Nigerian Journal of Mathematics and Applications
Volume **24**, (2015), 184 – 192
 http : //www.kwsman.com

Some Results on Properties of Alternating Semigroups

G. N. Bakare* and S. O. Makanjuola

Abstract

Let C_n be the symmetric inverse semigroup on $X_n = \{1, 2, \cdots, n\}$, A_n^c be the alternating semigroups of X_n. In this paper, we discuss some combinatorial properties in A_n^n. and obtained some triangles of numbers for it.

1. Introduction and Preliminaries

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. A transformation of X is a function from X to itself. Transformation semigroups is one of the most fundamental mathematical objects that occurs in theoretical computer science; where properties of language depend on algebraic properties of various transformation semigroups related to them.
Let $X_n = \{1, 2, \cdots, n\}$, then a (partial) transformation $\alpha : Dom\alpha \subseteq X_n \rightarrow Im\alpha \subseteq X_n$ is said to be full or total, if Dom $\alpha = X_n$, otherwise it is called strictly partial. The set of all partial transformation on n-object form a semigroup under the usual composition of transformation. It is denoted by P_n when it is partial, T_n when it is full or total and C_n when it is partial one-to-one. The elements C_n are usually called Charts. Partial one-to-one transformations are also called subpermutations, see came79. These are the three fundamental transformation semigroups which was introduced by [14].The semigroup C_n is the main object of the study in this research work.A transposition is a circuit/cycle of length two denoted by $(i\ j)$, while a semitransposition is a proper path of length

Received October 29, 2015. * Corresponding author.
2010 *Mathematics Subject Classification.* 20Mxx & 20D06.
Key words and phrases. Symmetric Inverse Semigroup and Alternating Semigroup.
Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
E-mail: bakare.ng@unilorin.edu.ng & somakanjuola@unilorin.edu.ng

two denoted by $(i\ j]$. A transpositional is a chart that is either a transposition $(i\ j)$ or a semitransposition $(i\ j]$.The idea of an even permutation has been generalized via path notation, to one-to-one partial transformation(charts) C_n. A transformation α in C_n said to be even if it can be expressed as a product of an even number of transpositional or a product of any number of circuits/paths of odd length. The set of all even charts on X_n, is called Alternating Semigroups usually denoted by A_n^c. This class of transformation was introduced by [21]. Many researchers have worked on different classes of transformation semigroups and obtained many results such as [17] who studied some algebraic and combinatorial properties of semigroup of injective partial contraction mappings and isometrics of a finite chain, [1] studied identity difference transformation semigroups and [2] also studied some semigroups of full contraction mapping on a finite chain. Combinatorial and algebraic properties of S_n (symmetric group),A_n (alternating group) and C_n have been studied over a long period and many interesting and delightful results have emerged (see for example [[9], [6], [7], [4], [19] and [5]]). Recently, inspired by the work of [21], [3] obtained some combinatorial result in the semigroup A_n^c. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations.Many sequence of numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers used by ([14] p.42 & 96), the factorial used by (munn57 and [24]), the binomial used by ([15] and [12]), the Fibonacci number used by [13], Catalan numbers used by ([10]) and Lah numbers used by [16] and [18], etc., have all featured in these enumeration problems. For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to [25]. This enumeration problems lead to many numbers in [23] but there are also others that are not yet in it.

2.0 Some Notation and Relevant Fact

In this section some basic terminology on C_n, $N(C_n)$ and A_n will be introduced and some known combinatorial result that shall be needed later are stated.
Let A_n^c be the Alternating Semigroup on X_n.
Definition 2.1: Let $\alpha \in A_n^c$. Then, the height of α is $h(\alpha) = |Im\alpha|$.
Definition 2.2: Let $\alpha \in A_n^c$. Then, the fix of α is $f(\alpha) = |F(\alpha)|$.
where $F(\alpha) = \{x \in Dom\alpha : x\alpha = x\}$.
Definition 2.3: $F(n, p, m) = |\{\alpha \in S : h(\alpha) = p \Lambda f(\alpha) = m\}|$.
Definition 2.4: $F(n; p) = |\{\alpha \in S : h(\alpha) = |Im\alpha| = p\}|$.
Definition 2.5: $F(n; m) = |\{\alpha \in S : f(\alpha) = m\}|$.
by [25] show that
$|A_n^c| = \sum_m F(n; m) = \sum_p F(n; p)$.

The following theorems and remarks are very crucial to this work.

Theorem 2.6: Let A_n be the alternating group on X_n. Then $|A_n| = \frac{n!}{2}(n \geq 2)$ where $|A_0| = 1 = |A_1|$.

Corollary 2.7: Let e_n be the number of even permutations without fixed point that is ,the number of even derangement on X_n. Then $e_n = \frac{n!}{2}\sum_{j=0}^{n-2}(-1)^j/j! + (-1)^{n-1}(n-1)$ which satisfy the following recurrence relations.
$e_n = (n-1)(e_{n-1} + e_{n-2}) + (-1)^{n-1}(n-1), e_0 = 1, e_1 = 0.$
$e_n = ne_{n-1} + (-1)^n(n-2)(n+1)/2, e_0 = 1.$

Corollary 2.8: Let $e(n;m) = |\{\alpha \in A_n : f(\alpha) = m\}|$, where $f(\alpha) = |\{x \in X_n : x\alpha = x\}|$. Then $e(n;m) = \binom{n}{m}e_{n-m}$ where $e(n,m)$ is the number of even permutations with exactly m fixed points.

Corollary 2.9: Let $e^{'}(n;m) = |\{\alpha \in A_n : f(\alpha) = m\}|$, where $f(\alpha) = |\{x \in X_n : x\alpha = x\}|$. Then $e^{'}(n;m) = \binom{n}{m}e^{'}_{n-m}$; where $e^{'}(n;m)$ is the number of odd permutations with exactly m fixed points.

Corollary 2.10: Let $e^{'}_n$ be the number of odd permutations without fixed points that is ,the number of odd derangement on X_n. Then $e^{'}_n = \frac{n!}{2}\sum_{j=0}^{n-2}(-1)^j/j!$ which satisfy the following recurrence relations.
$e^{'}_n = (n-1)(e^{'}_{n-1} + e^{'}_{n-2}) + (-1)^n(n-1), e^{'}_0 = 1, e^{'}_1 = 0.$
$e^{'}_n = ne^{'}_{n-1} + (-1)^n(n-1)(n-2)/2, e^{'}_0 = 0.$

Proposition 2.11: Let e_n and $e^{'}_n$ be as defined in corollary (2.3.3 and (2.3.6) respectively. Then
(i) $e_n = \frac{1}{2}[d_n - (-1)^n(n-1)], d_0 = 1$
(ii) $e^{'}_n = \frac{1}{2}[d_n + (-1)^n(n-1)], d_0 = 1$

Proposition 2.12: [[25] Corollary 2.3] Let $S = C_n$. Then
$F(n;p) = \binom{n}{p}^2 p!, (n \geq p \geq 0)$

Lemma 2.13: [[19] Theorem 2.4] Let $S = C_n$. Then
$F(n;p,m) = \frac{n!}{m!(n-p)!}\sum_{j=0}^{p-m}\binom{n-m-i}{p-m-i}\frac{(-1)^i}{i!}$ for $n \geq p \geq m \geq 0$.

Proposition 2.14: [[11] Theorem 2.5.1,p.22] Let $S = C_n$. Then $|C_n| = a_n = \sum_{p=0}^{n}\binom{n}{p}^2 p!$ which satisfied the recurrence relation

$a_n = 2na_{n-1} - (n-1)^2 a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$.

Theorem 2.15: [[21], Theorem 25.1] Let $\alpha \in C_n$ have $rank(n-1)$. Then $\alpha \in A_c^n$ if and only if its completion α^- is an even permutation of N.

Theorem 2.16: [[21], Theorem 25.2] If $\alpha \in C_n$ has rank at most $n-2$, then $\alpha \in A_c^n$.(All charts of $rank < n-1$ are even)

Theorem 2.17: [[21], Theorem 25.3] Let A_c^n be set of alternating semigroup on X_n. Then
$|A_c^n| = \frac{n!}{2} + \frac{n^2(n-1)!}{2} + \sum_{p=0}^{n-2} \binom{n}{p}^2 p!$ for $n \geq 2$

Remark 2.18. . Its observed from the listed elements C_n that at height $n-1$ half of the elements are even charts(A_n^c).

3.0 Results

Lemma 3.1: Let $S = A_n^c$. Then, $F(n; p_{n-1}) = \frac{n^2(n-1)!}{2}$
Proof: First notice from [Theorem 2.6] that $h(\alpha) = p = n$ then, $|A_n^c| = \frac{n!}{2}$ see Bash08, the order of the alternating group is $\frac{n!}{2}$. Also from [Remark 2.18 and Theorem 2.15] it is clear that exactly half of C_n are members of A_n^c and when $p \leq\ n-1$ all the charts are members of A_n^c. From [Proposition 1.7] it is know that $F(n; p) = \sum_{p=0}^{n} \binom{n}{p}^2 p!$
Moreover, by [Theorem 2.17] that the order of A_n^c is given by
$|A_c^n| = \frac{n!}{2} + \frac{n^2(n-1)!}{2} + \sum_{p=0}^{n-2} \binom{n}{p}^2 p!$
$|A_c^n| = \sum_{p=0}^{n} F(n; p) = F(n; p_{n-2}) + F(n; p_{n-1}) + F(n; p_n)$
$= \sum_{p=0}^{n-2} \binom{n}{p}^2 p! + \frac{n^2(n-1)!}{2} + \frac{n!}{2}$
$F(n; p_{n-1}) = \frac{n^2(n-1)!}{2}$.
Hence, the result follows.

Theorem 3.2: Let $S = A_n^c$. Then,

$$F(n; p) = \begin{cases} \frac{n!}{2}, \text{ if p = n} \\ \frac{n^2(n-1)!}{2} \text{ if p = n-1} \\ \binom{n}{p}^2 p! \text{ for } 0 \leq p \leq n-2 \end{cases}$$

Proof: It follows from Theorem 2.6, Lemma 3.1 and Proposition 2.11 respectively.
Remark 3.3: For some computed values of $F(n; p)$ for $|A_n^c|$, See Table 1.1

Table 1.1: Element of Alternating Semigroups by Height

n/p	0	1	2	3	4	5	6	$\sum F(n;p) = \lvert A_n^c \rvert$
0	1							01
1	1	1						02
2	1	2	1					04
3	1	9	9	3				22
4	1	16	72	48	12			149
5	1	25	200	600	300	60		1186
6	1	36	450	2400	5400	2160	360	10807

Theorem 3.4: Let $S = A_n^c$. Then

$$F(n; p_{n-1}, m_0) = \sum_{i-odd} \binom{n}{i} e_{n-i}\ i! + \sum_{i-even\geq 2} \binom{n}{i} e'_{n-i}\ i!,\ i \geq 0$$

Proof: Recall from [20] that subpermutation α of X_n can be pictured as a digraph on n vertices with $i\ j$ and edge of α if $i\alpha = j$. Each component of such a digraph is called an orbit, and they are of two types: Cycles (including 1 cycles or fixed points) and simple paths. Moreover, since $h(\alpha) = n - 1$ then there can be at most one simple path. Thus, we can decompose even subpermutation (even chart) without fixed points into a cycle component (without fixed points) and a simple path component.
Suppose there are i points in the path and so $n - i$ points for the cycle parts. (Note that $i > 0$). There are $\binom{n}{i}$ choices (for the path points) and $i!$ different ways of arranging the points in the path.
However, note that if i is odd then, the complexion α^- on X_i is an even permutation so we must have an even cycle part of which these are e_{n-i}. Similarly, if i is even then the complexion α^- on X_i is odd permutation so we must have an odd cycle part of which these are e'_{n-i}. Hence the result follows.

Theorem 3.5: Let $S = A_n^c$. Then
$F(n; p_{n-1}, m) = \binom{n}{m} F(n - m, p_{n-m-1}, m_0)$

$$= \binom{n}{m} \left\{ \sum_{i-odd} \binom{n-m}{i} e_{n-m-i}\ i! + \sum_{i-even} \binom{n-m}{i} e'_{n-m-i}\ i!, \text{ for } i \geq 0 \right.$$

Proof: First observe that m fixed points of α can be choosen from X_n in $\binom{n}{m}$ ways, on the remaining $n - m$ points there should be no fixed points.
Moreover, we can decompose any subpermutation with exactly m fixed points into a cycle and simple path components. Since $h(\alpha) = n - 1$ then there can be at most one simple path. Suppose there are i points in the path and so $n - m - i$

points for the cycle part (Note that $i > o$). There are $\binom{n-m}{i}$ choices(for the path points) and i different ways of arranging the points in the path. However, note that if i is odd. Then the completion α^- on X_i is an even permutation so we must have an even cycle part of which these are e_{n-m-i}. Similarly, if i is even then the complexion α^- on X_i is an odd permutation so we must have an odd cycle part of which these are $e^{'}_{n-m-i}$. Hence the result follows.

Remark 3.6: For some computed values of $F(n;p_{n-1},m)$ for $F(n;p_{n-1},m)$, See table 1.2

Table 1.2: Values of A_n^c in term of three parameter $F(n,p_{n-1},m)$

n/m	0	1	2	3	4	5	6	$\sum F(n,p_{n-1},m)$
0	0							00
1	1	0						01
2	0	2	0					02
3	6	0	3	0				09
4	20	24	0	4	0			48
5	135	100	60	0	5	0		300
6	924	810	300	120	0	6	0	2160

Theorem 3.7: Let $S = A_n^c$. Then

$$F(n;p,m) = \begin{cases} \binom{n}{m} e_{n-m}, \text{ if p = n} \\ \binom{n}{m}[\sum_{i-odd}\binom{n-m}{i} e_{n-m-i}\ i! + \sum_{i-even}\binom{n-m}{i} e^{'}_{n-m-i}\ i!,\ for\ i \geq 0] \text{ if p=n-1} \\ \frac{n!}{m!(n-p)!}\sum_{i=0}^{p-m}\binom{n-m-i}{p-m-i}\frac{(-1)^i}{i!} \text{ for } 0 \leq p \leq n-2 \end{cases}$$

Proof: It follow from corollary 2.8, theorem 3.5 and lemma 2.13 respectively.
Remark 3.8: The triangular of arrays of numbers $F(n;p,m)$ are not yet listed in NJA@. For some selected values of $F(n;p,m)$ for $F(n;p_{n-1},m)(m = 0,1,2,)$, See Table $1.3, 1.4$ and 1.5 respectively.

Table 1.3: values of A_n^c in terms of three parameters$F(n,p;0)$

n/p	0	1	2	3	4	5	6	$\sum F(n,p,0)$
0	1							01
1	1	0						01
2	1	0	0					01
3	1	6	6	2				15
4	1	12	42	20	3			78
5	1	20	130	320	135	24		630
6	1	30	315	1420	2715	924	130	5535

Table 1.4: Values of of A_n^c in terms of three parameters $F(n, p, 1)$

n/p	0	1	2	3	4	5	6	$\sum F(n,p,1)$
0	0							00
1	0	1						01
2	0	2	0					02
3	0	3	0	0				03
4	0	4	24	24	8			60
5	0	5	60	210	100	15		390
6	0	6	120	780	1920	810	144	3780

Table 1.5: Values of of A_n^c in terms of three parameters $F(n, p; 2)$

n/p	0	1	2	3	4	5	6	$\sum F(n,p,2)$
0	0							00
1	0	0						00
2	0	0	1					01
3	0	0	3	0				03
4	0	0	6	0	0			06
5	0	0	10	60	60	20		150
6	0	0	15	180	630	300	45	1170

4.0 Conclusion

In this paper, The combinatorial function $F(n; p)$ and $F(n, p, m)$ were used to derived some triangles of numbers for A_n^n and some of it combinatorial properties have been studied. It is hereby recommended that the work should be extended to other subsemigroups of alternating semigroups such as: order-preserving, order-decreasing, order preserving or order reversing, orientation preserving alternating semigroups,and other semigroups such as isometrics of injective mapping, identity difference transformation semigroups, semigroup of contraction injective mapping etc. The theory of semigroup has its scope widened to embrace many aspects of theoretical computer sciences, such as: automata theory, coding theory, computational theory and formal languages as well as applications in the sciences. It can also assist in sorting data and designing better networks.

References

[1] Adeniji, A. O. (2012). *Identity difference transformation semigroups.* University of Ilorin, Ilorin, Nigeria.

[2] Adeshola, A. (2013). *Some semigroups of full contraction mappings of a finite chain.* University of Ilorin, Ilorin, Nigeria.

[3] Bakare, G. N. and Makanjuola, S. O. (2013). *Some Combinatorial Results on Alternating Semigroups.* Journal of Science, Technology, Mathematics and Education. JOSTMED, *9* 98-102.

[4] Balakrishnan, V. K. (1995). *Combinatorics: Including concepts of graph theory.* Schuam's outline series. McGraw Hill Inc.

[5] Bashir, A. and Umar, A. (2008). *Some Combinatorial Properties of the alternating group* Southeast Asian Bulletin of Mathematics, *32* 823-830.

[6] Cameron, P. J. (1994). *Combinatorics: Topics, Techniques, Algorithms.* Cambridge University Press, New York

[7] Cameron, P. J. (2000). *Sequences realized by oligomorphic permutation groups.* J. Integer Seq., *03*, 00.1.5.

[8] Cameron, P. J. and Deza, M. (1979). *On Permutation geometrics.* Journal of London Math. Soc., *20*, 373-386.

[9] Comtet, L. (1974). *Advanced combinatorics: The art of finite and infinite expansions.* D. Reidel publishing company, Dordrecht Holland.

[10] Ganyushkin, O. and Mazorchuk, V. (2003). *On the structure of* IO_n. Semigroup Forum, *66*, 455-483.

[11] Ganyushkin, O. and Mazorchuk, V. (2009). *An Introduction to classical finite transformation semigroups.* Spinger-Verlag, London.

[12] Garba, G. U. (1994). *Nilpotents in semigroups of partial one-to-one order-preserving mappings.* Semigroup Forum, *48*, 37-49.

[13] Howie, J. M. (1971). *Products of idempotents in certain semigroups of transformations.* Proc. Edinb. Math. Soc., *17*, number 3, 223-236.

[14] Howie, J. M. (1995). *Introduction to fundamental of semigroup theory.* Oxford University Press, USA.

[15] Howie, J. M. and McFadden, R. B. (1990). *Idempotent rank in finite full transformation semigroups.* Proceeding of the Royal Society of Edinburgh Sect., *A114*, 161-167.

[16] Janson, S. and Mazorchuk, V. (2005). *Some remarks on the combinatorics of* IS_n. Semigroup Forum, *109*, 391-405.

[17] Kehinde, R. (2012). *Some algebraic and combinatorial properties of semigroups of injective partial contraction mappings and isometries of a finite chain.* University of Ilorin, Ilorin, Nigeria.

[18] Laradji, A. and Umar, A. (2004). *On the number of nilpotents in the partial symmetric semigroup.* Comm. Algebra, *32*, 3017-3023.

[19] Laradji, A. and Umar, A. (2007). *Combinatorial results for the symmetric inverse semigroup.* Semigroup Forum, *75*, 221-236.

[20] Laradji, A. and Umar, A. (2013). *On the number of subpermutations with fixed orbit size.* ARS Combinatoria, *109*, 447-460

[21] Lipscomb, S. (1996). *Symmetric inverse semigroups, Mathematical surveys and monographs.* Amer. Math. Soc. *46* (Providence R. L.) 43-51.

[22] Munn, W. D. (1957). *The characters of the symmetric inverse semigroup.* Proc.Cambridge Philos, *53*, 13-18.

[23] Sloane, N. J. A. *The on-line encyclopedia of integer sequences.* Schaum's outline series @http://www.research.att.com/ njas/sequences/.

[24] Umar, A. (1993). *On the semigroup of partial one-to-one order-decreasing finite transformations.* Proc. Roy. Soc., *123A*, 355-363.

[25] Umar, A. (2010). *Some combinatorial problems in the theory of symmetric inverse semigroups.* Algebra Discrete Math, *9*, number 2, 115-126.