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Modified Variational Homotopy Perturbation Method for Nonlinear Volterra Integro-differential Equations

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Abstract

In this paper, we apply the Modified Variational Homotopy Method (MVHPM) for the numerical solution of nonlinear Volterra integro-differential equations. The method is an elegant mixture of Variational Iteration and Homotopy Perturbation Methods. The modification is effected at the approximation stage. This modification is the addition of the coefficients of the least and highest powers of the homotopy parameter, p, instead of the addition of all the coefficients. The method is applied directly without discretization, linearization or perturbation. Two numerical examples are considered to demonstrate the effectiveness and performance accuracy of the method. The resulting numerical evidences show the method is highly reliable, effective and accurate as compared to existing methods the Variational Homotopy Method (VHPM) and the Variational Iteration Method (VIM), available in the literature.

1. Introduction

In this paper, we considered the Volterra nonlinear integro-differential equations of the form

$$y_n^{(m)}(x) = f(x) + \int_0^s f_n(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt \tag{1}$$

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where m is the order of derivatives and f_n, $n = 1, 2, \ldots, n$ are given, y_i are the solutions to be determined in the prescribed auxiliary boundary conditions.
This type of problem arises in the mathematical modelling of nanohydronamics, drop wish processes, glass- forming processes, etc. (see [1-3]). Over the years, several numerical techniques have being developed to tackle the problem of nonlinear integro-differential equations. The use of He's polynomials in the solution of fifth-order boundary value problems in variational iteration approach was studied by Aslam et al. [4]. Mamadu and Njoseh [5] explored the solution of linear and nonlinear Fredholm integro-differential equations using certain orthogonal polynomials as trial functions via orthogonal collocation approach. Also, Mamadu and Njoseh [6] studied the convergence analysis of the nonlinear first order Volterra integro-differential equations using Variational Integration Method (VIM). He [7] proposed the Homotopy Perturbation Method (HPM) and applied it to bifurcation of linear problems, limit cycle, boundary value problems, nonlinear oscillation, etc. Inspired by He's work, Othman et al. [8] employed the homotopy perturbation method for the numerical solution of 12^{th} order boundary value problems.
The variational iteration method has been explored by several researchers because of its applicability to wide range of nonlinear problems with approximates that converges rapidly and accurately, see [3]. He in [2] and [9] explored the relationship between the variational iteration and homotopy perturbation method. The homotopy perturbation method in real sense is the merging of the component parts - the standard homotopy and the perturbation. Thus, the merging of variational iteration and homotopy perturbation method results to variational homotopy perturbation method (VHPM) and has been applied to linear and nonlinear ordinary or partial differential equations, see [2],[9].
Motivated by the explosive research in this area, we proposed a modified version of the variational homotopy perturbation method for the solution of integro-differential equations. In this method, the correction functional is constructed from which the Lagrange multiplier is calculated, see [3]. He's polynomials are introduced and comparison of like powers of the homotopy parameter, p, gives the solution of various order. Finally, the final approximation solution is the addition of the coefficients of the least and highest powers of the homotopy parameter, p. This method requires no discretization, linearization or perturbation. The numerical evidences show the method is highly reliable and effective as compared with existing methods- the variational iteration method, available in the literature. Section 2 gives a brief review of variation iteration method. Section 3 presents the variational iteration treatment of the modified homotopy perturbation method. Section 4 presents some numerical applications. Finally, the conclusion is presented in section 5.

2. Preliminaries

Variational Iteration Method
To illustrate the basic ideas of the method, we consider the nonlinear differential equation of the form:

$$Ly(x) + Ny(x) = f(x) \tag{2}$$

with prescribed auxiliary conditions, where y is an unknown function, L is a linear operator, N a non linear term and f is an non-homogeneous forcing term

Using the VIM [3, 5-6], we can construct correction functional for equation (2) as:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda \left[Ly_k(\xi) + N\tilde{y}_k(x,\xi) - f(\xi)\right] d\xi, k \geq 0 \tag{3}$$

where $\lambda$is a general Lagrange multiplier, $\tilde{y}_k = 0$, i.e., $\tilde{y}_k$is a restricted variable.
Let the general Volterra integro-differential equations be given as:

$$y_n^{(m)}(x) = f(x) + \int_0^s f_n(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt, \tag{4}$$

where m is the order of derivatives and f_n, $n = 1, 2, \ldots, n$ are given, y_i are the solutions to be determined.
Let the correction functional be given as

$$\begin{aligned} y_{i,k+1}(x) = y_{i,k}(x) + \int_0^x \lambda_i(s) \Big[& y_{i,k}^{(m)}(s) - f(x) \\ & - \int_0^s f_i(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt \Big] ds \end{aligned} \tag{5}$$

for $i = 1, 2, \ldots, n$. A generalized value of Lagrange multiplier for VIM was proposed by [3] as

$$\lambda_m(s) = \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)}.$$

Thus, equation (5) becomes

$$\begin{aligned} y_{i,k+1}(x) = y_{i,k}(x) + \int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)} \Big[& (y_{i,k}^{(m)})(s) - f(x) \\ & - \int_0^s f(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt \Big] ds \end{aligned} \tag{6}$$

Equation (6) is the iteration formula for equation (4).

3. Modified Variational Homotopy Perturbation Method (MVHPM)

The Modified Homotopy Perturbation Method (MHPM) takes the usual Mathematical formulation of Homotopy Perturbation Method (HPM) with little modification in the solution format. To illustrate the basic ideas of the MVHPM, we will consider equation (2) as our general equation with usual definition of terms.
We defined convex homotopy H(y, p) by

$$H(r,p) = (1-p) * [L(r) - L(y_0)] + p[L(y) + N(y) - f(x)] = 0 \tag{7}$$

where $p \in [0, 1]$ is an embedding parameter. The MHPM involves expanding the Homotopy Parameter p to obtain

$$y = \sum_{i=0}^{\infty} p^i r_i \tag{8}$$

By comparing the like powers of p gives solutions of various order. Thus the MVHPM present the best approximation as the addition of the coefficients of the least and highest power of p only, that is,

$$y = r_{p^{(0)}} + r_{p^{(n)}} \tag{9}$$

where $r_{p^{(0)}}$ and $r_{p^{(n)}}$ is the coefficients of the least and highest occurring powers of p.
The convergence analysis of MVHPM can be treated as that of HPM, see [2] for more details.
Following from section (2) we have the correction functional as

$$y_{i,k+1}(x) = y_{i,k}(x) + \int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)} \Big[(y_{i,k}^{(m)})(s) - f(x) - \int_0^s f(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt\Big] ds,$$

Applying the variational iteration scheme, we have

$$\sum_{i=0}^{\infty} p^i r_i = y_0(x) + p\int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)} \left[\sum_{i=0}^{\infty} p^i \left((y_{i,k}^{(m)})(s) - f(x) - \int_0^s f(t, y_1(t), ..., y_1^{(m)}(t), ..., y_n(t), ..., y_n^{(m)}(t))dt\right)\right] ds$$

Such that the approximate solution of MVHPM is given as

$$y = r_{p^{(0)}} + r_{p^{(n)}}$$

where $r_{p^{(0)}}$ and $r_{p^{(n)}}$ are the coefficients of the least and highest occurring powers of p respectively.

4. Numerical Applications

Here, two nonlinear integro-differential equations are considered as examples to access the advantages and accuracy of MVHPM. For the sake of comparison, we take the same examples as used in [3] and [6].
Example 4.1: Consider the nonlinear integro-differential equation [6]

$$u'(x) = -1 + \int_0^x u^2(t)dt, 0 \leq x \leq 1, u(0) = 0. \tag{10}$$

Let our initial approximation be $u_0(x) = -x$ and then the correction functional for (9) is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[\frac{du_n}{ds} + 1 - \int_0^s (u_n(t))^2\, dt\right] ds].$$

Applying the VHPM, we have

$$\sum_{i=0}^{\infty} p^i r_i = -x - p\int_0^x \left[\left(\frac{du_0}{ds} + p\frac{du_1}{ds}\right) + 1 - \int_0^s (u_0(t) + pu_1(t))^2\, dt\right] ds.$$

Comparing the like powers of p, we have the following:

$$p^{(0)} : u_0 = -x$$

$$p^{(1)}: u_1 = -\frac{1}{12}x^4 + x$$
$$p^{(2)}: u_2 = x - \frac{1}{4}x^4 + \frac{1}{252}x^7$$
$$p^{(3)}: u_3 = -\frac{1}{12960}x^{10} - \frac{1}{12}x^4 + \frac{1}{252}x^7.$$

Our approximate solution by the MVHPM is given as

$$u(x) = u_0 + u_3 = \ -x - \frac{1}{12960}x^{10} - \frac{1}{12}x^4 + \frac{1}{252}x^7.$$

Results are shown in **Table 1** for $n = 1$. The maximum absolute error obtained is in order of 10^{-6} for $n = 1$. The results obtained are compared with that of VHPM (first order approximation) and VIM (Second order approximation in [6]). It is obvious that the approximate solution will converge absolutely to the exact solution as n increases.

Example 4.2: Consider the following nonlinear system of two integro-differential equation [3]

$$u'(x) = 1 - \frac{1}{2}v'^2(x) + \int_0^x ((x-t)v(t) + v(t)u(t)dt, \quad (11)$$
$$v'(x) = 2x + \int_0^x ((x-t)u(t) - v^2(t) + u^2(t)dt,$$
$$u(0) = 0, \quad v(0) = 0,$$

with the exact solution

$$u(x) = \sinh(x), \ \ v(x) = \cosh(x).$$

Let our initial approximation be $u_0(x) = 0$ and $v_0(x) = 1$.
The correction functional for (11) is given as

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 1 + \frac{1}{2}v_n'^2(s) - \int_0^s ((s-t)v_n(t) + u_n(t)v_n(t))dt\right] ds,$$
$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v_n'(s) - 2s - \int_0^s ((s-t)u_n(t) - v_n^2(t) + u_n^2(t))dt\right] ds$$

Applying the VHPM, we have

$$\sum_{i=0}^{\infty} p^i r_i = -x - p\int_0^x \left[\left(\frac{du_0}{ds} + p\frac{du_1}{ds}\right) - 1 + \frac{1}{2}\left(\frac{dv_0}{ds} + p\frac{dv_1}{ds}\right)^2\right.$$
$$\left. - \int_0^s (s-t)\,(v_0(t) + pv_1(t)) + (u_0(t) + pu_1(t))(v_0(t) + pv_1(t))dt\right] ds.$$

Comparing the like powers of p, we have the following:

$$p^{(1)}: u_1 = \frac{1}{6}x^3 + x$$
$$p^{(2)}: u_2 = \frac{1}{60}x^5 + \frac{1}{6}x^3 - x$$
$$p^{(3)}: u_3 = \frac{1}{504}x^7 + \frac{1}{30}x^5.$$

Our approximate solution by the MVHPM is given as

$$u(x) = u_1 + u_3 = \frac{1}{504}x^7 + \frac{1}{30}x^5 + \frac{1}{6}x^3 - x.$$

Similarly,

$$p^{(0)} : v_0 = 1$$
$$p^{(1)} : v_1 = \frac{1}{2}x^2$$
$$p^{(2)} : v_2 = \frac{1}{720}x^6 - \frac{1}{24}x^4 - \frac{3}{2}x^2$$
$$p^{(3)} : v_3 = \frac{1}{2016}x^8 + \frac{1}{360}x^6 + \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

Our approximate solution by the MVHPM is given as

$$v(x) = v_0 + v_3 = \frac{1}{2016}x^8 + \frac{1}{360}x^6 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + 1.$$

Results are shown in **Tables 2** and **3** for $n = 1$. The maximum absolute errors obtained is in order of 10^{-11} and 10^{-9} for u and v respectively. The results obtained are compared with results obtained using VHPM (first order approximation) and VIM (Second order approximation in [3]). We see again that the MVHPM is very effective and highly promising.

5. Conclusion

In this paper, we have succeeded in implementing MVHPM on nonlinear integro-differential equations. The initial approximation for example 4.1 was selected arbitrary in form of the exact solution. In example 4.2, the initial approximation was arbitrary not in form of the exact solution with unknown coefficients. The method is applied in direct way without discretization, linearization or perturbation. The numerical evidence clearly reflects the effectiveness and performance of MVHPM. It has more accuracy than the VHPM and variational iteration method as obviously seen in **Tables 1, 2** and **3**. In addition, first iterate approximation of MVHPM correspond to second iterate approximate of VIM. This obviously shows MVHPM has been more effective and efficient.

Table 1: Results obtained using MVHPM compared with results obtained using VHPM and VIM in [6]

X	**Exact** Solution	**First-order** approximation, MVHPM	**First order** approximation VHPM	**Second-order** approximation, VIM	**Error** MVHPM	**Error** VHPM	**Error** VIM
0.0000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000	0.0000	0.0000
0.0938	-0.0938000000	-0.0938064508	-0.0938387027	-0.0937935492	6.4508E-06	3.8703E-05	6.4508E-06
0.2188	-0.2188000000	-0.2189908936	-0.2199445996	-0.2186091064	1.9089E-04	1.1446E-03	1.9089E-04
0.3125	-0.3125000000	-0.3132935744	-0.3172522097	-0.3117064256	7.9357E-04	4.7522E-03	7.9357E-04
0.4062	-0.4062000000	-0.4084614758	-0.4197109676	-0.4039385242	2.2615E-03	1.3511E-02	2.2615E-03
0.5000	-0.5000000000	-0.5051774067	-0.5308167257	-0.4948225933	5.1774E-03	3.0817E-02	5.1774E-03
0.6250	-0.6250000000	-0.6375685304	-0.6992313573	-0.6124314696	1.2569E-02	7.4231E-02	1.2569E-02
0.7188	-0.7188000000	-0.7406553750	-0.8467962584	-0.6969446250	2.1855E-02	1.2800E-01	2.1855E-02
0.8125	-0.8125000000	-0.8478992591	-1.0175134130	-0.7771007409	3.5399E-02	2.0501E-01	3.5399E-02
0.9062	-0.9062000000	-0.9604346130	-1.2157913990	-0.8519653870	5.4235E-02	3.0959E-01	5.4235E-02
1.0000	-1.0000000000	-1.0794422400	-1.4452160490	-0.9205577601	7.9442E-02	4.4522E-01	7.9442E-02

Table 2: Results obtained using MVHPM compared with results obtained using VHPM and VIM in [3]

X	**Exact**	**First order** approximation MVHPM	**First order** approximation VHPM	**Second order** approximation VIM	**Error** MVHPM	**Error** VHPM	**Error** VIM
0.01	0.0100001667	0.0100001667	0.1000033333	0.0100001667	0.0000E+00	1.6667E-07	0.0000E+00
0.02	0.0200013334	0.0200013334	0.0200026668	0.0200013335	8.0000E-11	1.3334E-06	1.1000E-10
0.03	0.0300045002	0.0300045008	0.3000900080	0.0300045010	6.1000E-10	4.5006E-06	8.1000E-10
0.04	0.0400106675	0.0400106701	0.0400213368	0.0400106709	2.5600E-09	1.0669E-05	3.4100E-09
0.05	0.0500208359	0.0500208438	0.0500416771	0.0500208464	7.8100E-09	2.0841E-05	1.0420E-08
0.06	0.0600360065	0.0600360259	0.0600720257	0.0600360324	1.9450E-08	3.6019E-05	2.5930E-08
0.07	0.0700571807	0.0700572227	0.0701143894	0.0700572367	4.2040E-08	5.7209E-05	5.6040E-08
0.08	0.0800853606	0.0800854426	0.0801707759	0.0800854699	8.1960E-08	8.5415E-05	1.0927E-07
0.09	0.0901215492	0.0901216969	0.0902431969	0.0901217461	1.4770E-07	1.2165E-04	1.9691E-07
0.10	0.1001667500	0.1001670002	0.1003336669	0.1001670835	2.5020E-07	1.6692E-04	3.3350E-07

Table 3: Results obtained using MVHPM compared with results obtained using VHPM and VIM in [3].

X	**Exact**	**First order** approximation MVHPM	**First order** approximation VHPM	**Second order** approximation VIM	**Error** MVHPM	**Error** VHPM	**Error** VIM
0.01	1.0000500000	1.0000500000	0.9999500000	1.0000500000	0.0000E+00	1.0000E-04	0.0000E+00
0.02	1.0002000070	1.0002000070	0.9998000000	1.0002000070	0.0000E+00	4.0001E-04	0.0000E+00
0.03	1.0004500340	1.0004500340	0.9995500000	1.0004500340	0.0000E+00	9.0003E-04	0.0000E+00
0.04	1.0008001070	1.0008001070	0.9992000000	1.0008001070	0.0000E+00	1.6001E-03	0.0000E+00
0.05	1.0012502600	1.0012502600	0.9987500001	1.0012502600	0.0000E+00	2.5003E-03	0.0000E+00
0.06	1.0018005400	1.0018005400	0.9982000002	1.0018005400	0.0000E+00	3.6005E-03	0.0000E+00
0.07	1.0024510010	1.0024510010	0.9975500005	1.0024510010	0.0000E+00	4.9010E-03	0.0000E+00
0.08	1.0032017070	1.0032017070	0.9968000011	1.0032017080	0.0000E+00	6.4017E-03	1.0000E-09
0.09	1.0040527340	1.0040527350	0.9959500022	1.0040527360	1.0000E-09	8.1027E-03	2.0000E-09
0.10	1.0050041680	1.0050041690	0.9950000042	1.0050041710	1.0000E-09	1.0004E-02	3.0000E-09

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