppose $\{x_n\}_{n=1}^{\infty}$ is a local null sequence in (X, τ). By **NOTE 1**, the sequence

$\{f(x_n)\}_{n=1}^{\infty}$ is local null in (Y,μ), and so, since a local null sequence is null and so bounded, there exists a real number $\lambda > 0$ such that $f(x_n) \in \lambda V$ for all n[2, p.108]. Hence, $(\frac{1}{\lambda})x_n \in f^{-1}(V)$ for all n. That is, $\lambda x_n \in f^{-1}(V)$, for all n and so again by [2, paragraph following **DEFINITION** 2.6.1, p.108]$f^{-1}(V)$ absorbs the sequence $\{x_n\}_{n=1}^{\infty}$.
Since $\{x_n\}_{n=1}^{\infty}$ was an arbitrary local null sequence, then by [7, Problem 8.6.114, p. 126]$f^{-1}(V)$ is a bornivore. And so our CLAIM is true. The absolutely convex bornivores of a *lcs* constitute a base of neighborhood of zero for its associated bornological topology [2, **Exercises** 3.7.8(a), p. 226]. By the CLAIM it follows[2, **PROPOSITION** 2.5.1, p.97] that f is (τ^b, μ^b)-continuous.

In **LEMMA A**, we have shown that bounded linear map $u : (X,\tau) \longrightarrow (Y,\mu)$ between $lcss(X,\tau)$ and (Y,μ) is (τ^b, μ^b)-continuous and so (τ^b, μ)-continuous. We show in the next lemma that the converse is true. Let B be a disc of $(X,\tau), X_B$ the linear span of B in X and q_B the gauge of B in X_B. If R denotes the collection of all discs of (X,τ) then clearly $\cup_{B\in R} X_B$ covers X. Following [7] we denote the topology of the seminorm σq_B by σq_B. Well-known is [5, **Definition** 6.2.4 and **Proposition** 6.2.5, p. 174] that the inductive limit topology oy the natural inclusions $i_B : (X_B, \sigma q_B) \longrightarrow X$ is the associated bornological topology of$(X,\tau), \tau^b$.

LEMMA B:[2, **Exercise** 3.7.8 (b), p. 226] Let (X,τ) and (Y,μ) be *lcss*. The bounded linear maps $f : (X,\tau) \longrightarrow (Y,\mu)$ are the (τ^b,μ)-continuous linear maps.

Proof :
Suppose linear map $f : (X,\tau) \longrightarrow (Y,\mu)$ are the (τ^b,μ)-continuous. Then, by [7, THEOREM 13.1.8, p. 210] and the preceding discussion, the restriction

$$f|XB : (X_B, \sigma q_B) \longrightarrow (Y,\mu)$$

of f to $(X_B, \sigma q_B)$ is continuous for each disc B of (X,τ), and noting that by [6, Theorem 12.2, p. 112] B is bounded in $(X_B, \sigma q_B)$, it follows that f is a bounded map.

Proof of THEOREM B If u is bounded, then clearly fou is bounded for all bounded linear functionals f on (Y,μ). This trivially establishes the implication $\Longrightarrow$. Now for the implication $\Longleftarrow$, suppose fou is bounded for all $f \in Y^b$. Let $u' : Y^b \longrightarrow X^{\#}, u'(f) = fou, f \in Y^b$. Clearly, by hypothesis, $u' : Y^b \longrightarrow X^b$, and so considering the dual pairs $\langle X, X^b\rangle$ and $\langle Y, Y^b\rangle$ it follows from [7, Lemma 11.1.1,p.164][2, **PROPOSITION** 3.12.1, p. 254] that

$$u : (X, \sigma(X, X^b)) \longrightarrow (Y, \sigma(Y, Y^b)$$

is continuous. By our **LEMMA A**, since continuous maps are bounded, therefore, u is $(\sigma(X,X^b)^b, \sigma(Y,Y^b)^b)$-continuous. But clearly $\tau, \sigma(X,X')$ and τ^b have

same bounded sets, and also $\sigma(X, X^b)$ and τ^b have same bounded sets [7, **THEOREM** 8.4.1,p.114]. Clearly, $(\mu^b)^b = \mu^b$, and all topologies of a dual pair have the same associated bornological topology. Hence, since (X, X^b) and X^b are topologies of the dual pair $\left\langle X, X^b \right\rangle$, $\sigma(X, X^b)^b = (\tau^b)^b = \tau^b$ and $\sigma(Y, Y^b)^b = (\mu^b)^b = \mu^b$. So, u is (τ^b, μ^b)-continuous, and so (τ^b, μ)-continuous, from which follows by **LEMMA B** that u is bounded.
We shall also apply **LEMMA B** in **NOTE 4**.
REMARK A :
The observation in the above proof that all topologies of a dual pair have the same associated bornological topology is a crucial fact to be employed in **NOTE 5** below to deduce that the associated bornological topology is duality invariant and so definable for any separated dual pair $\langle X, Y \rangle$.
REMARK B:
The role of the Hellinger-Toeplitz [7, Definition 11.1.5, p.165 and Section 11.2, p.176.169] property of the associated bornological topology (established in our **LEMMA 1**) in proving our theorem is worth noting. The Hellinger-Toeplitz property of the associated bornological topology is the main thing in **NOTE 4**.
REMARK C:
LEMMA B: is a result common to a number of associated topologies of which we mention some two here. For $lcs(X, \tau)$,
let τ^+ = the finest locally convex topology on X having same convergent sequences as τ, and
τ^{ub} =the finest locally convex topology on X having same Banach discs as τ. (It is also the coarsest ultrabornological topology on X finer than τ). Disc B is called a Banach disc if the normed space (X_B, q_B) is Banach.
(X, τ) is called C-Sequential [7, Problem $58.4.127, 128, 201$, p.118] if$\tau = \tau^+$, bornological if $\tau = \tau^b$ and ultrabornological if $\tau = \tau^{ub}$. And for C-Sequential / bornological / ultrabornological $lcs(X, \tau)$ and arbitrary $lcs(Y, \mu)$, the sequentially continuous linear maps/ the bounded linear maps / the linear maps bounded on Banach discs of $(X, \tau), f : (X, \tau) \longrightarrow (Y, \mu)$, are the continuous linear maps. It can be shown that:

THEOREM E For arbitrary $lcss(X, \tau)$ and (Y, μ)
(a) the sequentially continuous linear maps $f : (X, \tau) \longrightarrow (Y, \mu)$ are the $(\tau^+, \mu)-$continuous linear maps,
(b) the bounded linear maps are the (τ^b, μ)-continuous linear maps [Our Lemma B], and
(c) the linear maps $f : (X, \tau) \longrightarrow (Y, \mu)$ bounded on Banach discs of (X, τ) are the (τ^b, μ)-continuous linear maps.

Observe that the result stated before the preceding theorem is immediately deducible from the Theorem.

REMARK D :

The proof of **Lemma A:** mutatis mutandis can also be used to establish **Lemma A:** for τ^+. That is,

THEOREM E

If for $lcss(X,\tau)$ and $(Y,\mu), u : (X,\tau) \longrightarrow (Y,\mu)$ is sequentially continuous then u is (τ^b, μ^b)-continuous.

We discuss these REMARKS in detail elsewhere.

NOTE 4 :

THE BANACH-STEINHAUS CLOSURE THEOREM IS TRUE FOR BOUNDED MAPS:

In this **NOTE 4** we show that the Banach-Steinhaus closure theorem [6, THEOREM 9.3.7, p.137][2, COROLLARY of PROPOSITION 3.6.5, p. 216] is also valid for bounded maps.

A locally complete lcs is what Willansky in [7, 10.4.3, p.158] calls a Banach-Mackey space. $Lcs(X,\tau)$ is locally complete [1][4, Definition 5.1.5, p.152, Proposition 5.1.6, p.152, Proposition 5.1.11,p.153][7, Definition 10.4.3, p.158] if $\beta^*(X,X') = \beta(X,X')$. Since $Lcs(X,\tau)$ is called quasibarrelled if $\tau = \beta^*(X,X')$ and barrelled if $\tau = \beta(X,X')$, and a bornological space is quasibarrelled, it follows, as is well-known, from the definition of local completeness given, that a locally complete borno- logical space is barrelled.

THEOREM F:(Banach-Steinhaus Closure Theorem[7, **THEOREM** 9.3., p.137][2, **COROLLARY** of **PROPOSITION** 3.6.5, p.216])

Let (X,τ) be a barrelled lcs and (Y,μ) an arbitrary lcs. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous linear maps $f_n : (X,\tau) \longrightarrow (Y,\mu)$ converging pointwise to the linear map $g : (X,\tau) \longrightarrow (Y,\mu)$. Then, g is continuous.

THEOREM G: Let (X,τ) be a locally complete lcs and (Y,μ) an arbitrary lcs. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded maps $f_n : (X,\tau) \longrightarrow (Y,\mu)$ converging pointwise to the linear map $f_n : (X,\tau) \longrightarrow (Y,\mu)$. Then,$g$ is bounded.

Proof Since τ and τ^b have same bounded sets and (X,τ) is locally complete, then τ^b is also a locally complete [4, Proposition 5.1.6 (i) $\Longleftrightarrow$ (iv), p.152] bornological 6.2.5 p. 174] space. The theorem now follows from **LEMMA 2** of **NOTE 3**, the Banach-Steinhaus Closure Theorem, and the observation preceding it, since now $Lcs(X,\tau^b)$ is a barrelled and bornological space and so the continuous maps

$$f : (X,\tau^b) \longrightarrow (Y,\mu)$$

are the bounded maps

$$f : (X,\tau) \longrightarrow (Y,\mu)$$

It follows from our definition of local completeness and barrelledness given above that a barrelled space is locally complete. For, generally,

$$\tau \leq \beta^*(X,X') \leq \beta(X,X).$$

Thus, we now have from **THEOREM G :**

THEOREM H: (Banach-Steinhaus Closure Theorem for bounded maps)
. Let $lcsX,\tau)$ be a barrelled space and (Y,μ) an arbitrary lcs. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded linear maps $f_n : (X,\tau) \longrightarrow (Y,\mu)$ converging pointwise to the linear map $g : (X,\tau) \longrightarrow (Y,\mu)$. Then, g is bounded.

NOTE 5:

HELLINGER-TOEPLITZ TOPOLOGIES : In this (X,τ) **NOTE 5** we furnish (i) an addition to the class of Hellinger-Toeplitz topologies and (ii) a method of obtaining new Hellinger-Toeplitz topologies from known ones. Let (X,τ) be a lcs with continuous dual τ^b. Then, the bornivores of (X,τ) are also the bornivores of (X,X') for any topology τ^* of the dual pair $\langle(X,X'\rangle$, since bounded sets are duality invariant. Hence, since the associated bornological topology τ^b has a base of neighborhoods of zero comprising the absolutely convex bornivores of (X,τ) it follows that τ^b is also duality $\langle X,X'\rangle$ invariant. We consequently here denote it by$b(X,X')$. Thus, all the topologies $\sigma(X,X'),\tau(X,X'),\beta(X,X')$ and $b(X,X')$ are all definable for the dual pair $\langle X,X'\rangle$. Is the finest locally convex topology having same convergent sequences as τ, τ^+[7, **Problem** 8.4.124, p.117] definable for $\langle X,X'\rangle$? We shall extend Wilanskys concept [7, **Definition** 11.1.5, p.165] of a Hellinger -Toeplitz topology by dispensing with the requirement of admissibility.

DEFINITION B:

A separated locally convex topology $\sigma(X,X')$, definable for any dual pair $\langle X,X'\rangle$ shall be called a Hellinger-Toeplitz topology if whenever $\langle X_1,X_1'\rangle$ and $\langle X_2,X_2'\rangle$ are separated dual pairs and $f : \langle X_1,\tau_1\rangle \longrightarrow \langle X_2,\tau_2\rangle$ is a continuous linear map for some topologies τ_1 and τ_2 of the dual pairs $\langle X_1,X_1'\rangle$ and $\langle X_2,X_2\rangle$ respectively, then f is also $(\sigma(X_1,X_1'))$ and $(\sigma(X_2,X_2'))$-continuous.

THEOREM I: Let $\langle X,X'\rangle$ be a dual pair. Then,$b(X,X')$is a Hellinger-Toeplitz topology.

Proof Let $\langle X_1,X_1\rangle$ and $\langle X_2,X_2\rangle$ be separated dual pairs and τ_1 and τ_2 compatible topologies on X_1,X_2, respectively, and $f : \langle X_1,\tau_1\rangle \longrightarrow \langle X_2,\tau_2\rangle$ a continuous, and so bounded, linear map. Then, by **LEMMA 1**, f is (τ_1^b,τ_2^b)-continuous; that is, by the discussion preceding the **DEFINITION** above, f is $(b(X_1,X_1'),b(X_2,X_2'))$-continuous.

THEOREM J:

Let $\langle X,X'\rangle$ be a dual pair and $H(X,X')$ and $H^*(X,X')$ Hellinger-Toeplitz topologies. Then, their supremum $H(X,X')VH^*(X,X')$ is a Hellinger-Toeplitz topology.

Proof Let $\langle X_1,X_1\rangle$ and $\langle X_2,X_2\rangle$ be dual pairs, τ_1 and τ_2 compatible topologies on X_1,X_2, respectively, and $f : (X_1,\tau_1) \longrightarrow (X_2,\tau_2)$ a continuous linear map. By

hypothesis, $f:(X_1,X_1')\longrightarrow(X_2,X_2')$-continuous and also $H^*(X_1,X_1'),H^*(X_2,X_2')$-continuous. Consider the intersection $U\cap V$ of a neighborhood of zero, U of $H(X_2,X_2')$ and a neighborhood of zero, V of $H^*(X_2,X_2')$. Then, $f^{-1}(U\cap V)=f^{-1}(U)\cap f^{-1}(V)$ is, by the Hellinger-Toeplitz property of H and H^*, a neighborhood of zero of $H^*(X_1,X_1')VH^*(X_2,X_2')$.
Hence, f is $(H^*(X_1,X_1')VH^*(X_1,X_1'),H^*(X_2,X_2')VH^*(X_2,X_2'))$-continuous. Therefore, HVH^* is Hellinger-Toeplitz.

COROLLARY 7:
Let $\langle X,X'\rangle$ be a dual pair. Then $b(X,X')V\beta(X,X')$ is a Hellinger-Toeplitz topology.

Given the dual pair $\langle X,X'\rangle$, denote the topology $b(X,X')V\beta(X,X')$ by $b(X,X')$. By **COROLLARY 7** and [711.2.3 p.168] $b\beta?$ is a Hellinger-Toeplitz topology.

QUESTION 1:
Can we located $b\beta?$ That is, descriptions, characterizations and properties of $b\beta??$

REMARK E: THEOREM J is true of an arbitrary collection of Hellinger-Toeplitz topologies.

APPLICATION :
Let (E,τ) be a lcs. Recall that (E,τ) is called bornological if $\tau=\tau^b$. Let $(E_\alpha,\tau_\alpha)\alpha\in I$ be a family of $lcss, E$ a linear space, $f_\alpha:(E_\alpha,\tau_\alpha)\longrightarrow E,\alpha\in I$, linear maps such that $\cup_{\alpha\in I}f_\alpha(E_\alpha)$ spans E . A locally convex topology τ on E is called a test topology for maps $\{f_\alpha\}_{\alpha\in I}$ [7 **Definiion** 13.1.1. p.209] if f_α is (τ_α,τ)-continuous for all $\alpha\in I$. The finest of all test topologies, which we denote here by $ind.lim(f_\alpha,\tau_\alpha)$ is called the inductive limit topology of the space (E_α,τ_α) by the linear maps $\{f_\alpha\}_{\alpha\in I}$. We establish the following well-known result.

THEOREM K: [7, **Theorem** 13.1.13, p.211] A separated inductive limit of bornological lcs spaces is bornological.

Proof:
Let $(E_\alpha,\tau_\alpha)\alpha\in I$, be $lcss, f_\alpha:(E_\alpha,\tau_\alpha)\longrightarrow E$ linear maps into the linear space E. Assume that $\cup_{\alpha\in I}f_\alpha(E_\alpha)$ spans E and that $ind.\lim(f_\alpha,\tau_\alpha)$ is separated. Then, the maps

$$f_\alpha:(E_\alpha,\tau_\alpha)\longrightarrow(E,ind.\lim(f_\alpha,\tau_\alpha))$$

are continuous. By **THEOREM H**, the linear maps

$$f_\alpha:(E_\alpha,\tau_\alpha^b)\longrightarrow(E,ind.\lim(f_\alpha,\tau_\alpha))^b$$

remain continuous. Hence, $ind.\lim(f_\alpha,\tau_\alpha)^b$ is a test topology for the maps

$$f_\alpha : (E_\alpha, \tau_\alpha^b)$$

and so by the definition of the inductive limit,

$$ind.\lim(f_\alpha,\tau_\alpha))^b \leq ind.\lim(f_\alpha,\tau_\alpha^b)$$

Thus, we have

$$ind.\lim(f_\alpha,\tau_\alpha^b) \leq ind.\lim(f_\alpha,\tau_\alpha))^b \leq ind.\lim(f_\alpha,\tau_\alpha^b) \qquad (3)$$

and so if all the spaces (f_α,τ_α^b) are bornological, that is $\tau = \tau_\alpha^b$ for all $\alpha \in I$, then, by (3) we have that

$$ind.\lim(f_\alpha,\tau_\alpha^b) = ind.\lim(f_\alpha,\tau_\alpha))^b$$

from which follows that (f_α,τ^b) is bornological if separated.
By [7COROLLARY 11.2.6, p.169] the Mackey and the strong topologies $\tau(X,Y), \beta(X,Y)$ of the dual pair $\langle X, Y\rangle$ are Hellinger-Toeplitz topologies. Similarly, by [7problem 11.2.101, p.169] the topology $\beta^*(X,Y)$ on X of uniform convergence on the $\beta(Y,X)$-bounded sets is a Hellinger-Toeplitz topology. Recall that $lcs(E,\tau)$ with dual E' is called Mackey or quasibarrelled or barrelled if $\tau = \tau(E,E')$ or $\tau = \beta * (E,E')$ or $\tau = \tau(E,E')$. We have, the proof of**THEOREM J** mutatis mutandi.

THEOREM L: A separated inductive limit of Mackey/quasibarrelled/ barreled spaces is Mackey/quasibarrelled/barreled.
QUESTION 2 : The topology τ^{ub} (see Remark C) is also definable for a dual pair $\langle X, X'\rangle$ and so one can denote it $ub(X,X')$. Is $ub(X,X')$ Hellinger-Toeplitz?

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