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A Modified Interior Penalty Function Method for Constrained Optimization Problems

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Abstract

A modified Interior penalty function method algorithm for solving non-linear constrained optimization problems is presented in this work. The modified scheme was achieved by introducing a new function to the existing method and the proof of convergence for the new function is established. The modified algorithm is used to solve numerical examples and the results obtained show that the proposed interior penalty function method is convergent to the exact solutions which shows that the method is effective and reliable in solving nonlinear constrained optimization problems.

1. Introduction

Nonlinear constrained optimization problems form an important class of problems with a broad range of engineering, scientific, and operational applications. Solving nonlinear problems is an important part of optimization. Optimization can be of constrained or unconstrained problems. The presence of constraints in a nonlinear programming creates more problems while finding the minimum as compared to unconstrained ones. Several situations can be identified depending on the effect of constraints on the objective function. The simplest situation is

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when the constraints do not have any influence on the minimum point. Here the constrained minimum of the problem is the same as the unconstrained minimum, that is, the constraints do not have any influence on the objective function. For simple optimization problems it may be possible to determine before hand, whether or not the constraints have any influence on the minimum point. However, in most of the practical problems, it will be extremely difficult to identify it. Thus, one has to proceed with general assumption that the constraints will have some influence on the optimum point. The minimum of a nonlinear programming problem will not be, in general, an extreme point of the feasible region and may not even be on the boundary.

Penalty function methods are methods that transform the basic optimization problem into alternative formulations such that numerical solutions are sought by solving a sequence of unconstrained minimization problems.The basic optimization problem with inequality constraints is given by

Problem 1.1

Find x which minimizes F(x)
subject to

$$g_i(x) \leq 0, i = 1, 2, ..., m \tag{1}$$

This problem is converted into an unconstrained minimization problem by constructing a function of the form

$$\theta_k = \theta(x, \mu_k) = f(x) + \mu_k \sum_{i=1}^{m} G_i[g_i(x)] \tag{2}$$

where G_i is some function of the constraint g_i and μ_k is a positive constant known as the penalty parameter.
If the unconstrained minimization of the $\theta - function$ is repeated for a sequence of values of the penalty parameter $\mu_k(k = 1, 2, ...)$ the solution may be brought to converge to that of the original problem. That is why the penalty function methods are also known as sequential unconstrained minimization techniques(SUMTS).
When solving a general nonlinear programming problem in which the constraints cannot easily be eliminated, it is necessary to balance the aims of reducing the objective function and staying inside the feasible region, in order to induce global convergence(that is convergence to a local solution from any initial approximation). The idea of penalty method is simple, you give a "‘fine” for violating the

constraints and obtain approximate solutions to your original problem by balancing the objective function and a penalty term involving the constraints. By decreasing the penalty term, the approximate solution is forced to approach the feasible region and consequently, the solution of the original constrained problem. The two main branches of penalty methods are;
1. Interior penalty and,
2. Exterior penalty
Some commonly used forms of the interior method are;

(1) $G_i = \frac{-1}{g_i(x)}$

(2) $G_i = log[-g_i(x)]$

Similarly, some commonly used forms of the exterior method are;

(1) $G_i = max[0, g_i(x)]$

(2) $G_i = max[0, g_i(x)]^2$

In the interior method, the unconstrained minima of θ_k all lie in the feasible region and converge to the solution of the original problem as μ_k is varied in a particular manner. In the exterior method, the unconstrained minima of θ_k all lie in the infeasible region and converge to the desired solution from the outside as μ_k is varied.

Definition 1.1. A function $B : \Re^n \to \Re$ is called an interior penalty (Barrier) function if B(x) satisfies:
(i) $B(x)$ is continuous on $\Re^n$
(ii) $B(x) \geq 0$ for all $x \in intS$, where S denotes the feasible region for Problem 1.1
(iii) $B(x) \to +\infty$ as x approaches the boundary of S.

The original form of the interior penalty function θ is given by (Kirsch, 1981).

$$\theta(x, \mu_k) = f(x) - \mu_k \left(\sum \frac{1}{g_i(x)} + \sum \frac{1}{h_j(x)} \right) \tag{3}$$

where μ_k reduces from a high value to 0 gradually.
Rao (1984) proposed the following function for selecting μ_k at the start of the optimization procedure:

$$\mu_1 = (\alpha_k)\left(\frac{f(x_1)}{-\sum \frac{1}{g(x_1)}}\right), \tag{4}$$

where $\alpha_k \in [0.1, 1]$, and x_1 is the initial point in the feasible region. The optimization procedure is similar to the exterior penalty function method except that μ_k reduces to 0 gradually.
The reduction follows Rao(1984):

$$\mu_{k+1} = c\mu_k \tag{5}$$

where c is a coefficient less than 1. Hence, ordinary penalty functions generally require the value of some coefficient to be specified at the beginning of the optimization. Besides the classical nonlinear optimization method, smart computational techniques, especially Genetic Algorithms, have been applied to find the minimum of the response surface θ.
Some special penalty functions such as the Death penalty, Static and Dynamic penalty and Maximum-violation penalty methods have been introduced and discussed by (Barbosa and Lemonge, 2003),(Wu and Walski, 2005) and (Gupta, et.al, 2007).
Joghataie and Takalloozadeh (2009) proposed a new penalty function defined as;

$$\theta(x, \mu_k) = f(x) + \mu_k\left(\sum \frac{1}{g_i(x)} + \sum \frac{1}{h_j(x)}\right) + \phi(\mu_k) \tag{6}$$

where they only presented the proof for convergence of the new interior penalty method. In this work we extend the work of Joghataie and Takalloozadeh (2009) to produce a new scheme and also prove the convergence of our method.

2. The Proposed New Interior Penalty Function

Given Problem 1.1, we propose a new interior penalty function in the form

$$\theta(x, \mu_k) = f(x) + \mu_k\left(\sum \frac{1}{g_i(x)} + \sum \frac{1}{h_j(x)}\right) + \phi(\mu_k) \tag{7}$$

where μ_k is the penalty parameter, $g_i(x)$ and $h_j(x)$ are constraints, ϕ is a function of μ_k of the form

$$\phi(\mu_k) = \alpha_k \mu_k^2, \tag{8}$$

where α_k is a constant.
The new function $\phi(\mu_k)$ has the following properties:

(9) $$\mu_k > 0$$

(10) $$\phi(\mu_k) \geq 0, \forall \mu_k \geq 0$$

(11) $$\lim_{\mu_k \to \infty} \phi(\mu_k) = 0$$

Also, $\phi(\mu_k)$ is monotonically decreasing with μ_k, i.e if

(12) $$\mu_{k_1} \leq \mu_{k_2} \ then \ \phi(\mu_{K_1}) \geq \phi(\mu_{k_2})$$

It is therefore expected that the new interior penalty function is minimized gradually with the advancement of the optimization, when μ_k increases towards very high values, forcing the constraints to be satisfied and $\phi(\mu_k)$ reduces to zero. Therefore any function satisfying the above criteria could be used for $\phi(\mu_k)$.
The general rule is that a large α_k should be selected for accuracy. A suitable set of convergence criteria is

(13) $$\|x_k - x_{k-1}\| < \epsilon,$$

For accuracy where $\|.\|$ is the length of vector and ϵ is a small positive number to be specified beforehand.

3. Proof of Convergence for the New Penalty Function

We show the convergence of the new penalty function by proving the following theorem.

Theorem 1. The minimization of the new θ function provides the optimal solution, whether the answer to the constrained optimization problem is a point x inside the feasible region or on the feasible surface and it never provides answer outside the feasible region.
Proof:
Assuming that f, g_i, i=1,2,...,m and h_j, $j = 1, 2, ..., k$ are continuous and that an optimum solution exists for the given problem (1.1) and also that the optimum of θ is a point x inside the feasible region, then

(14) $$\sum \frac{1}{g_i(x)} + \sum \frac{1}{h_j(x)} = 0$$

(15) $$\therefore \ \theta(x, \mu_k) = f(x) + 0 + \alpha_k \mu_k^2$$

Recall that f and $\alpha_k\mu_k^2$ are independent. Therefore θ is minimized when both functions are minimized independently. The minimum of $\alpha_k\mu_k^2$ is zero when $\mu_k \to \infty$ and the minimum of f inside the feasible region is obtained for $x = x^*$. Suppose x^* is the optimal solution to θ and is inside the feasible region. Then we have that:

$$\theta(x^*, \mu_k) = f(x^*) + \mu_k \left(\sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} \right) + \phi(\mu_k) \tag{16}$$

where

$$\phi(\mu_k) = \alpha_k \mu_k^2 \tag{17}$$

and θ is a function of μ_k
The optimum is obtained by equating the first derivative to zero;

$$\therefore \ \frac{\partial\theta}{\partial\mu_k} = 0 \Rightarrow \sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} + 2\alpha_k\mu_k = 0 \tag{18}$$

Solving for μ_k , we have

$$\mu_k = \frac{-1}{2\alpha_k} \left(\sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} \right) \tag{19}$$

Substituting (19) into (16) gives

$$\begin{aligned}\theta(x^*, \mu_k) = f(x^*) + &\left[\frac{-1}{2\alpha_k} \left(\sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} \right) \left(\sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} \right) \right] \\ &+ \phi\left(\frac{-1}{2\alpha_k} \left(\sum \frac{1}{g_i(x^*)} + \sum \frac{1}{h_j(x^*)} \right) \right)\end{aligned} \tag{20}$$

Simplifying (20) we get

$$\begin{aligned}\theta(x*, \mu_k) = f(x*) + &\left[\frac{-1}{2\alpha_k} \left(\sum \frac{1}{g_i(x*)^2} - \frac{1}{\alpha_k} \left(\sum \frac{1}{g_i(x*)} \sum \frac{1}{h_j(x*)^2} \right) - \frac{1}{2\alpha_k} \sum \frac{1}{h_j(x*)^2} \right) \right] \\ &+ \phi \left[\frac{-1}{2\alpha_k} \left(\sum \frac{1}{g_i(x*)} - \frac{1}{2\alpha_k} \sum \frac{1}{h_j(x*)} \right) \right]\end{aligned} \tag{21}$$

If α_k is assigned a sufficiently large value, the second and third terms become zero.
Therefore as $\alpha_k \to \infty$, $\theta(x^*, \mu_k) = f(x^*)$, and hence x^* is the solution to the original constrained optimization problem.

3.1. **The Modified Interior Penalty Function Algorithm:** Step 1. Choose a suitable interior(Barrier) function B(x). In practice we take $B(x) = -\sum_{i=1}^{m} \frac{1}{g_i(x)}$ or $-\sum_{i=1}^{m} log(g_i(x))$.
Step 2. Choose a starting point $x^{(o)} \in intS$ and construct the unconstrained minimization problem as:
$Min\ \theta(x, \mu_k) = f(x) + \mu_k B(x) + \phi(\mu_k)$ and choose $\{\mu_k\}_{k=1}^{\infty}$ such that for all k, $\mu_k > 0$, $\mu_{k+1} < \mu_k$ and $\mu_k \to 0$ as $k \to \infty$.
In practice we take $\mu_1 = 10, \mu_2 = 1, \mu_3 = 0.1, \mu_4 = 0.01$ etc. Next, solve by a suitable unconstrained minimization technique starting with the point $x^{(o)}$. Let $\bar{x}^{(1)}$ be an optimal solution . Set $k = 1$.
Step 3. Construct the following unconstrained minimization problem for μ_{k+1}
$Min\ \theta(x, \mu_{k+1}) = f(x) + \mu_{k+1} B(x) + \phi(\mu_{k+1})$
and solve by a suitable unconstrained minimization technique starting with $\bar{x}^{(k)}$, where $\bar{x}^{(k)}$, is the solution of μ_k as obtained at the preceding step.
Step 4. Continue till $\mu_k B(\bar{x})^{(k)}$ is close to zero or equivalently $f(\bar{x}^{(k)})$ is close to $\theta(\bar{x}^{(k)}, \mu_k)$ i.e $\mu_k B(\bar{x})^{(k)} < \epsilon$ or $\theta(\bar{x}^{(k)}), \mu_k) - f(\bar{x}) < \epsilon$ for a tolerance $\epsilon > 0$

4. Numerical Problems

The modified interior function algorithm is used to solve the following problems and the results are presented in Tables 4.1- 4.4

Problem 4.1. *minimize* $f(x_1, x_2) = \frac{1}{3}(x_1 + 1)^3 + x_2$
subject to

$$g_1(x_1, x_2) = -x_1 + 1 \leq 0$$

$$g_2(x_1, x_2) = -x_2 \leq 0$$

solution: The corresponding unconstrained minimization problem is
$\theta(x, \mu_k) = f(x) - \mu \sum \frac{1}{g_1(x)} + \sum \frac{1}{h_j(x)} + \phi(\mu_k)$
$\theta(x, \mu_k) = \frac{1}{3}(x_1 + 1)^3 + x_2 - \mu_k \left[\frac{1}{x_1 - 1} + \frac{1}{x_2}\right] + \phi(\mu_k)$
To find the unconstrained minimum, we use the necessary condition and we have that

$$x_1^*(\mu_k) = (\mu_k^{\frac{1}{2}} + 1)^{\frac{1}{2}}$$

$$x_2^*(\mu_k) = \mu_k^{\frac{1}{2}}$$

and we have that

$$\theta_{min}(\mu_k) = \left\{ \frac{1}{3}[(\mu_k^{\frac{1}{2}}+1)^{\frac{1}{2}}+1]^3 + 2\mu_k^{\frac{1}{2}} - \frac{1}{\frac{1}{\mu_k} - \left(\frac{1}{\mu_k^{\frac{3}{2}}} + \frac{1}{\mu_k^2}\right)^{\frac{1}{2}}} \right\}$$

To obtain the solution of the original problem, we know that

$$\theta_{min} = \lim_{\mu_k \to 0} \theta_{min}(\mu_k)$$

$$x_1^* = \lim_{\mu_k \to 0} x_1^*(\mu_k)$$

$$x_2^* = \lim_{\mu_k \to 0} x_2^*(\mu_k)$$

Hence, the exact solution is $x_1^* = 1$ and $x_2^* = 0$
The values of x_1^*, x_2^* and θ corresponding to the decreasing sequence of μ_k, $\mu_1 = 1000$ are shown in the Table below.

Table of Results for Problem 4.1.

μ_k	x_1^*	x_2^*	$\theta_{min}(\mu_k)$	$\theta(\mu_k)$
1000	5.71164	31.62278	376.2636	132.4003
100	3.31662	10.00000	89.9772	36.8109
10	2.04017	3.16228	25.3048	12.5286
1	1.41421	1.00000	9.1046	5.6904
0.1	1.14727	0.31623	4.6117	3.6164
0.01	1.04881	0.10000	3.2716	2.9667
0.001	1.011569	0.03162	2.8569	2.7615
0.0001	1.00499	0.01000	2.7267	2.6967
0.00001	1.00158	0.00316	2.6856	2.6762
0.000001	1.00050	0.00100	2.6727	2.6697
0.0000001	1.00015	0.0003	2.6672	2.6678
0.00000001	1.00005	0.0001	2.6667	2.6670

Exact solution is 2.66667 Rao (2009)

Problem 4.2

Consider the optimization problem *minimize* $f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$ subject to

$$x_1^2 - x_2 \leq 0$$

$$e^{-x_1} - x_2 \leq 0$$

$$-x_1 + 2x_2 - 2 \leq 0$$

solution: We consider the sequence of problems:
$\theta(x, \mu_k) = (x_1 - 3)^2 + (x_2 - 4)^2 - \mu_k \left[\frac{1}{x_2 - x_1^2} + \frac{1}{x_2 - e^{-x_1}} + \frac{1}{x_1 - 2x_2 + 2}\right] + \alpha_k \mu_k^2$

The iteration step using MATLAB for penalty method and $x = [10 : 10]$, $\mu = 1$; $\alpha = 0.01$ $\beta = 0.1$ $tol1 = 1.0e^{-5}$; $tol2 = 1.0e^{-5}$, $h = 0.1$ and $N = 10$

Table of Result for Example 4.2.

μ_k	x_1^*	x_2^*	θ_{min}
1	10.00000	10.00000	85.00000
0.1	1.762313	2.438395	3.970477
0.01	1.376889	1.774431	7.58764
0.001	1.2919403	1.655287	8.415144
0.0001	1.2819129	1.6418969	8.51247
0.00001	1.2808902	1.6405392	8.52239
0.000001	1.2807877	1.6404033	8.523387
0.0000001	1.2807775	1.640389	8.523486
0.00000001	1.2807765	1.6403883	8.520314

Exact solution is at $\theta^* = 8.523502$ Hailay (2012)

Problem 4.3

$$Minimize\ f(x) = x_1^2 + 2x_2^2$$

subject to

$$1 - x_1 - x_2 \leq 0$$

Solution. The define barrier function is

$$B(x) = -log(-g(x)) = -log(x_1 + x_2 - 1)$$

The constrained problem is transformed to an unconstrained problem as:

$$Minimize\ x_1^2 + 2x_2^2 - \mu_k log(x_1 + x_2 - 1) + \alpha_k \mu_k^2$$

Using the iterative method, starting with $\mu_1 = 1$,$\alpha_1 = 0.01$, $\epsilon = 0.005$, $x^{(0)} = (0, 0)$

Table of Results for Problem 4.3.

μ_k	x_1^*	x_2^*	$\theta(x,\mu_k)$	$\mu_k\theta(x,\mu_k)$
1.0	1.0	0.5	-0.5	0.693
0.1	0.7140	0.3570	-0.071	0.265
0.01	0.6720	0.3360	-0.008	0.048
0.001	0.6672	0.3336	-0.0008	0.007
0.0001	0.6666	0.3333	-0.0001	0.0009

The Exact solution is $x_1^* = 0.6666667$ and $x_2^* = 0.333333$ Rao (2009)

Problem 4.4

$$minimize\ 4\left[\frac{1}{3}(x_1+1)^3 + x_2\right]$$

subject to

$$g_1(x) = 2 - 2x_1 \leq 0$$

$$g_2(x) = -2x_2 \leq 0$$

Solution.

$$\theta(x,\mu_k) = f(x) + \mu_k \sum_{i=1}^{2} \frac{-1}{g_i(x)}$$

$$= 4\left[\frac{1}{3}(x_1+1)^3 + x_2\right] - \frac{\mu_k}{2(1-x_1)} + \frac{\mu_k}{2x_2} + \phi(\mu_k)$$

Using the necessary condition, we have

$$x_1^* = \sqrt{1 + \sqrt{\frac{\mu_k}{8}}}$$

and

$$x_2^* = \sqrt{\frac{\mu_k}{8}}$$

The optimum solution is obtained as $\mu_k \to 0$ and $x_1^* = 1$, $x_2^* = 0$ Using the iterative method with $\mu_1 = 1000$ the results are shown in the table below.

Table of Results for Problem 4.4.

μ_k	x_1^*	x_2^*	$\theta(x^{(k)}, \mu_k)$	$\theta(x^{(k)})$
1000	3.4900	11.1803	410.9367	165.4129
100	2.1296	3.5355	113.4177	55.0100
10	1.4553	1.180	39.6600	24.2100
1	1.1634	0.3535	19.3889	14.9145
0.1	1.0544	0.1118	13.3744	12.0081
0.01	1.0175	0.0354	11.5176	11.0905
0.001	1.0055	0.0111	10.9353	10.7993
0.0001	1.0017	0.0035	10.7514	10.7078
0.00001	1.0005	0.0011	10.6935	10.6790
0.000001	1.0002	0.0003	10.6760	10.6694
0.0000000	1.0001	0	10.6666	10.6666

The Exact Solution is $x_1^* = 1$, $x_2^* = 0$ and the optimal solution is 10.6666 Ray (2014)

5. Conclusion

In this work, the interior penalty function method which is among the most useful class of algorithms available for solving nonlinear constrained optimization problems is considered. We proposed a new interior penalty function by adding a new term to the existing function and established the proof of convergence of the new function. Our results showed that the method converges to approximately the exact solution in most cases regardless of the convexity characteristics of the objective function and constraints. The problems considered in this work are nonlinear constrained optimization problems which exist in real life situations whereby we place penalties for violating some laws or procedures. We can therefore conclude that the modified algorithm is effective and reliable in solving nonlinear constrained optimization problems.

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