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ON AN INTEGRAL INEQUALITIES OF OPIAL-TYPE

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Abstract

In this paper, we generate some new integral inequalities which are extensions of Sinnamon's, Anthonio and Rauf Opial-type inequalities by means of convexity.

1. Introduction

Since its discovery more than five decades ago, Opial's inequality has found and would continuous to find many interesting applications.
Opial's inequality and its several generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial value problems.
A recent trend in inequality is to establish, mainly by Jensen's inequality and its generalization due to Steffenson, some general inequalities that include some special cases of independent interest and that were originally proved by quite different methods. For recent work, see ([1, 2 & 9]).
The following inequality, which is of wide applications, is due to Opial ([3, 7, 8, 11 & 12]): Let $x\,(t) \in C'\,[0, b]$ be such that $x(0) = x(b) = 0$ and $x(t) > 0$ for all $t \in (0, b)$. Then, the following inequality holds

$$\int_0^b |x(t)x'(t)|dt \leq \frac{b}{4}\int_0^b (x'(t))^2 dt \tag{1}$$

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and the constant $\frac{b}{4}$ is the best possible.

The following results were also established in ([4, 6, & 10]):

([4]) Let g be continuous and non-decreasing on $[a,b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}} \left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}} \tag{1.1a}$$

where

$$C(a,b,p,q,\delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}} g(b)^{p+\delta q} \left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{q}{p}(p-1)} > 0 \tag{1.1b}$$

([6]) Let g be a continuous function which is non-decreasing on $[a,b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Suppose that $p \geq q \geq 1, 0 < q+s \leq p, \delta > 0$ and $f(x)$ is non-negative, non-decreasing and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Then

$$\left[\int_a^b \left[\int_x^b f(t)dg(t)\right]^q f(x)^s \upsilon_1(x)dg(x) + B(p,q,s,\delta,g(a))\right]^{\frac{1}{q+s}}$$

$$\leq C(p,q,s,\delta)^{\frac{1}{q+s}} \left[\int_a^b f(t)^p u_1(t)dg(t)\right]^{\frac{1}{p}}$$

where

$$\upsilon_1(x) = g(x)^{\frac{\delta(q+s)}{p}-1} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{(q+s)-pq}{p}}$$

$$u_1(t) = g(t)^{\frac{pq(1+\delta)}{q+s}-1}$$

$$C(p,q,s,\delta) = [\delta^{-1}]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right]$$

and

$$B(p,q,s,\delta,g(a)) = C(p,q,s,\delta)g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}$$

([10]) Suppose that $0 < l < \infty, 0 < m < k < \infty, k > 1$ and $P(t), q(t) : (\alpha,\tau) \longrightarrow [0,\infty]$ are measurable functions. Further, suppose that for all non-negative measurable functions y(t) on (α,τ) the following Hardy's inequality holds:

(2)
$$\left(\int_{\alpha}^{\tau}\left(q^{k}(t)p^{-m}(t)\right)^{\frac{1}{k-m}}\left(\int_{\alpha}^{t}y(s)ds\right)^{\frac{kl}{k-m}}dt\right)^{\frac{k-m}{kl}}\leq L\left(\int_{\alpha}^{\tau}P(t)|y(t)|^{k}dt\right)^{\frac{1}{k}}$$

where L is a positive constant. Then, for $x(t)\in AC_{o}(\alpha,\tau)$ the following inequality holds

$$\int_{\alpha}^{\tau}q(t)|x(t)|^{l}|x'(t)|^{m}dt\leq L^{l}\left[\int_{\alpha}^{\tau}P(t)|x'(t)|^{k}dt\right]^{\frac{l+m}{k}} \tag{3}$$

In this article, we consider some special cases of our recent work and Sinnamon's results which extends some known results in literature.

2. Main Result

In this section, we shall give our main result which uses the well known Jensen's inequality for convex function. First we need the following Lemma:
Lemma 2.1: (Jensen's inequality)
Let φ be continuous and convex and let $\psi(s)$ be non negative, $s\geq 0$ and λ be non decreasing. Let$-\infty\leq\xi(t)\leq\eta(t)<\infty$, and suppose φ has a continuous inverse φ^{-1}(which is necessarily concave). Then,

$$\varphi^{-1}\left[\frac{\int_{\xi(t)}^{\eta(t)}\psi(s)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)}d\lambda(s)}\right]\int_{\xi(t)}^{\eta(t)}d\lambda(s)\leq\int_{\xi(t)}^{\eta(t)}\varphi^{-1}(\psi(s))d\lambda(s) \tag{2.1a}$$

Then, the sign of inequality (2.1a) is reversed if φ is concave.
The inequality (2.1a) above is known as Jensen's inequality for functions.

Remark. : By setting $\varphi(u)=u^{l},\xi(t)=0\ \eta(t)=t$, then (2.1a) yield for $\mu\geq 1$ that

$$\int_{0}^{t}d\lambda(s)\left[f\left(\frac{\int_{0}^{t}\psi(s)d\lambda(s)}{\int_{0}^{t}d\lambda(s)}\right)\right]^{\frac{1}{\mu}}\leq\int_{0}^{t}f(\psi(s))^{\frac{1}{\mu}}d\lambda(s) \tag{2.1b}$$

Let x(t) and $\lambda(t)$ be absolutely continuous functions [a,b] with $\lambda(t)>0,t\in[a,b]$. Let Q(t) and R(t) be non-negative and measurable function on [a, b] such that either $k>0,l>0,m>0,l>\mu m$ or $l<0$, $m>0$ and $l<\mu m$ where

$l < |m-k|$, then, the following inequality holds:

$$\int_a^b \left[Q(t)^{\frac{kl}{k-m}}\lambda'(t)^l R^{\frac{-klm}{k-m}}\lambda(t)^{\frac{kl}{k-m}}dt\right]^{\frac{k-m}{kl}} \times \left[\left(\left(\left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l dt\right)^{\frac{1}{\omega}}\right]^{\omega} dt$$

$$(1) \qquad \leq \int_a^b \left[y'(t)^{\frac{-m}{k}}y(t)^{\frac{l}{k}}dt\right]^{\frac{1}{k(l-m)}} \times \left[\left(Q(t)\lambda'(t)\lambda(t)^{\frac{l(k-m)}{k}}|x'(t)|^m dt\right)^{\frac{1}{\omega}}\right]^{\omega} dt$$

Proof:

In (2.1b), putting $\xi(t) = 0, \eta(t) = t$ and $\psi(s) = x'(s)R(s)$, we have

$$(2) \qquad f\left(\frac{\int_0^t x'(t)R(t)d\lambda(t)}{\int_0^t d\lambda(t)}\right)^{\frac{1}{\mu}} \leq \frac{\int_0^t f(x'(t)R(t))^{\frac{1}{\mu}}d\lambda(t)}{\int_0^t d\lambda(t)}$$

That is,

$$(3) \qquad f\left(\frac{\int_0^t x'(t)R(t)d\lambda(t)}{\int_0^t d\lambda(t)}\right) \leq \left(\frac{\int_0^t f(x'(t)R(t))^{\frac{1}{\mu}}d\lambda(t)}{\int_0^t d\lambda(t)}\right)^{\mu}$$

Using the fact that $f(u) = u^l$, implying

$$(4) \qquad \left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right)^l \leq \lambda^{l-\mu}(t)\left(\int_0^t x'(t)^l R(t)^l)^{\frac{1}{\mu}}d\lambda(t)\right)^{\mu}$$

Setting $\mu = \frac{l}{k}$ then, inequality (2.4) becomes

$$\left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l \leq \lambda^{\frac{kl-l}{k}}(t)\left(\int_0^t (x'(t)^l R(t)^l)^{\frac{k}{l}}d\lambda(t)\right)^{\frac{l}{k}}$$

$$\leq \lambda^{\frac{l(k-1)}{k}}(t)\left(\int_0^t x'(t)^k R(t)^k d\lambda(t)\right)^{\frac{l}{k}}$$

Putting $y(t) = \int_0^t x'(t)^k R(t)^k d\lambda(t)$ to obtain

$$(5) \qquad \left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l \leq \lambda(t)^{\frac{l(k-1)}{k}}y(t)^{\frac{l}{k}}$$

Now that

$$y(t) = \int_0^t x'(t)^k R(t)^k d\lambda(t)$$

$$y'(t) = x'(t)^k R(t)^k \lambda'(t)$$

$$y'(t)^{\frac{m}{k}} = x'(t)^m R(t)^m \lambda'(t)^{\frac{m}{k}}$$

$$(6) \qquad R(t)^{-m}\lambda'(t)^{\frac{-m}{k}} = x'(t)^m y'(t)^{\frac{-m}{k}}$$

Then, multiplying both sides of (2.6) with Q(t)$\lambda'(t)$ to have

$$Q(t)\lambda'(t)^{\frac{k-m}{k}}R(t)^{-m} = Q(t)\lambda'(t)|x'(t)|^m y'(t)^{-\frac{m}{k}} \tag{7}$$

Combining both (2.5) and (2.7) to obtain

$$Q(t)R(t)^{-m}\lambda'(t)^{\frac{k-m}{k}}\left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l \leq Q(t)\lambda'(t)\lambda^{\frac{l(k-1)}{k}}(t)y'(t)^{-\frac{m}{k}}y(t)^{\frac{l}{k}}x'(t)^m \tag{8}$$

By integration both sides of inequality (2.8) over $[a,b]$ with respect to t and by using Hölder's inequality with indices $\frac{kl}{k-m}$ and $\frac{1}{\omega}$ on left hand side with $(l-m)$ and $\frac{1}{\omega}$ on the right hand side to obtain:

$$\int_a^b \left[Q(t)^{\frac{kl}{k-m}}\lambda'(t)^l R^{\frac{-klm}{k-m}}\lambda(t)^{\frac{kl}{k-m}}dt\right]^{\frac{k-m}{kl}} \times \left[\left(\left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l dt\right)^{\frac{1}{\omega}}\right]^{\omega} dt$$

$$\leq \int_a^b \left[y'(t)^{\frac{-m}{k}}y(t)^{\frac{l}{k}}dt\right]^{\frac{1}{(l-m)}} \times \left[\left(Q(t)\lambda'(t)\lambda(t)^{\frac{l(k-m)}{k}}x'(t)^m dt\right)^{\frac{1}{\omega}}\right]^{\omega} \tag{9}$$

This completes the proof of the Theorem.

Remark. Putting $R(t) = P(t)^{\frac{1}{l(k-1)}}, Q(t) = q(t)^{\frac{1}{l}}P(t)^{\frac{1}{l(k-1)}}, \lambda'(t) = P(t)^{-\frac{1}{l(k-1)}}$

in integral inequality (2.9), we obtain another useful inequality:

$$\left[\int_a^b q(t)^{\frac{k}{k-m}}P(t)^{\frac{k}{(k-1)(k-m)}-\frac{1}{k-m}-\frac{km}{(k-1)(k-m)}}\left(\int_0^t P(t)^{-\frac{1}{l(k-1)}}dt\right)^{\frac{kl}{k-m}}dt\right]^{\frac{k-m}{kl}} \times \left[\left(\int_a^b\int_0^t x'(t)dt\right)^{\frac{l}{\omega}}\right]^{\omega}$$

$$\leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t)\right)^{\frac{-m}{k}}\left(|x'(t)|^k R(t)^k d\lambda(t)\right)^{\frac{l}{k}}dt\right]^{\frac{1}{(l-m)}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}}|x'(t)|^m\int_0^t P(t)^{-\frac{k-m}{k-1}}dt\right)^{\frac{1}{\omega}}\right]^{\omega} \tag{10}$$

$$\left[\int_a^b q(t)^{\frac{k}{k-m}}P(t)^{\frac{k-(k-m)-km}{(k-1)(k-m)}}\left(\int_0^t P(t)^{-\frac{1}{l(k-1)}}dt\right)^{\frac{kl}{k-m}}dt\right]^{\frac{k-m}{kl}} \times \left[\left(\int_a^b x(t)dt\right)^{\frac{l}{\omega}}\right]^{\omega}$$

$$\leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t)\ dt\right)^{\frac{l-m}{k}}\right]^{\frac{1}{(l-m)}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}}|x'(t)|^m\int_0^t P(t)^{-\frac{k-m}{k-1}}dt\right)^{\frac{1}{\omega}}\right]^{\omega} \tag{11}$$

$$\left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{-m}{(k-m)}} \left(\int_0^t P(t)^{-\frac{1}{l(k-1)}} dt\right)^{\frac{kl}{k-m}} dt\right]^{\frac{k-m}{kl}} \times \left[\left(\int_a^b x(t)dt\right)^{\frac{l}{\omega}}\right]^{\omega} dt$$

$$(12) \quad \leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t)\right) dt\right]^{\frac{1}{k}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}} |x'(t)|^m \int_0^t P(t)^{-\frac{k-m}{k-1}} dt\right)^{\frac{1}{\omega}}\right]^{\omega}$$

$$\left[\int_a^b \left(q(t)^k P(t)^{-m}\right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{l(k-1)}}\right)^{\frac{kl}{k-m}} dt\right]^{\frac{k-m}{kl}} \times \left[\left(\int_0^t x'(t)dt\right)^{\frac{l}{\omega}}\right]^{\omega}$$

$$(13) \quad \leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t)dt\right)\right]^{\frac{1}{k}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}} |x'(t)|^m \int_0^t P(t)^{-\frac{k-m}{k-1}} dt\right)^{\frac{1}{\omega}}\right]^{\omega}$$

$$\left[\int_a^b \left(q(t)^k P(t)^{-m}\right)^{\frac{1}{k-m}} \left(\int_\alpha^t P(t)^{-\frac{1}{l(k-1)}}\right)^{\frac{kl}{k-m}} dt\right]^{\frac{k-m}{kl}} \times \left[\left(\int_0^t x'(t)dt\right)^{\frac{l}{\omega}}\right]^{\omega}$$

$$(14) \quad \leq \left[\left(\int_a^b x'(t)^k P(t)^{\frac{1}{l}} dt\right)\right]^{\frac{1}{k}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{-\frac{k-m}{k-1}} dt\right)^{\frac{1}{\omega}}\right]^{\omega}$$

Remark. Putting $r(t) = P(t)^{-\frac{1}{l(k-1)}} = x'(t), P(t) = P(t)^{\frac{1}{l}}, \omega = 0$ and $L = \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{-\frac{k-m}{k-1}} dt\right)^{\frac{1}{\omega}}\right]^{\omega}$ in inequality (2.14) to obtain

$$(15) \quad \left[\int_a^b \left(q(t)^k P(t)^{-m}\right)^{\frac{1}{k-m}} \left(\int_\alpha^t r(s)ds\right)^{\frac{kl}{k-m}} dt\right]^{\frac{k-m}{kl}} \leq L \times \left[\int_a^b P(t) r(t)^k dt\right]^{\frac{1}{k}}$$

Let takes all assumptions of Theorem 2.1. Then, the following inequality also holds:

$$\int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)^l R'(t) d\lambda(t)\right)^l$$

$$(16) \quad \leq \int_a^b Q(t)(\lambda'(t))^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-m)}{k}} R^{-m}(t) y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}}$$

Proof:
Rearrange equation (2.6) and simplify to obtain the following

$$(17) \quad x'(t)^m = R(t)^{-m} \lambda'(t)^{-\frac{m}{k}} y'(t)^{\frac{m}{k}}.$$

Then, multiplying both sides of (2.17) with $\lambda'(t)$ to have

$$\lambda'(t)x'(t)^m = \lambda'(t)^{\frac{k-m}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}} \tag{18}$$

Combining both (2.4) and (2.18) to obtain:

$$\lambda'(t)x'(t)^m \times \left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l \leq \lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} \times y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} \tag{19}$$

Multiplying both sides of (2.19) with $Q(t)$ then integration over $[a,b]$ with respect to t to obtain:

$$\int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l dt$$

$$\leq \int_a^b Q(t)\lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} dt \tag{20}$$

Next, by applying the Hölder's inequality with indices $\frac{k}{m}$ and $\frac{k}{k-m}$ to the right hand side of (2.20), we have:

$$\int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l dt$$

$$\leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt\right]^{\frac{k-m}{k}} \times \left[\int_a^b y'(t)y(t)^{\frac{l}{m}} dt\right]^{\frac{m}{k}} \tag{21}$$

Hence,

$$\int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t)\right)^l dt \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt\right]^{\frac{k-m}{k}}$$

$$\times \left[\int_a^b \left(\int_0^t |x'(t)^k| R(t)^k d\lambda(t)\right)^{\frac{l}{m}} \left(\int_0^t |x'(t)^k| R(t)^k d\lambda(t)\right) dt\right]^{\frac{m}{k}} \tag{22}$$

$$\Rightarrow \int_a^b Q(t)\lambda'(t)x'(t)^m \left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right)^l dt$$

$$\leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt\right]^{\frac{k-m}{k}} \times \left[\int_a^b \left(\int_0^t x'(t)^k R(t)^k d\lambda(t)\right)^{\frac{l+m}{m}} dt\right]^{\frac{m}{k}} \tag{23}$$

and the proof is complete.

Theorem 2.3 : Under the assumptions of Theorem 2.2, then the following inequality holds:

$$\int_a^b q(t)x'(t)^m x(t)^l dt \leq L^l \left[\int_a^b P(t)x'(t)^k dt\right]^{\frac{l+m}{k}} \tag{24}$$

Proof:

In Theorem 2.1, by setting $\lambda'(t) = P(t)^{-\frac{1}{k-1}}, Q(t) = q(t)P(t)^{\frac{1}{k-1}}, R(t) = P(t)^{\frac{1}{k-1}}$ and $\lambda(t) = \int_0^t P(t)^{-\frac{1}{k-1}} dt$, then inequality (2.24) yields

$$\int_a^b q(t)P(t)^{\frac{1}{k-1}} P(t)^{-\frac{1}{k-1}} x(t)^m \left(\int_0^t x'(t)dt\right)^l dt$$
$$\leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_a^b P(t)^{-\frac{1}{k-1}} dt\right)^{\frac{l(k-1)}{k-m}}\right]^{\frac{k-m}{k}}$$
$$\times \left[\left(\int_0^t |x'(t)|^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt\right)^{\frac{l+m}{m}}\right]^{\frac{m}{k}} \tag{25}$$

That is

$$\int_a^b q(t)|x(t)|^m \left(\int_0^t |x'(t)|dt\right)^l dt$$
$$\leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_a^b P(t)^{-\frac{1}{k-1}} dt\right)^{\frac{l(k-1)}{k-m}}\right]^{\frac{k-m}{k}}$$
$$\times \left[\left(\int_a^b |x'(t)|^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt\right)^{\frac{l+m}{m}}\right]^{\frac{m}{k}} \tag{26}$$

Further simplification yeilds,

$$\int_a^b q(t)|x'(t)|^m |x(t)|^l dt \leq \left[\int_a^b \left(q(t)^k P(t)^{-m}\right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt\right)^{\frac{l(k-1)}{k-m}} dt\right]^{\frac{k-m}{k}}$$
$$\times \left[\int_a^b P(t)|x'(t)|^k dt\right]^{\frac{l+m}{k}} \tag{27}$$

By setting

$$L^l = \left[\int_a^b \left(q(t)^k P(t)^{-m}\right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt\right)^{\frac{l(k-1)}{k-m}} dt\right]^{\frac{k-m}{k}}$$

then, inequality (2.27) becomes

$$\int_a^b q(t)|x'(t)|^m|x(t)|^l dt \leq L^l \left[\int_a^b P(t)|x'(t)|^k dt\right]^{\frac{l+m}{k}} \tag{28}$$

Taking the $(l+m)^{th}$ root of both sides of inequality (2.28) to obtain

$$\left[\int_a^b q(t)x'(t)^m x(t)^l dt\right]^{\frac{1}{l+m}} \leq L^{\frac{1}{l+m}} \left[\int_a^b P(t)x'(t)^k dt\right]^{\frac{1}{k}} \tag{29}$$

Remark 2.4 :
When $w(t) = q(t)$, $m = 0$, $C = L^{\frac{1}{l}}$, $q = l$, $k = p$, $u(t) = x'(t)$, $u'(t) = x(t)$ and $v(t) = P(t)$ inequality (2.29) reduces to the form of (1.1a) in [4] and refine [6].
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References

[1] Anthonio Y. O., Salawu S. O. and Sogunro S. O., (2014). Dual Results of Opial's inequality, *IOSR J. Math.* **10**, 01-04.

[2] Anthonio Yisa Oluwatoyin and Rauf Kamilu, (2015). On New Variations of Opial-type Integral Inequalities, *Global J. of Math.* **3**(1), 226-231.

[3] Beesack, P. R., (1962). On an Integral Inequality of Opial, *Trans. Amer. Math. Soc.* **104**, 470-475.

[4] Adeagbo-Sheikh, A. G. and Imoru, C. O., (2006). An Integral Inequality of the Hardy's -type. *Krag. J. Math.* **29**, 57-61.

[5] Imoru, C. O. and Adeagbo-Sheikh, A. G., (2013). On an Integral Inequality of the Hardy's-type, Austral J. of Math. Ana. and App. **5**(4), 56-64.

[6] Adeagbo-Sheikh, A. G. and Fabelurin. O. O., (2011). On a Bessack's Inequality related to Opial's and Hardy's. *Krag. J. Math.* **35** (1), 145-150.

[7] Olech, C., (1960). A Simple Proof of a certain result of Opial Z. *Ann., Polon. Math.* **8**, 61-63.

[8] Opial, Z., (1960). Sur une intégalité, *Ann., Polon. Math.* **8**, 29-32.

[9] Fabelurin, O. O., Adeagbo-Sheikh, A. G. and Anthonio, Y. O., (2010). On an inequality related to Opial, *Octogon Math Magazine*, **18** (1), 32-41.

[10] Sinnamon, G. J., (1991). Weighted Hardy and Opial-type inequalities, *J. Math. Anal. Appl.* **160**, 434-445.

[11] Shum, D. T., (1974). A general and sharpened form of Opial's inequality, *Canad. Math. Bull.* **17**, 385-389.

[12] Shum, D. T., (1975). On a class of new inequalities, *Trans. Amer. Math. Soc.* **204**, 299-341.