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Some Hankel Determinants for Functions satisfying Re $f(z)/z > 0$

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Abstract

We obtain sharp bounds on some Hankel determinants with Fekete-Szego parameter for analytic mappings of the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ satisfying Re $f(z)/z > 0$ in U. Our results extend some known ones.

1. Introduction

Let A denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. In [7], Noonan and Thomas defined the qth Hankel determinants of f for $q \geq 1$, $n \geq 0$ by:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

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This determinant has been considered for specific choices of q and n by several authors which include, in particular, the case $\lambda = 1$ of the well known Fekete-Szego functional $|a_3 - \lambda a_2^2|$ where λ is any real number, as a special case $q = 2$ and $n = 1$.

The second Hankel determinant defined by $H_2(2) = a_2a_4 - a_3^2$ also received a lot of attention by researchers among which is the notable work of Janteng et-al [2]. They considered the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part . In their work, they have shown that if $f \in RT$, then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. They also obtained in [3], the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively.

Babalola [1] observed that for function classes defined by other function classes (for example the classes of close-to-star, close-to-convex, quasi-convex, α-starlike, α-convex, α-close-to-star, α-close-to-convex whose definitions involve other function classes), coefficient functionals of the form $|a_2a_3 - \lambda a_4|$ and $|a_2a_4 - \lambda a_3^3|$ (and possibly more) for the defining function classes have frequently appeared to be resolved in the investigations of Hankel determinants for the desired classes of functions. He therefore defined Hankel determinants with such Fekete-Szego parameters as follows:

Definition 1.1. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the qth-Hankel determinants with Fekete-szego parameter λ, $H_q^\lambda(n)$ as:

$$H_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & \lambda a_{n+q-1} \\ a_{n+1} & \cdots & \dots & \vdots \\ \vdots & \cdots & \dots & \vdots \\ a_{n+q-1} & \cdots & \dots & a_{n+2(q-1)} \end{vmatrix}$$

He also defined another coefficient determinant associated also with function classes of analytic mappings of the unit disk as follows:

Definition 1.2. Let λ be a nonnegative real number. Then for integers $n \geq 1$ and $q \geq 1$, we define the $B_q^\lambda(n)$ determinants as:

$$B_q^\lambda(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \cdots & \dots & \vdots \\ \vdots & \cdots & \dots & \vdots \\ a_{n+q-1} & \cdots & \dots & \lambda a_{n+2(q-1)} \end{vmatrix}$$

In this work, we obtain the best-possible bounds on the determinants,

$$H_2^{\lambda}(2) = \begin{vmatrix} a_2 & \lambda a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - \lambda a_3^2$$

and

$$B_2^{\lambda}(1) = \begin{vmatrix} 1 & a_2 \\ a_3 & \lambda a_4 \end{vmatrix} = a_2 a_3 - \lambda a_4,$$

for the class of functions S_0 satisfying Re $f(z)/z > 0$ studied by Yamaguchi [8].

2. Preliminary lemmas

Let P denote the class of Caratheodory functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic and satisfy $p(0) = 1$, Re $p(z) > 0$ in open unit disk U. To prove the main results in the next section, we need the well known Caratheodory inequality $|c_k| \leq 2$ and the following lemma.

Lemma 2.1. [5, 6] *Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.1}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.2}$$

for some value of x,z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

Our main results are:

Theorem 3.1. *Let $f \in S_0$. Then*

$$|a_2 a_4 - \lambda a_3^2| \leq \begin{cases} 4(1-\lambda) & \text{if } \lambda \leq 0, \\ \frac{3+4\lambda-8\lambda^2}{2(1-\lambda)} & \text{if } 0 < \lambda \leq \frac{3}{4}, \\ 4\lambda & \text{if} \lambda \geq \frac{3}{4}. \end{cases}$$

Proof. Let $f \in S_0$. Then there exists a $p \in P$ such that $f(z)/z = p(z)$. Equating coefficients of $f(z)$ and $zp(z)$, we find that $a_2 = c_1$, $a_3 = c_2$ and $a_4 = c_3$. Therefore,

$$|a_2 a_4 - \lambda a_3^2| = |c_1 c_3 - \lambda c_2^2|. \tag{3.1}$$

Suppose λ is negative, then take $\lambda = -\sigma$ for some $\sigma > 0$. Then

$$|a_2 a_4 - \lambda a_3^2| = |c_1 c_3 + \sigma c_2^2| \leq 4(1 + \sigma) = 4(1 - \lambda).$$

Next suppose λ is nonnegative, then substitute for c_2 and c_3 using Lemma 1, we obtain

$$|a_2a_4 - \lambda a_3^2| = \left| \frac{(1-\lambda)c_1^4}{4} + \frac{(1-\lambda)c_1^2(4-c_1^2)x}{2} - \frac{(c_1^2+\lambda(4-c_1^2))(4-c_1^2)x^2}{4} + \frac{zc_1(1-|x|^2)(4-c_1^2)}{2} \right| \tag{3.2}$$

Using the Caratheodory inequality, $c_1 \leq 2$, and letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$.

Suppose also $(1-\lambda)$ is nonnegative. Applying the triangle inequality on equation (3.2) with $\rho = |x|$, we have

$$\begin{aligned} |a_2a_4 - \lambda a_3^2| &\leq \frac{(1-\lambda)}{4}c^4 + \frac{c(4-c^2)}{2} + \frac{(1-\lambda)c^2(4-c^2)}{2}\rho \\ &+ \frac{[(1-\lambda)c^2 - 2c + 4\lambda](4-c^2)}{4}\rho^2 \\ &= F(\rho) \end{aligned}$$

so that

$$F'(\rho) = \frac{(1-\lambda)c^2(4-c^2)}{2} + \frac{[(1-\lambda)c^2 - 2c + 4\lambda](4-c^2)}{2}\rho$$

and we notice that $F'(\rho) \geq F'(1) > 0$ since the first term is nonnegative, so that

$$F'(\rho) \geq \frac{(4-c^2)[2(1-\lambda)c^2 - 2c + 4\lambda]}{2} \geq (4-c^2)(2-c) > 0.$$

Thus $F(\rho)$ is increasing on $[0, 1]$ so that $F(\rho) \leq F(1)$. That is

$$F(\rho) \leq 4\lambda + (3-4\lambda)c^2 - \frac{(1-\lambda)}{2}c^4 = G(c).$$

If $(3-4\lambda)$ is negative that is $\lambda \geq \frac{3}{4}$, then $G(c)$ is decreasing on $[0, 2]$ so that $G(c) \leq G(0) = 4\lambda$. Otherwise (that is $\lambda \leq \frac{3}{4}$), we have

$$G'(c) = 2(3-4\lambda)c - 2(1-\lambda)c^3$$

Then the maximum of $G(c)$ on $[0, 2]$occurs at $c = \sqrt{\frac{3-4\lambda}{1-\lambda}}$ and is given by

$$G\left(\sqrt{\frac{3-4\lambda}{1-\lambda}}\right) = \frac{3+4\lambda-8\lambda^2}{2(1-\lambda)}.$$

Next we consider that $(1-\lambda)$ is negative. Then we write equation (3.2) as

$$|a_2a_4 - \lambda a_3^2| = \left| \frac{(\lambda-1)}{4}c_1^4 + \frac{(\lambda-1)c_1^2(4-c_1^2)}{2}x + \frac{[c_1^2+\lambda(4-c_1^2)](4-c_1^2)}{4}x^2 \right.$$
$$\left. - \frac{c_1(1-|x|^2)(4-c_1^2)}{2}z \right|$$

Following the same argument as above, we obtain

$$\begin{aligned} |a_2a_4 - \lambda a_3^2| &\le \frac{(\lambda-1)}{4}c^4 + \frac{c(4-c^2)}{2} + \frac{(\lambda-1)c^2(4-c^2)}{2}\rho \\ &+ \frac{[4\lambda - 2c - (\lambda-1)c^2](4-c^2)}{4}\rho^2 \\ &= F(\rho) \end{aligned}$$

$$F(\rho) \le F(1) = 4\lambda - c^2 = G(c)$$

and G(c) is decreasing on $[0,2]$ so that

$$G(c) \le G(0) = 4\lambda.$$

□

Corollary 3.2. *Let $f \in S_0$, then $|a_2a_4 - a_3^2| \le 4$.*

Theorem 3.3. *Let $f \in S_0$. Then*

$$|a_2a_3 - \lambda a_4| \le \begin{cases} 4 - 2\lambda & \text{if } \lambda \le 1, \\ \frac{6\lambda-4}{3}\sqrt{\frac{2-3\lambda}{3(1-\lambda)}} & \text{if } 1 < \lambda \le 2, \\ \frac{6\lambda-4}{3}\sqrt{\frac{2(3\lambda-2)}{3\lambda}} & \text{if } \lambda \ge 2. \end{cases}$$

Proof. If $f \in S_0$, then $a_2 = c_1, a_3 = c_2$ and $a_4 = c_3$. Thus

$$|a_2a_3 - \lambda a_4| = |c_1c_2 - \lambda c_3|.$$

First, suppose λ is negative, then take $\lambda = -\sigma$ for some $\sigma > 0$. Then

$$|a_2a_3 - \lambda a_4| = |c_1c_2 + \sigma c_3| \le 4 + 2\sigma = 4 - 2\lambda.$$

Now substituting for c_2 and c_3 using Lemma 1, we have

$$\begin{aligned} |a_2a_3 - \lambda a_4| = \left| \frac{(2-\lambda)c_1^3}{4} + \frac{(1-\lambda)c_1(4-c_1^2)x}{2} + \frac{\lambda c_1(4-c_1^2)x^2}{4} \right. \\ \left. - \frac{\lambda(4-c_1^2)(1-|x|^2)z}{2} \right| \end{aligned} \tag{3.3}$$

Using the Caratheodory inequality, $|c_1| \le 2$, and letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Suppose $(1-\lambda)$ is nonnegative and applying the triangle inequality on equation (3.3) with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - \lambda a_4| &\leq \frac{(2-\lambda)c^3}{4} + \frac{\lambda(4-c^2)}{2} + \frac{(1-\lambda)c(4-c^2)\rho}{2} \\ &+ \frac{\lambda(c-2)(4-c^2)\rho^2}{4} \\ &= F(\rho) \end{aligned}$$

The extreme values of $F(\rho)$ are at $\rho = 0$, $\rho = 1$ and ρ such that

$$F'(\rho) = \frac{(1-\lambda)c(4-c^2)}{2} + \frac{\lambda(c-2)(4-c^2)\rho}{2} = 0$$

Now, let

$$G_1(c) = F(0) = \frac{(2-\lambda)c^3}{4} + \frac{\lambda(4-c^2)}{2}$$
$$G_2(c) = F(1) = (2-\lambda)c$$

and

$$G_3(c) = F'(\rho) = \frac{(1-\lambda)c(4-c^2)}{2} + \frac{\lambda(c-2)(4-c^2)\rho}{2} = 0$$

implies

$$\rho = \frac{(1-\lambda)c}{\lambda(2-c)}$$

$$F\left(\frac{(1-\lambda)c}{\lambda(2-c)}\right) = \frac{(2-\lambda)c^3}{4} + \frac{\lambda(4-c^2)}{2} + \frac{(1-\lambda)^2c^2(c+2)}{4\lambda}$$

By elementary calculus, we find that $G_1(c) \leq G_1(0) = 2\lambda$, $G_2(c) \leq G_2(2) = 4-2\lambda$ and $G_3(c) \leq G_3(0) = 2\lambda$ for all admissible c. Hence $G(c) \leq G_2(2) = 4 - 2\lambda$.

Next suppose $1-\lambda$ is negative while $2-\lambda$ is nonnegative, that is $(1 < \lambda \leq 2)$. Then we write equation (3.3) as

$$\begin{aligned} |a_2a_3 - \lambda a_4| = \Bigg| &\frac{(2-\lambda)c_1^3}{4} - \frac{(\lambda-1)c_1(4-c_1^2)x}{2} + \frac{\lambda c_1(4-c_1^2)x^2}{4} \\ &- \frac{\lambda(4-c_1^2)(1-|x|^2)z}{2}\Bigg| \end{aligned} \tag{3.4}$$

as above, with $c_1 = c \in [0,2]$ and $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - \lambda a_4| &\leq \frac{(2-\lambda)c^3}{4} + \frac{\lambda(4-c^2)}{2} + \frac{(\lambda-1)c(4-c^2)\rho}{2} \\ &+ \frac{\lambda(c-2)(4-c^2)\rho^2}{4} \\ &= F(\rho) \end{aligned}$$

Applying the same extreme value technique, we find that the extreme value of $F(\rho)$ yielding the best possible bound for the functional is

$$G(c) = F(1) = (1-\lambda)c^3 + (3\lambda - 2)c$$

and that

$$G'(c) = 3(1-\lambda)c^2 + (3\lambda - 2).$$

Thus the maximum of $G(c)$ on [0,2] occurs at $c = \sqrt{\frac{2-3\lambda}{3(1-\lambda)}}$ and is given by

$$G(c) = \frac{6\lambda - 4}{3}\sqrt{\frac{2-3\lambda}{3(1-\lambda)}}.$$

Finally, we suppose $2-\lambda$ is negative, that is $\lambda \geq 2$. Then we write equation (3.3) as

$$\begin{aligned}|a_2a_3 - \lambda a_4| = \Bigg| &\frac{(\lambda-2)c_1^3}{4} + \frac{(\lambda-1)c_1(4-c_1^2)x}{2} - \frac{\lambda c_1(4-c_1^2)x^2}{4} \\ &+ \frac{\lambda(4-c_1^2)(1-|x|^2)z}{2}\Bigg|\end{aligned} \tag{3.5}$$

As we have shown above, with $c_1 = c \in [0,2]$ and $\rho = |x|$, we have

$$\begin{aligned}|a_2a_3 - \lambda a_4| &\leq \frac{(\lambda-2)c^3}{4} + \frac{\lambda(4-c^2)}{2} + \frac{(\lambda-1)c(4-c^2)\rho}{2} \\ &+ \frac{\lambda(c-2)(4-c^2)\rho^2}{4} \\ &= F(\rho).\end{aligned}$$

Using the same extreme value technique, we obtain

$$G(c) = F(1) = (3\lambda - 2)c - \frac{\lambda c^3}{2}$$

and

$$G'(c) = (3\lambda - 2) - \frac{3\lambda c^2}{2}.$$

Thus the maximum of $G(c)$ on $[0,2]$ occurs at $c = \sqrt{\frac{2(3\lambda-2)}{3\lambda}}$ and is given by

$$G\left(\sqrt{\frac{2(3\lambda-2)}{3\lambda}}\right) = \frac{6\lambda-4}{3}\sqrt{\frac{2(3\lambda-2)}{3\lambda}}.$$

□

Corollary 3.4. *Let $f \in S_0$, then $|a_2a_3 - a_4| \leq 2$.*

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