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Crank-Nicolson Galerkin Error Estimates for Linear Stochastic Wave Equation driven by Space-Time White Noise

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Abstract

We considered the linear stochastic wave equation driven by space-time white noise. The Galerkin finite element method was used to derive the approximate solution. For the time discretization, we applied the Crank-Nicolson finite difference approximation scheme and the error estimates were proved both in the L_2 and maximum norms.

1. Introduction

We study the Crank-Nicolson Galerkin approximation of linear stochastic wave equation driven by space-time white noise

$$\begin{aligned} u_{tt} - \delta u = dW \text{ in } \Omega \times [0, T] \\ u(., t) = 0 \text{ on } \partial\Omega \\ u(0, .) = x_0,\ u(0, .) = x_1 \text{ in } \Omega \times [0, T] \end{aligned} \tag{1}$$

where Ω is a bounded domain in R^d; $d \leq 3$, with smooth boundary $\partial\Omega$. $-\Delta = A$ denote the Laplacian. A is self-adjoint, positive definite linear elliptic operator of second order with smooth coefficients. $\{W(t)\}_{t\geq 0}$ is an $L_2(\Omega)$-valued Wiener process defined on a filtered probability space $\left(\Omega, F, P, \{F_t\}_{t\geq 0}\right)$ with respect to the normal filtration $\{F_t\}_{t\geq 0}$.

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The properties of Stochastic Partial Differential Equations (SPDEs) have been thoroughly studied in [1], [2], [3] and the references therein. Numerical approximations of stochastic wave equation have not received enough attention until recently; see [4], [5], [6], [7], [8], [9] for existing results on finite element approximations where the backward Euler time step was applied in the time discretization. For more recent literature on Crank-Nicolson approximation and Numerical Solutions of Stochastic Wave Equation, see the works of [15], [16], [17], [18] and the references therein.
In this study, we shall be extending the works of [10], [11], [12] and [13] by applying the Crank-Nicolson time stepping technique to the time discretization of the linear wave equation. We shall assume that Ω is a bounded convex polygonal domain with $\partial\Omega$ smooth. The outline of the remaining part of this paper is as follows: In section 2, we present some preliminaries on the meaning of the Wiener process and other theoretical framework within which we shall derive our error estimates. The finite element analysis for the given problem is in section 3 and in section 4, we derive strong convergence estimates for the finite element approximations of the given problem (1) in the L_2-norm while the error estimates in the maximum norm will be derived in section 5.

2. Preliminaries

We now present definitions and notations that will be used throughout this work. Let $H = L_2(\Omega)$ with inner product $(u, v) = \int_\Omega uvdx$ and corresponding norm $\|.\| = (.,.)^{\frac{1}{2}}$.
Let $-\Delta = A$ with domain $D(A) = H_0^1 \cap H^2$ where the spaces H^2 and H_0^1 are as defined below.

$$H^s = \{v \in L_2 : D^\alpha v \in L_2,\ |\alpha| \leq s\}$$

and

$$H_0^1 = \left\{v \in H^1 : D^\alpha v = 0,\ \text{on } \Gamma = \partial\Omega\right\}$$

The space H^s has the inner product (See [12])

$$(v, w)_s = \sum_{|\alpha|\leq s} \int_\Omega D^\alpha v D^\alpha w dx$$

and a corresponding norm $\|v\|_s = (v, v)_s^{1/2}$.
Define the space $H^s = H^s(\Omega) = D(A^{\frac{s}{2}})$ with norm $|v|_s = \left\|A^{\frac{s}{2}}v\right\|$ for any $s \in \mathbb{R}$.
Hence from Parseval's relation,

$$|v|_s^2 = \left\|A^{\frac{s}{2}}v\right\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 \hat{v}_j^2$$

where λ_j are eigenvalues of A and $\hat{v}_j = (v, \phi_j)$ with ϕ_j an orthornomal basis of corresponding eigenfunctions.
For any Hilbert space, H, we define

$$L_2(\Omega, H) = \left\{ v : E\left\|v\right\|_H^2 = \int_\Omega \left\|v(w)\right\|_H^2 dP(w) < \infty \right\}$$

with norm $\left\|v\right\|_{L_2(\Omega,H)} = \left(E\left\|v\right\|_H^2\right)^{1/2}$
Let HS denote the space of Hilbert Schmidt operators from H to H, i.e.

$$HS = \left\{ \psi \in L(H) : \sum_{j=1}^{\infty} \left\|\psi\phi_j\right\|^2 < \infty \right\}$$

with norm

$$\left\|\psi\right\|_{HS} = \left(\sum_{j=1}^{\infty} \left\|\psi\phi_j\right\|^2 \right)^{1/2}$$

where $L_2 = H$ and $\{\phi_j\}$ is an arbitrary orthonormal basis for H. Let E denote expectation and $\psi(s) \in HS$, then $\int_0^t \psi(s)dW(s)$ can be define to have the Ito Isometry

$$E\left\|\int_0^t \psi(s)dW(s)\right\|^2 = \int_0^t \left\|E\psi(s)\right\|_{HS}^2 ds$$

We assume that $W(t)$ is a Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series. Suppose that Q has eigenvalues $\gamma_i > 0$ and corresponding eigenfunctions ξ_i. Then

$$W(t) = \sum_{j=1}^{\infty} \gamma_i^{\frac{1}{2}} \xi_i \beta_i(t) \tag{2}$$

where $\beta_i,\ i = 1, 2, \ldots$ is a sequence of real-valued independence identically distributed Brownian motions and $\{(\gamma_i, \xi_i)\}_{i=1}^{\infty}$ are eigenpairs of Q with orthonormal eigenvectors. The series in (2) converges in $L_2(\Omega, H)$, since for $t \geq 0$,

$$\left\|W(t)\right\|_{L_2(\Omega,H)}^2 = E\left(\left\| \sum_{j=1}^{\infty} \gamma_i^{\frac{1}{2}} \xi_i \beta_i(t) \right\|_H^2 \right) = \sum_{j=1}^{\infty} \gamma_i E(\beta_i(t)) = t\sum_{j=1}^{\infty} \gamma_i = tTr(Q) < \infty$$

For more on these definitions, see [12], [13] and [14].

3. Finite Element Analysis

First we formulate equation (1) as a first order system in time by setting $A = -\Delta$,

$$y = \begin{bmatrix} u \\ u_t \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \tag{3}$$

where the functions lie in the domain $D(A) = H_0^1 \cap H^2$, and thus the components of those in $D(B)$ vanishes on $\partial\Omega$. We then have

$$\begin{aligned} y_t + By &= dW,\ t > 0 \\ y(0) &= y_0 = (x_0, x_1)^T \end{aligned} \tag{4}$$

which has a unique solution

$$y(t) = e^{-tB} y_0 \text{ for any } y_0 \in L_2 \times L_2$$

with $E(t) = e^{-tB}$

3.1. **Galerkin Discretization.** Let S_h be a family of finite elements spaces, which consists of continuous piecewise linear finite elements that vanish on the boundary with respect to the triangulation T_h of $\Omega_h \subset \Omega$ with boundary nodes of h on Ω_h on $\partial\Omega$. We also have that $\{S_h\} \subset H_0^1$. Then the semidiscrete problem is to find $u_h(t) \in S_h$, such that,

$$u_{h,tt} + A_h u_h = P_h dW,\ t > 0,\ u_h(0) = x_{0h},\ u_t(0) = x_{1h} \tag{5}$$

and (4) becomes

$$y_{h,t} + B_h y_h = P_h dW,\ t > 0,\ y_h(0) = y_{0h} = (x_{0h}, x_{1h})^T \tag{6}$$

For the fully discrete scheme, let $r(z)$ be a rational function approximating e^{-z} to order p, i.e., such that (for more on rational function approximation, see [13])

$$r(z) = e^{-z} + O(z^{p+1}), \quad \text{for } z \to 0, \quad \text{where } p \geq 1 \tag{7}$$

and which is A-stable, so that

$$|r(z)| \leq 1,\ \text{for } \mathrm{Re}(z) \geq 0 \tag{8}$$

We define an approximation $Y_n = (U_n, V_n)^T$ to the solution of (1) at time $t_n = nk$; where k is the time step, by

$$Y_n = r(kB_h) Y_{n-1},\ \text{for } n \geq 1 \text{ with } Y_0 = y_{0h} = (x_{0h}, x_{1h})^T \tag{9}$$

Applying the Crank-Nicolson approximations which corresponds to the rational function $r(z) = \left(1 - \frac{1}{2}z\right)\left(1 + \frac{1}{2}z\right)$ gives

$$\left(1 + \frac{1}{2}kB_h\right) Y_n = \left(1 - \frac{1}{2}kB_h\right) Y_{n-1},\ \text{for } n \geq 1 \text{ with } \ Y_0 = y_{0h} = (x_{0h}, x_{1h})^T$$

This is written as

$$\left(1+\frac{1}{4}k^2A_h\right)V_n=\left(1+\frac{1}{4}k^2A_h\right)V_{n-1}-kA_hU_{n-1}+\frac{1}{2}\int_{t_{n-1}}^{t_n}P_hdW(s),$$

$$U_n=U_{n-1}+\frac{1}{2}k(V_n+V_{n+1}),\ \text{for } n\geq 1 \text{ with }\ U_0=x_{0h},\ V_0=x_{1h}$$

when express in terms of the components of $Y_n=(U_n,V_n)^T$.
Eliminating V_n we find that, with $\bar{\partial}U_n=(U_n-U_{n-1})/2k$ and $\hat{U}_n=\frac{1}{2}(U_n+2U_{n-1}+U_{n-2})$,

$$(\bar{\partial}U_n,\chi)+A(\hat{U}_n,\chi)=P_h\Delta\hat{W},\ \forall\ \chi\in V_h,\ n\geq 2 \tag{10}$$

$$U_0=x_{0h},\ \bar{\partial}U_1=\frac{1}{2}x_{1h}+\frac{1}{2}\left(1+\frac{1}{2}k^2A_h\right)^{-1}\left(x_{1h}-kA_hx_{0h}+\int_{t_{n-1}}^{t_n}P_hdW(s)\right)$$

4. Convergence results

We let $P_h:L_2\to\to V_h$ and $R_h:H_0^1\to V_h$ as L_2 and Ritz projections respectively, then $\|P_hv\|\leq C\|v\|$ and $\|R_hv\|\leq C\|v\|$ and by the standard finite element analysis

$$\|P_hv-v\|\leq Ch^\beta\|v\|_\beta\ \text{ for } v\in H^\beta\cap H_0^1 \tag{11}$$

and

$$\|R_hv-v\|\leq Ch^\beta\|v\|_\beta\ \text{ for } v\in H^\beta\cap H_0^1 \tag{12}$$

To present our error estimate, we need the following results from [6].

Lemma 4.1. Let $x_0,x_1\in H^\beta$ and assume that $\|x_{0h}-x_0\|\leq Ch^\beta\|x_0\|_\beta$ and $\|x_{1h}-x_1\|\leq Ch^\beta\|x_1\|_\beta$ Then there is a constant C such that

$$\|u_h(t)-u(t)\|+\|u_{h,t}(t)-u_t(t)\|\leq Ch^\beta\left(|x_0|_\beta+|x_1|_\beta+\left\|A^{-(1-\beta)/2}\right\|_{L_2}\right)$$

and

$$\|x_h(t)-x(t)\|\leq Ch^\beta\left(|x_0|_{\beta\times\beta}+\left\|A^{-(1-\beta)/2}\right\|_{L_2\times L_2}\right)$$

Lemma 4.2. Let $E_h(t)=e^{-tA_k}$ be the analytic semigroup generated by A_h. Let $F_h(t)=E_h(t)P_h-E(t)$.
Then

$$\|F_h(t)\|_{L_\infty([0,T],H)}\leq Ch^\beta|v|_\beta,\ \text{for } v\in H^\beta,\ 0\leq\beta\leq 1$$

and

$$\|F_h(t)\|_{L_2([0,T],H)}\leq Ch^\beta|v|_{\beta-1},\ \text{for } v\in H^{\beta-1},\ 0\leq\beta\leq 2$$

Theorem 4.1. Assume that $r(\lambda)$ satisfies (7) and (8). Let $x_0, x_1 \in H^s$, $s = \max(\beta, p)$, and assume that $\|x_{0h} - x_0\| \leq Ch^\beta \|x_0\|_\beta$ and $\|x_{1h} - x_1\| \leq Ch^\beta \|x_1\|_\beta$. Then for the solution of (1) and (10) we have

(13)
$$\begin{aligned}\|U^n - u(t_n)\| + \|V^n - u_t(t_n)\| \leq & Ch^\beta \left(|x_0|_\beta + |x_1|_\beta + \left\| A^{-(1-\beta)/2} \right\|_{L_2} \right) \\ & + Ck^p \left(|x_0|_p + |x_1|_p + \left\| A^{-(1-p)/2} \right\|_{L_2} \right), \ t_n \geq 0\end{aligned}$$

This is equivalent to

(14)
$$\|X_n - x(t)\| \leq Ch^\beta \left(|x_0|_\beta + \left\| A^{-(1-\beta)/2} \right\|_{L_2} \right) + Ck^p \left(|x_0|_p + \left\| A^{-(1-p)/2} \right\|_{L_2} \right)$$

Proof: We have

$$\|X_n - x(t)\| \leq \|X^n - x_h(t_n)\| + \|x_h(t_n) - x(t)\|$$

By Lemma (4.1)

$$\|x_h(t) - x(t)\| \leq Ch^\beta \left(|x_0|_{\beta\times\beta} + \left\| A^{-(1-\beta)/2} \right\|_{L_2\times L_2} \right)$$

Hence we need to estimate

$$\begin{aligned} X^n - x_h(t_n) &= \left[r(kB_h)^n x_{0h} + r(kB_h)^n x_{0h} P_h \Delta\hat{W}(t) - e^{-(t_n B_h)} x_{0h} \right] \\ &= \left[r(kB_h)^n x_{0h} - e^{-(t_n B_h)} x_{0h} + r(kB_h)^n x_{0h} P_h \Delta\hat{W}(t) \right] \end{aligned}$$

We assume $x_{0h} = P_h x_0$, where P_h is also the orthogonal projection of $L_2 \times L_2$ onto $V_h \times V_h$ and if we let

$$F_n = F_n(kB_h) = r(kB_h)^n - e^{-(t_n B_h)}$$

such that

$$\|X^n - x_h(t_n)\| \leq \|F_n(kB_h) P_h x_0\| + \left\| r(kB_h)^n P_h \Delta\hat{W}(t) \right\| \equiv \|I\| + \|II\|$$

The bounds for (I) we will get from Lemma 4.1 and Lemma 4.2

$$\|I\| = \|F_n(kB_h) P_h x_0\| \leq Ch^\beta \left(|x_0|_\beta + |x_1|_\beta \right)$$

Also from Lemma 4.1, 4.2, Ito isometry and Paserval's relation, we obtain

$$\begin{aligned}\|II\| = \left\|r(kB_h)^n P_h \Delta\hat{W}(t)\right\| &= \left\|r(kB_h)^n \int_{t_{n-1}}^{t_n} P_h dW(s)\right\| \\ &\le C\left\|\int_{t_{n-1}}^{t_n} P_h dW(s)\right\| \\ &= C\left\|P_h \sum_{j=1}^{\infty} \gamma_j^{1/2}\left(\beta_j(t_n) - \beta_j(t_{n-1})\right)\xi_j\right\| \\ &\le Ch^\beta \left\|A^{-(1-\beta)/2}\right\|_{L_2}^2 + Ck^p \left\|A^{-(1-p)/2}\right\|_{L_2}^2\end{aligned}$$

This completes the proof.

5. Maximum-norm error estimates

5.1. **Resolvent Estimates:** Here we consider A as a densely defined operator in the Banach space $C_0(\Omega)$ of continuous functions in $\bar{\Omega}$ vanishing on $\partial\Omega$, with norm

$$\|v\| = \max_{x\in\Omega} |v(x)| \tag{15}$$

and throughout this work, when the space is not specified as a subscript, the norm denotes the maximum-norm (15). The spectrum $\sigma(A)$ of A is located in a segment $\{\lambda : \lambda \ge C_0 > 0\}$ of the positive real axis, with C_0 the smallest eigenvalue of A. The following is then a special case of a result shown by [14].

Lemma 5.1. For any $\epsilon > 0$ there is a constant $C = C_0$ such that

$$\left\|(zI - A)^{-1}\right\| \le C(1+|z|)^{-1}, \text{ for } z \notin \sum_{\epsilon} = \{z : \arg z\} < \epsilon \tag{16}$$

We require also the following results from [13] which we shall use in the proof of our main result here.

Lemma 5.2. Assume that A satisfies the resolvent estimate (16), and let B be the operator on $X \times X$ defined by

$$\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}$$

Then there exists $\theta \in (0, \frac{\pi}{2})$ such that

$$\left\|(zI - B)^{-1}\right\| \le C(1+|z|)^{-1}, \text{ for } z \notin \sum_{\theta} = \{z : \arg z\} < \epsilon$$

For the semidiscrete schemes Njoseh and Ayoola in [6] proved the following result which we shall apply in the proof of our main result.

Lemma 5.3. With $y_{0h} = (R_h x_0, R_h x_1)^T$ we have for the solutions of (1) and (5)

$$\|u_h(t) - u(t)\| + \|u_{h,t}(t) - u_t(t)\| \leq Ch^2 l_h \left(||Ax_0||_\beta + ||Ax_1||_\beta\right) + Ch^\beta \left\|A^{-(1-\beta)/2}\right\|_{L_2},$$

$$0 \leq \beta \leq 1, \; t > 0$$

Using the notations for the semidiscrete problem

$$\tilde{y}_h(t) := \begin{bmatrix} I & 0 \\ 0 & A_h^{-1} \end{bmatrix} y_h(t) = F_h y_h(t) = \begin{bmatrix} u_h(t) \\ A_h^{-1} u_{h,t}(t) \end{bmatrix} \tag{17}$$

where

$$\tilde{B}_h = \begin{bmatrix} 0 & -A_h \\ -I & 0 \end{bmatrix}$$

we have a result which yields an error estimate for $u_h(t)$ of the same order as Lemma 5.3 above under weaker regularity assumptions on x_1.

Lemma 5.4. With $y_{0h} = (R_h x_0, R_h x_1)^T$ we have for the solutions of (1) and (5)

$$\|u_h(t) - u(t)\| \leq Ch^2 l_h \left(||Ax_0||_\beta + ||Ax_1||_\beta\right) + Ch^\beta \left\|A^{-(1-\beta)/2}\right\|_{L_2}, \; t > 0$$

5.2. **Convergence Result:** First let us state these important results from [13].

Lemma 5.5. Let $-B$ generate an analytic semigroup $E(t) = e^{-tB}$ in a Banach Space X with norm $\|.\|$, then

$$\|r(kB)^n v\| \leq C\|v\| \tag{18}$$

and

$$\left\|\left(r(kB)^n - e^{-t_n B}\right) v\right\| \leq Ck^p \|B^p v\|, \text{ for } v \in D(B^p), n \geq 0 \tag{19}$$

Lemma 5.6. If we define $P_h : L_2 \to V_h$ and $R_h : H_0^1 \to V_h$ as L_2 and Ritz projections respectively, then we have

$$\|P_h v - v\| \leq Ch^2 l_h \|Av\|, \text{ for } v \in C^2(\bar{\Omega}) \cap H_0^1(\Omega)$$

and

$$\|R_h v - v\| \leq Ch^2 l_h \|Av\|, \text{ for } v \in C^2(\bar{\Omega}) \cap H_0^1(\Omega)$$

The error estimate in the maximum norm for the fully discrete scheme is

Theorem 5.1. For $y_{0h} = (P_h x_0, P_h x_1)^T$ we have for the solutions of (1) and (10)

(20)

$$\|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| \leq Ch^2 l_h^2 \left(\|Ax_0\| + \|Ax_1\| + \left\|A^{-(1-\beta)/2}\right\|\right) + Ck^2 l_h^2 \left(\left\|A^2 x_0\right\| + \left\|A^2 x_1\right\| + \left\|A^{-(1-\beta)/2}\right\|\right), \; 0 \leq \beta \leq 1, \; t_n > 0$$

and

(21)

$$\|U_n - u(t_n)\| \leq C(h^2 l_h^2 + k^2)\left(\|Ax_0\| + \|Ax_1\| + \left\|A^{-(1-\beta)/2}\right\|\right), \; 0 \leq \beta \leq 1, \; t_n > 0$$

Proof:

Using Parsevals relation and Ito isometry, we have

$$\left\| P_h \Delta \hat{W}(s) \right\| \leq Ch^2 \left\| A^{-(1-\beta)/2} \right\|$$

By lemma 5.5,

$$\begin{aligned} \|Y_n - y_h(t_n)\| &= \left\| r(kB_h)^n y_{h0} + P_h \Delta \hat{W}(s) - e^{-nkB_h} y_{h0} \right\| \\ &= \left\| \left(r(kB_h)^n - e^{-nkB_h} \right) y_{h0} + P_h \Delta \hat{W}(s) \right\| \\ &\leq Ck^2 \left(\|B_h y_{h0}\| + \left\| A^{-(1-\beta)/2} \right\| \right) \end{aligned}$$

Next we show that

$$\|B_h R_h y_0\| \leq Cl^2 \left(\left\| A^2 x_0 \right\| + \left\| A^2 x_1 \right\| \right)$$

With A_h as defined earlier, we have $A_h R_h v = P_h A v$. Hence

$$B_h R_h y_0 = \begin{pmatrix} -P_h A x_0 \\ A_h P_h A x_0 - P_h A x_1 \end{pmatrix}$$

To estimate the norm of $A_h P_h A x_0 = (A_h P_h A - A_h R_h A) x_0 + A_h R_h A x_0$, we note that the last term equals $P_h A^2 x_0$. Using the global uniformity of the triangulation, i.e.,

$$\|A_h \phi\| \leq Ch^{-2} \|\phi\|, \text{ for } \phi \in S_h$$

and Lemma 5.6, we have

$$\|A_h (P_h - R_h) A x_0\| \leq Ch^{-2} \|(P_h - R_h) A x_0\| \leq Cl_h^{-2} \left\| A^2 x_0 \right\|$$

Hence,

$$\|A_h P_h A x_0\| \leq Cl_h^2 \left\| A^2 x_0 \right\|$$

Since $\|P_h A x_0\| \leq C \|A x_0\| \leq C \left\| A^2 x_0 \right\|$, and similar argument for terms in x_1. This proves (20).

For the prove (21), by Lemma 5.5,

$$\left\| \tilde{Y}_n - \tilde{y}_h(t_n) \right\| \leq Ck^2 \left(\left\| \tilde{B}_H^2 \tilde{y}_{0h} \right\| + \left\| A^{-(1-\beta)/2} \right\| \right)$$

and using the notations of (17), we have

$$\tilde{B}_H^2 \tilde{y}_{0h} = \begin{pmatrix} 0 & -A_h \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R_h x_0 \\ R_h x_1 \end{pmatrix} \begin{pmatrix} -P_h A x_0 \\ A_h P_h A x_0 - R_h x_1 \end{pmatrix}$$

and

$$\|R_h x_1\| \leq \|x_1\| + Ch^2 l_h^2 \|A x_1\| \leq C \|A x_1\|$$

we have

$$\left\| \tilde{B}_H^2 \tilde{y}_{0h} \right\| = Ch^2 l_h^2 \left(\|A x_0\| + \|A x_1\| \right)$$

This concludes the proof of the theorem.

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