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Variation Iteration Decomposition Method for the Numerical Solution of Integro-Differential Equations

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Abstract

This paper seeks the numerical solution of integro-differential equations via the variation iteration decomposition method (VIDM). The mode of convergence of this method as applied to integro-differential equations is determined by the step size parameter, h. The approximate solution converges for $h \leq 10^{5+n}$ where n is the number of iterations. Similarly, the method requires no discretization, linearization or perturbation. We apply the method in the stimulation of numerical examples for the approximate solution of linear and nonlinear Volterra and Fredholm integro-differential equations via maple 18 software. The resulting numerical evidences show the method is reliable, effective and efficient for the numerical solution of integro-differential equations.

1. Introduction

We consider the standard integro-differential equation of the form

$$y^{(n)}(x) = f(x) + \lambda \int_{r(x)}^{g(x)} k(x,s)u(s)ds, \tag{1}$$

where $r(x)$ *and* $g(x)$ are the limits of integration, λ is a constant parameter and $k(x,s)$is the nucleus of the integral. If the limits of integration are constants, then we have Fredholm integro-differential equation. Similarly, if the limit $g(x)$

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is replaced with a variable of integration x, then it is called Volterra integro-differential equation. In this paper, we seek the numerical solution of integro-differential equations via the variation iteration decomposition method (VIDM). The mode of convergence of this method as applied to integro-differential equations is determined by the step size parameter, h. The approximate solution converges for $h \leq 10^{5+n}$ where n is the number of iterations.

In recent years, there has been growing interest in the integro-differential equations due to its applicability in the mathematical modelling of electric circuit, stochastic processes, damping process, exhibitory and inhibitory interactions, etc. Several numerical methods for solving linear and nonlinear integro-differential equations have been given and available in literature. Hossein [1] used Tau method for an error estimation of the integro-differential equations. Biazer [2] solved systems of integro-differential equations by the Adomian decomposition method. Manafianheris [3] employed the modified Laplace Adomian decomposition for the numerical solution of integro-differential equations. Arikoglu and Ozkol [4] seeks the solution of integro-differential equations via the differential transform method. He [5] developed the variation iteration method (VIM) for linear and nonlinear boundary value problems. Subsequently, the variation iteration method has been applied to seek the solutions of both linear and nonlinear integro-differential equations [6-8]. Also Mamadu and Njoseh [9] constructed their own orthogonal polynomials and applied then in the use of orthogonal collocation method to solve Fredholm integro-differential equations, while in [10] they proved the convergence of VIM for numerical solution of nonlinear integro-differential equations.

The variation iteration decomposition method (VIDM) was proposed by Noor and Mohyud-Din [6] for solving fifth-order boundary value problems. The method is a combination of the variation iteration method and the decomposition method. The variation iteration decomposition method gives the solution in a compact series which converges rapidly. The VIDM requires no discretization, linearization or perturbation. To the best of our knowledge, the method has not been applied to solve integro-differential equations. We formulate the correction functional for the given integro-differential equation and determine the Lagrange multiplier optimally. A generalized value of the Lagrange multiplier has been proposed by Abbasbandy and Shivanian [7]. The Adomian polynomials, $A_n, \quad n \geq 0$, are introduced in the correction functional and estimated by using the specified algorithm [2], [3], [11] and [12-15].

Thus, the approximate solution is evaluated by introducing the Lagrange multiplier and the Adomian polynomials, $A_n, \quad n \geq 0$. The initial approximations are estimated by the modified Laplace decomposition method [3]. Moreover, the

VIDM as applied to the integro-differential equations produces solutions that converge for $h \leq 10^{5+n}$, where n is the number of iterations. Thus, three numerical examples are given to test the effectiveness and reliability of the method.

2. Variation Iteration Method

Consider the general differential equation

$$Ly + Ny = f(x) \tag{2}$$

with prescribed auxiliary conditions, where y is an unknown function, L is a linear operator, N, a non linear term, and f, the source term. We can construct correction functional for equation (2) as [1-5]:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda \left[Ly_n(\tau) + N\tilde{y}_n(\tau) - f(\tau) \right] d\tau, \; n \geq 0 \tag{3}$$

where λ is a general Lagrange multiplier, $\tilde{y}_n = 0$, i.e., $\tilde{y}_k$ is a restricted variable. Abbasbandy and Sivanian [7] proposed a generalized value of the Lagrange multiplier for the variation iteration method as

$$\lambda(s) = \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)} \tag{4}$$

where m is the order of the derivatives. See [5-8] for more constructive study of the variation iteration method and its applications.

3. The Adomian Decomposition Method

Consider the standard operation [2], [3], [11-15]

$$Ly + Ry + Ny = G, \tag{5}$$

with prescribed auxiliary conditions, where y is the unknown function, L is the highest order derivative which is assumed to be invertible, Ny is the nonlinear term, and G is the source term. Applying the inverse operator L^{-1} to both sides of equation (5), and using the prescribed conditions, we obtain

$$y = L^{-1}(G - Ry - Ny) = L^{-1}(G) - L^{-1}(Ry) - L^{-1}(Ny), \tag{6}$$

where the function y is the term arising from integrating the source term and from using the auxiliary conditions.
The standard Adomian defines the solution y as

$$y = \sum_{n=0}^{\infty} y_n, \tag{7}$$

and the nonlinear term as

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{8}$$

where A_n are the Adomian polynomials determined normally from the relation [2], [3] and [11-15].

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i\right)\right]_{\lambda=0} \tag{9}$$

If the non-linear term is a non-linear function $F(y)$, the Adomian polynomials are arranged into the form:

$$\begin{aligned} A_0 &= F(y_0) \\ A_1 &= y_1 F'(y_0) \\ A_2 &= y_2 F'(y_0) + \frac{y_1^2}{2!} F''(y_0) \\ A_3 &= y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{y_1^3}{3!} F'''(y_0) \end{aligned} \tag{10}$$

The component $y_n,\ n \geq 0$, are determined recursively. An n-component truncated series is thus obtained as

$$S_n = \sum_{i=0}^{n} y_i \tag{11}$$

4. Variation Iteration Decomposition Method

For detail illustration of the concept of variation iteration decomposition method, we consider equation (2) and its correction functional (3).
Let the unknown function $y(x)$ be defined as

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

The component $y_n,\ \ n \geq 0$ are determined recursively. The decomposition method [9] involves finding the components $y_n,\ n \geq 0$, individually.
Also, we define the nonlinear term as

$$Ny = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials. Thus, the approximate solution can be obtained using

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(-1)^m (r-x)^{m-1}}{(m-1)!}\left[L y_n(r) + \sum_{n=0}^{\infty} A_n - f(r)\right] dr, n \geq 0 \tag{12}$$

Equation (12) is called the variation iteration decomposition method and is highly efficient.

The approximate solution obtained from using equation (12) converges for $h \leq 10^{5+n}$, where n is the number of iterations, and $x = i/h,\ i = 0,\ 1,\ 2, \cdots m$. ($m$ is any positive integer)
The absolute error for this formulation is defined as $|y(x) - y_n(x)|$, where $y(x)$ is the analytic solution and $y_n(x)$ is the approximate solution.

5. Numerical Examples

In this section, we apply VIDM to solve linear and nonlinear Volterra and Fredholm integro-differential equations. The main objective is to solve these examples for various values of h for $n = 1$.
Example 5.1: We consider the nonlinear integro-differential equation [8]

$$u'(x) = -1 + \int_0^x u^2(t)dt, u(0) = 0, \quad 0 \leq x \leq 1. \tag{13}$$

The exact solution is $u(x) = -x$.
The correction functional for equation (13) is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[\frac{du_n(s)}{ds} + 1 - \int_0^x u_n^2(t)dt\right] ds.$$

Take initial guess as $u_0(x) = -x$.
Now applying the variation iteration decomposition method we have

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[\frac{du_n(s)}{ds} + 1 - \int_0^s \sum_{n=0}^{\infty} A_n(t)dt\right] ds,$$

where A_n are the Adomian polynomials for $Ny = u^2(t)$. Using the algorithm (10), we have

$$A_0 = u_0^2(t)$$

$$A_1 = 2u_0^2\,(t)\,u_1(t)$$

$$A_2 = 2u_0^2\,(t)\,u_2\,(t) + u_1^2(t)$$

Using the above relations for $n = 1$ the approximate solution is given as

$$y\,(x) = -x - \frac{1}{252}x^7 + \frac{1}{12}x^4$$

.
(See **Table 1** for computational results.)
Example 5.2: We consider the linear Fredholm integro-differential equation [8]

$$u'''(x) = e^x - 1 + \int_0^1 tu(t)dt, u(0) = 1,\ u'(0) = 1,\ u''(0) = 1. \tag{14}$$

The exact solution is $u(x) = e^x$.

The correction functional for equation (14) is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{(s-x)^2}{2}\left[\frac{d^3u_n(s)}{ds^3} - e^s + 1 - \int_0^1 tu_n(t)dt\right]ds.$$

Take initial guess as $u_0(x) = 1 + x + \frac{1}{2!}x^2$.
Now apply the variation iteration decomposition method to have

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{(s-x)^2}{2}\left[\frac{d^3u(s)}{ds^3} - e^s + 1 - \int_0^s t\sum_{n=0}^{\infty} u_n(t)dt\right]ds,$$

Using the above relations for $n = 1$,the approximate solution is given as

$$u\left(x\right) = e^x - \frac{1}{144}x^3$$

(See **Table 2** below for computational results.)
Example 5.3: We consider the linear Volterra integro-differential equation [6]

(15) $$u''(x) = \cos x + \frac{1}{2}x^2 - \int_0^x u(t)dt - \int_0^x u''(t)dt, u(0) = -1,\ \ u'(0) = 1.$$

The exact solution is $u(x) = x - \cos x$.
The correction functional for equation (15) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[\frac{d^2u_n(s)}{ds^2} - \cos s - \frac{1}{2}s^2 + \int_0^s (u_n(t) + \frac{d^2u_n(t)}{dt^2})dt\right]ds.$$

Take initial guess as $u_0(x) = -1 + x$.
Now applying the variation iteration decomposition method we have

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[\frac{d^3u(s)}{ds^3} - \cos s - \frac{1}{2}s^2 + \int_0^s \sum_{n=0}^{\infty} u_n(t)dt + \int_0^s \frac{d^2}{dt^2}\left(\sum_{n=0}^{\infty} u_n(t)\right)dt\right]ds,$$

Using the above relations for $n = 1$ the approximate solution is given as

$$u\left(x\right) = x - \frac{1}{720}x^6 - \frac{1}{12}x^4 + \frac{1}{6}x^3 - cosx.$$

.(See **Table 3** for computational results.)

5.1. **Discussion of Results.** We have successively employed the variation iteration decomposition method for the numerical solution of linear and nonlinear integro-differential equations. It is evident that the method showed an excellent rate of convergent, which can be seen in **Tables 1 - 3**. It is obviously seen that the method converges for $|h| \leq 10^5$, and not necessarily dependent on the number of iterates. Also, the approximate solution coincides absolutely with the analytic solution whenever $h = 10^5$ and $h = 10^4$ as shown in the **Tables 1 - 3**, respectively, with reference to the first iterate, $n = 1$.

6. Conclusion

In this paper, the variation iteration decomposition method has been initiated for the numerical solution of linear and nonlinear integro-differential equations. The mode of convergence of the method is determined by the step size parameter, h, even at few iterates. If the number of iterates increases, then the approximate solution converges for $h \leq 10^{5+n}$, where n is the number of iterations. Thus, the method is highly reliable, effective and efficient for the numerical stimulation of integro-differential equations. The variation iteration decomposition method can be extended to other fields of Mathematics such as; higher order boundary value problems, stochastic integro-differential equations, etc.

Table 1: Shows the numerical Results for various values of h for **Example 5.1**.

$x = i/h,\ i = 0(1)10$				
Absolute Error $h = 10$	**Absolute Error** $h = 100$	**Absolute Error** $h = 1000$	**Absolute Error** $h = 10000$	**Absolute Error** $h = 100000$
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8.3329E-06	8.3300E-10	1.0000E-13	0.0000E+00	0.0000E+00
1.3328E-04	1.3330E-08	1.0000E-12	0.0000E+00	0.0000E+00
6.7413E-04	6.7500E-08	7.0000E-12	0.0000E+00	0.0000E+00
2.1268E-03	2.1333E-07	2.1000E-11	0.0000E+00	0.0000E+00
5.1773E-03	5.2083E-07	5.2000E-11	0.0000E+00	0.0000E+00
1.0689E-02	1.0800E-06	1.0800E-10	0.0000E+00	0.0000E+00
1.9682E-02	2.0008E-06	2.0000E-10	0.0000E+00	0.0000E+00
3.3301E-02	3.4132E-06	3.4100E-10	0.0000E+00	0.0000E+00
5.2777E-02	5.4673E-06	5.4700E-10	1.0000E-13	0.0000E+00
7.9365E-02	8.3329E-06	8.3300E-10	1.0000E-13	0.0000E+00

Table 2: Shows the numerical Results for various values of h for **Example 5.2**.

$x = i/h,\ i = 0(1)10$			
Absolute Error $h = 10$	**Absolute Error** $h = 100$	**Absolute Error** $h = 1000$	**Absolute Error** $h = 10000$
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
6.9440E-06	7.0000E-09	0.0000E+00	0.0000E+00
5.5556E-05	5.6000E-08	0.0000E+00	0.0000E+00
1.8750E-04	1.8800E-07	0.0000E+00	0.0000E+00
4.4444E-04	4.4400E-07	0.0000E+00	0.0000E+00
8.6806E-04	8.6800E-07	1.0000E-09	0.0000E+00
1.5000E-03	1.5000E-06	2.0000E-09	0.0000E+00
2.3819E-03	2.3820E-06	2.0000E-09	0.0000E+00
3.5556E-03	3.5560E-06	4.0000E-09	0.0000E+00
5.0625E-03	5.0620E-06	5.0000E-09	0.0000E+00
6.9444E-03	6.9440E-06	7.0000E-09	0.0000E+00

Table 3: Shows the numerical Results for various values of h for **Example 5.3**.

$x = i/h,\ i = 0(1)10$				
Absolute Error $h = 10$	**Absolute Error** $h = 100$	**Absolute Error** $h = 1000$	**Absolute Error** $h = 10000$	**Absolute Error** $h = 100000$
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
1.5833E-04	1.6580E-07	2.0000E-10	0.0000E+00	0.0000E+00
1.1999E-03	1.3200E-06	1.3000E-09	0.0000E+00	0.0000E+00
3.8240E-03	4.4325E-06	4.5000E-09	0.0000E+00	0.0000E+00
8.5276E-03	1.0453E-05	1.0600E-08	0.0000E+00	0.0000E+00
1.5603E-02	2.0312E-05	2.0800E-08	0.0000E+00	0.0000E+00
2.5135E-02	3.4920E-05	3.5900E-08	0.0000E+00	0.0000E+00
3.6995E-02	5.5166E-05	5.7000E-08	1.0000E-10	0.0000E+00
5.0836E-02	8.1920E-05	8.5000E-08	1.0000E-10	0.0000E+00
6.6087E-02	1.1603E-04	1.2100E-07	1.0000E-10	0.0000E+00
8.1944E-02	1.5833E-04	1.6580E-07	2.0000E-10	0.0000E+00

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