Nigerian Journal of Mathematics and Applications
Volume **25**, (2016), 65 − 79. *Printed by Unilorin press*

http://www.njmaman.com

Combinatorial Properties of Order-Preserving and Order-Decreasing Semigroup of Partial Injective Contraction Mapping

B. M. Ahmed and S. O. Makanjuola*

Abstract

Let CI_n be the semigroup of partial injective contraction mappings. The focus of this paper is to provide some properties subsemigroups of CI_n which are semigroup of order preserving partial injective contraction mapping (OCI_n) and semigroup of order decreasing partial injective contraction mapping (DCI_n). The results on this work are combinatorial properties of subsemigroups of CI_n which are both order-preserving and order decreasing denoted by $ODCI_n$.

1. Introduction

2. PRELIMINARY IDEAS

2.1. **TRANSFORMATION SEMIGROUP.** Every finite semigroup can be embedded in a full transformation semigroup T_n on the set $\{1, 2, \cdots, n\}$, so also every finite inverse semigroup is embeddable in a symmetric inverse semigroup I_n on the set $\{1, 2, \cdots, n\}$. It is necessary to mention some subsemigroups of I_n that are of interest i.e subsemigroups of order-preserving, order-decreasing and of order-preserving and order-decreasing.

Received: 09/03/2016, Accepted: 29/06/2016, Revised: 13/07/2016.
2015 *Mathematics Subject Classification.* 47Jxx & 35A23. * Corresponding author.
Key words and phrases. Order-Reversing Partial Transformation, Order-Decreasing Semigroup & Partial Injective Contraction Mapping.
Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
E-mail: @unilorin.edu.ng

Let X_n denote the set $\{1, 2, \cdots, n\}$ and let $\alpha : \ dom(\alpha) \subseteq X_n \rightarrow X_n$ be a map in I_n. A partial mapping α is said to be contraction if, for all $x, y \in dom(\alpha)$, $|x\alpha - y\alpha| \leq |x - y|$ and is said to be an isometry if
$|x\alpha - y\alpha| = |x - y|$.
Let

(1) $IO_n = \{\alpha \in I_n : (\forall x, y \in dom(\alpha)) \ x \leq y \Rightarrow x\alpha \leq y\alpha\}$
(2) $OCI_n = \{\alpha \in IO_n : (\forall x, y \in dom(\alpha)) \ |x\alpha - y\alpha| \leq |x - y|\}$
(3) $ID_n = \{\alpha \in I_n : (\forall x, y \in dom(\alpha)) \ x\alpha \leq x\}$
(4) $DCI_n = \{\alpha \in ID_n : (\forall x, y \in dom(\alpha)) \ |x\alpha - y\alpha| \leq |x - y|\}$

be the subsemigroups of I_n consisting of order-preserving, order-preserving contraction mappings, order-decreasing and order-decreasing contraction mappings respectively. Many researchers have worked on semigroup of contraction mapping including Howie (1995), Umar (1992),Kehinde (2012), Adeniji (2012), Adeshola (2013) and others. In this paper, we study the combinatorial properties of order-preserving and order-decreasing semigroup of partial injective mappings. We start by presenting some preliminary ideas and basic definitions about the study in his section. Section 2 contains review of some past literature while section 3 is on the method and procedure. In the final section, we stated the combinatorial results of OCI_n, DCI_n and $ODCI_n$.

2.2. BASIC DEFINITIONS.

(a) Full Transformation Semigroup:-
A map α : Dom $(\alpha) \subseteq X_n \rightarrow Im(\alpha) \subseteq X_n$ is called full (or total) transformation of X_n if Dom $(\alpha) = X_n$; otherwise it is called strictly partial. The set T_n of all full transformation of X_n forms a semigroup under composition of mapping called the full transformation semigroup.

(b) Partial Transformation Semigroup:-
Let A and B be two subsets of X_n, a mapping $\alpha : A \rightarrow B$ is called partial transformation of X_n. A and B are called domain and range respectively and are denoted by Dom (α) and Im (α).The set of all partial transformation is denoted by P_n and the set P_n is a semigroup under composition called the partial transformation semigroup.

(c) Partial one-one Transformation:-
A partial transformation semigroup α: Dom $\alpha \subseteq$ Im α is said to be partial one-one transformation if Dom $\alpha \subseteq$ Im $\alpha \rightarrow$ Im $\alpha \subseteq X_n$ in the domain is mapped to one and only one element in the set X_n.

(d) Injective Contraction Mapping
A transformation α in I_n for which (for all $x, y \in X_n$)

$$|x\alpha - y\alpha| \leq |x - y|$$

is said to be a contraction and is said to be isometry if (for all $x, y \in Dom \ \alpha$)

$$|x\alpha - y\alpha| = |x - y|$$

(e) Order-Preserving or Order-Reversing Partial Transformation:-

A transformation $\alpha \in P_n$ is said to be order preserving if (for all $x, y \in Dom\alpha$) $x \leq y \Rightarrow xa \leq ya$. The semigroup of order-preserving full and partial transformation of X_n will be denoted by O_n and PO_n respectively.

A transformation $\alpha \in P_n$ for which (for all $x, y \in Dom\alpha$) $x \leq y \Rightarrow xa \geq ya$ is said to be order reserving.The semigroup of order-preserving or reversing full and partial transformation of X_n will be denoted by OD_n and POD_n respectively.

(f) Order-Decresing Transformation:-

A transformation α in P_n for which $xa \leq x$ (for all $x, y \in Dom\alpha$) is said to be order decreasing.Order increasing is defined analogously.The semigroups of order-decreasing full and partial transformation of X_n will be denoted by D_n and PD_n respectively.

(g) Order-Preserving and Order-Decreasing Partial Transformation:-

We define $C_n = O_n \cap D_n$ and $PC_n = PO_n \cap PD_n$ as the semigroup of order-preserving and order decreasing full and partial transformation of X_n respectively.The monoid C_n is also known as the catalan monoid because $|C_n|$ is also known as the nth catalan number.

TYPES OF TRANSFORMATION	FULL	PARTIAL	PARTIAL ONE-TO-ONE
Partial transformation	T_n	P_n	I_n
Order preserving	O_n	PO_n	IO_n
Order preserving or reversing	OD_n	POD_n	IO_n
Order decreasing	D_n	PD_n	ID_n
Order preserving and decreasing	C_n	PC_n	IC_n

3. LITERATURE REVIEW

3.1. **BACKGROUND RESULTS.** The main focus of this research work is combinatorial properties of semigroup of partial one-one contraction mapping to take care of open problems created by Kehinde (2012), Adeniji (2012) and Adeshola (2013).

The work of Adeshola (2013) centered on full contraction mapping and its subsemigroup like order-preserving (OCT_n), order-preserving or order reversing ($ORCT_n$) and order decreasing ($ODCT_n$) of full contraction mapping. His results are summarized below:

S	$\lvert S\rvert$	E(S)
OCT_n	$(n+1)2^{n-2}$	$\binom{n+1}{2}$
$ORCT_n$	$(n+1)2^{n-1}-n$	$\binom{n+1}{2}$
$ODCT_n$	2^{n-1}	$\binom{n-1}{p-1}$

$$F(n;.p,m,k) = \binom{n-m-1}{n-p-1}$$

$$F(n;.p,k) = \begin{cases} 2\binom{n-1}{p-1}, & \text{if } p>1 \\ & 1 \text{ otherwise} \end{cases}$$

$$F(n;.p,k,m) = \begin{cases} \binom{n-m-1}{p-m}, & \text{if } p=k \\ & 0 \text{ otherwise} \end{cases}$$

Kehinde (2012) also started work on semi-group of partial injective contraction mapping. He later got his results on semigroup of isometry mapping of finite chain. A mapping is said to be isometry if $\forall\ x,y \in \text{Dom }(\alpha)\ |x\alpha - y\alpha| = |x-y|$. His results are summarized below:

S	$\lvert S\rvert$	E(S)
DP_n	$3\times 2^{n+1}-(n+2)^2-1$	2^n
ODP_n	$3\times 2^n - 2(n+1)$	2^n
DDP_n	$3a_{n-1}-2a_{n-2}-2^{\frac{n}{2}}+n+1$	2^n
$ODDP_n$	$2^{n+1}-(n+1)$	2^n

Adeniji (2012) worked on an identity difference on full and partial one-one transformed semigroup.The identity difference is defined to be transformation such that $\forall\ x,y \in \text{Dom }(\alpha)$:$|w^+(\alpha)-w^-(\alpha)| \le 1$ which can be seen as semigroup of contraction mappings. Her results are summarized below

S		$\lvert N_n\rvert$	$\lvert L_n\rvert$	$\lvert F_n\rvert$	E(S)	$\lvert E_n\rvert$
IDT_n	$n+(n-1)(2^n-2)$	-	$n+(n-1)2^{n-2}$	C_n	L_n	L_n
IDI_n	n^3-2n^2+2n+1	A	$n(n+1)$	$C_n - N_n$	$n+1$	E(s)
IDR_n	B	?	$1+n^2$	?	F	L_n
$OIDT_n$	n^2-n+1	-	n^2	n^2	$2n-1$	n^2
$OIDI_n$	G	H	$n(n-1), n\ge 2$	n^2-n+2	$n+1$	E(S)
$OIDP_n$	C	D	$(n+1)+n(n-1)$	E	I	L_n

4. METHOD AND PROCEDURE

4.1. Semigroup of Order-preserving Partial One-One Contaction Mapping. A transformation $\alpha \in I_n$ is said to be order-preserving if

$$(\forall x, y \in Dom\alpha) x \leq y \Rightarrow x\alpha \leq y\alpha$$

The semigroup of order-preserving partial one-one transformation of X_n denoted by IO_n while the semigroup of order preserving partial one-one Contraction mapping is denoted by OCI_n. The elements of OCI_n in Domain/image of α are listed below for $n = 1, 2, 3, 4, 5$.

For n = 1, OCI_n has 2 elements

$|Im\alpha| = 1$

Dom α / Im α	(1)
(1)	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\emptyset$

For $n = 2$, OCI_n has 6 elements

$|Im\alpha| = 2$

Dom α/ Im α	(1 , 2)
(1 , 2)	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	(1)	(2)
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
2	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

and $\emptyset$

For n = 3, OCI_n has 18 elements

$|Im\alpha| = 3$

Dom α/ Im α	(1, 2, 3)
(1, 2, 3)	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$|Im\alpha| = 2$

Dom α/ Im α	(1, 2)	(2, 3)	(1, 3)
(1, 2)	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$	
(2,3)	$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$	
(1,3)	$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	(1)	(2)	(3)
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\3\end{pmatrix}$
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$

and $\emptyset$

For $n = 4$, OCI_nhas 53 elements
$|Im\alpha| = 4$

Dom α/ Im α	(1 , 2, 3, 4)
(1 , 2, 3, 4)	$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$

$|Im\alpha| = 3$

Dom α/ Im α	(1 2 3)	(2 3 4)	(1 2 4)	(1 3 4)
(1, 2, 3)	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&4\end{pmatrix}$		
(2, 3, 4)	$\begin{pmatrix}2&3&4\\1&2&3\end{pmatrix}$	$\begin{pmatrix}2&3&4\\2&3&4\end{pmatrix}$		
(1, 2, 4)	$\begin{pmatrix}1&2&4\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&4\\2&3&4\end{pmatrix}$	$\begin{pmatrix}1&2&4\\1&2&4\end{pmatrix}$	
(1, 3, 4)	$\begin{pmatrix}1&3&4\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&3&4\\2&3&4\end{pmatrix}$		$\begin{pmatrix}1&3&4\\1&3&4\end{pmatrix}$

$|Im\alpha| = 2$

Dom α/ Im α	(1 2)	(2 3)	(3 4)	(1 3)	(2 4)	(1 4)
(1, 2)	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$\begin{pmatrix}1&2\\2&3\end{pmatrix}$	$\begin{pmatrix}1&2\\3&4\end{pmatrix}$			
(2, 3)	$\begin{pmatrix}2&3\\1&2\end{pmatrix}$	$\begin{pmatrix}2&3\\2&3\end{pmatrix}$	$\begin{pmatrix}2&3\\3&4\end{pmatrix}$			
(3, 4)	$\begin{pmatrix}3&4\\1&2\end{pmatrix}$	$\begin{pmatrix}3&4\\2&3\end{pmatrix}$	$\begin{pmatrix}3&4\\3&4\end{pmatrix}$			
(1, 3)	$\begin{pmatrix}1&3\\1&2\end{pmatrix}$	$\begin{pmatrix}1&3\\2&3\end{pmatrix}$	$\begin{pmatrix}1&3\\3&4\end{pmatrix}$	$\begin{pmatrix}1&3\\1&3\end{pmatrix}$	$\begin{pmatrix}1&3\\2&4\end{pmatrix}$	
(2, 4)	$\begin{pmatrix}2&4\\1&2\end{pmatrix}$	$\begin{pmatrix}2&4\\2&3\end{pmatrix}$	$\begin{pmatrix}2&4\\3&4\end{pmatrix}$	$\begin{pmatrix}2&4\\1&3\end{pmatrix}$	$\begin{pmatrix}2&4\\2&4\end{pmatrix}$	
(1, 4)	$\begin{pmatrix}1&4\\1&2\end{pmatrix}$	$\begin{pmatrix}1&4\\2&3\end{pmatrix}$	$\begin{pmatrix}1&4\\3&4\end{pmatrix}$	$\begin{pmatrix}1&4\\1&3\end{pmatrix}$	$\begin{pmatrix}1&4\\2&4\end{pmatrix}$	$\begin{pmatrix}1&4\\1&4\end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	(1)	(2)	(3)	(4)
1	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}1\\3\end{pmatrix}$	$\begin{pmatrix}1\\4\end{pmatrix}$
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}2\\3\end{pmatrix}$	$\begin{pmatrix}2\\4\end{pmatrix}$
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}3\\4\end{pmatrix}$
4	$\begin{pmatrix}4\\1\end{pmatrix}$	$\begin{pmatrix}4\\2\end{pmatrix}$	$\begin{pmatrix}4\\3\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$

For $n = 5, OCI_n$ has 154 elements, its table is constructed in the same way.

4.2. **Semigroup of Order Decreasing Partial one-one contraction mapping.** A transformation α in I_n for which $x\alpha \leq x$ (for all x $\in Dom\alpha$) is said to be order decreasing.The semigroups of order decreasing partial one-one transformation of X is denoted by ID_n while the semigroup of order-decreasing partial one-one contraction mapping is denoted by DCI_n.

The elements of DCI_n in Domain/ Image of α are listed below for n= 1, 2, 3, 4, 5.

For n=1 DCI_n has 2 elements

Dom α/ Im α	1
1	$\begin{pmatrix}1\\1\end{pmatrix}$

and $\emptyset$

For n=2 DCI_n has 5 elements

$|Im\alpha| = 2$

Dom α/ Im α	(1, 2)
(1, 2)	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	(1)	(2)
1	$\begin{pmatrix}1\\1\end{pmatrix}$	
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$

and $\emptyset$

For n=3 DCI_n has 14 elements

$|Im\alpha| = 3$

Dom α/ Im α	(1 2 3)
(1 2 3)	$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$

$|Im\alpha| = 2$

Dom α/ Im α	(1, 2)	(2, 3)	(1, 3)
(1, 2)	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$		
(2, 3)	$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$ $\begin{pmatrix}2 & 3\\2 & 1\end{pmatrix}$	$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$	
(1, 3)	$\begin{pmatrix}1 & 3\\1 & 2\end{pmatrix}$		$\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	1	2	3
1	$\begin{pmatrix}1\\1\end{pmatrix}$		
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$

for n =4 , DCI_n has 39 elements and for n = 5 has 110 elements. Their tables are constructed in the same way.

4.3. **Semigroup of Order-preserving and Order-decreasing Partial one-one Contraction Mapping.** We define $IC_n = IO_n \cap ID_n$ as the semigroup of order preserving and order-decreasing partial one-one transformation. We now define
$ODCI_n = OCI_n \cap DCI_n$ as the semigroup of order-preserving and odrer-decreasing partial one-one contraction mapping of X_n.

The elements of $ODCI_n$ in Domain/ Image of α are listed below for n = 1, 2, 3, 4, 5.

For n= 1 , $ODCI_n$ has 2 elements

Dom α/ Im α	1
1	$\begin{pmatrix}1\\1\end{pmatrix}$

and $\emptyset$

For n=2 $ODCI_n$ has 5 elements

$|Im\alpha| = 2$

Dom α/ Im α	(1, 2)
(1, 2)	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	(1)	2
1	$\begin{pmatrix}1\\1\end{pmatrix}$	
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$

and $\emptyset$

For n=3 $ODCI_n$ has 13 elements
$|Im\alpha| = 3$

Dom α/ Im α	(1 2 3)
(1, 2, 3)	$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$

$|Im\alpha| = 2$

Dom α/ Im α	(1 2)	(2 3)	(1 3)
(1, 2)	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$		
(2, 3)	$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}$	$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$	
(1, 3)	$\begin{pmatrix}1 & 3\\1 & 2\end{pmatrix}$		$\begin{pmatrix}1 & 3\\1 & 3\end{pmatrix}$

$|Im\alpha| = 1$

Dom α/ Im α	1	2	3
1	$\begin{pmatrix}1\\1\end{pmatrix}$		
2	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	
3	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$

and $\emptyset$

For n =4 , $ODCI_n$ has 34 elements and for n = 5, $ODCI_n$ has 89 elements. Their tables are constructed in the same way.

5. COMBINATORIAL RESULTS

In this section, we draw tables for each subsemigroups under consideration. For natural number $n \geq p \geq m \geq 0$, we define:

(1) F(n;p) = $|\{\alpha \in S : h(\alpha) = |Im\alpha| = p\}|$
(2) F(n;m) = $|\{\alpha \in S : f(\alpha) = m\}|$
(3) F(n;k) = $|\{\alpha \in S : W^{+}(\alpha) = max(Im\alpha) = k\}|$
(4) F(n;p,m) = $|\{\alpha \in S : h(\alpha) = p \wedge f(\alpha) = m\}|$
(5) F(n;p,k) = $|\{\alpha \in S : h(\alpha) = p \wedge w^{+}(\alpha) = k\}|$

and

$$|S| = \sum_k F(n;k) = \sum_m F(n;m) = \sum_p F(n;p)$$

Where S is a subsemigroup of partial one-one contraction mappings of a finite chain.

5.1. **4.1 Semigroup of Order-preserving Partial one-one contraction mapping.** 4.1 Semigroup of Order-preserving Partial one-one contraction mapping The tables under this consideration are drawn below:

5.2. **Semigroup of Order Decreasing Partial one-one Contraction Mapping.** The tables under this consideration are drawn below:

Table 4:2:1 Number of Elements in Terms of Height Elements F(n,p) in DCI_n

n/p	0	1	2	3	4	5	$\sum$F(n,p)
0	01						01
1	01	01					02
2	01	03	01				05
3	01	06	06	01			14
4	01	10	19	08	01		39
5	01	15	46	37	10	01	110

Table 4:2:2 Number of Elements in Terms of Fixed Elements F(n,m) in DCI_n

n/m	0	1	2	3	4	5	$\sum$F(n,m)
0	01						01
1	01	01					02
2	02	02	01				05
3	05	05	03	01			14
4	14	12	08	04	01		39
5	40	31	21	12	15	01	110

Table 4:2:3 Number of Elements in Terms of Waist Elements F(n,k) in DCI_n

n/p	0	1	2	3	4	5	$\sum$F(n,w^+)
0	01						01
1	01	01					02
2	01	02	02				05
3	01	03	06	04			14
4	01	04	12	14	08		39
5	01	05	20	36	32	16	110

5.3. **Semigroup of Order-preserving and order-decreasing Partial one-one Contraction Mapping.** The tables under this consideration are drawn below:

Table 4:3:1 Number of Elements in Terms of Height Elements F(n,p) in $ODCI_n$

n/p	0	1	2	3	4	5	$\sum$F(n,p)
0	01						01
1	01	01					02
2	01	03	01				05
3	01	06	05	01			13
4	01	10	15	07	01		34
5	01	15	35	28	09	01	89

Table 4:3:2 Number of Elements in Terms of Fixed Elements F(n,m) in $ODCI_n$

n/m	0	1	2	3	4	5	6	7	$\sum$F(n,m)
0	01								01
1	01	01							02
2	02	02	01						05
3	05	04	03	01					13
4	13	09	07	04	01				34
5	34	22	16	11	05	01			89
6	89	56	38	27	16	06	01		233
7	233	145	94	65	43	22	07	01	610

Table 4:3:3 Number of Elements in Terms of Waist Elements F(n,k) in $ODCI_n$

n/k	0	1	2	3	4	5	$\sum$F(n,w^+)
0	01						01
1	01	01					02
2	01	02	02				05
3	01	03	05	04			13
4	01	04	09	12	08		34
5	01	05	14	25	28	16	89

5.4. **MAIN RESULTS.**

5.4.1. *LEMMA.* Let $S = ODCI_n$. Then

$$F(n;p,k) = F(n-1;p,k) + \sum_{j=1}^{k} F(n-j;p-1,k-j)$$

Proof

Let $\alpha \in ODCI_n$ be such that h(α) = p and w(α) = k.Then for all such α we either have n $\notin$ Dom α or n $\in$ Dom α. In the former, it is clear that there would be F(n-1;p,k) such maps while in the latter there would be F(n-j;p-1.k-j) for i$\leq$j $\leq$k, since nα = k and so $(k-j)\alpha^{-1} \leq$ n-j, by contraction property.Thus taking the sum over j yields the required result.

5.4.2. *THEOREM.* Let $S = ODCI_n$.Then

$$F(n; p, k_p) = \binom{n}{p} \, and F(n; p, k_n) = \binom{n-1}{p-1}$$

Proof

If K = p then Im $\alpha = \{1, 2, ...p\}$ which admits any subset of size p as a possible domain and so there are $\binom{n}{p}$ such maps. If k = n then nα = n and so x$\alpha = x$ for all x $\in$ Dom α, these correspond to partial identities of height p-1 on $X_n/\{n\}$, of which there are $\binom{n-1}{p-1}$. Hence the results.

5.4.3. *THEOREM.* Let $S = ODCI_n$.Then F(n; p, k) = $\binom{k-1}{p-1}\binom{n+p-k}{p}$

Proof

By double induction Basis step:

$$F(n; p, k_p) = \binom{p-1}{p-1}\binom{n+p-p}{p} = \binom{n}{p} \, and$$

$$F(n; p, k_n) = \binom{n-1}{p-1}\binom{n+p-n}{p} = \binom{n-1}{p-1}$$

are true by theorem above. Induction Hypothesis.

suppose F(n; p, k) = $\binom{k-1}{p-1}\binom{n+p-k}{p}$ is true for all n $\geq$ k $\geq$ p.

Inductive Step: [Goal: to show that

F(n+1, p, k) = $\binom{k-1}{p-1}\binom{n+1+p-k}{p}$ is true for all n+1 $\geq$ k $\geq$ p . Now,

$$F(n+1; p, k) = F(n; p, k) + \sum_{j=1}^{k} F(n+1-j, p-1, k-j)$$

$$= \binom{k-1}{p-1}\binom{n+p-k}{p} + \sum_{j=1}^{k}\binom{k-j-1}{p-2}\binom{n+p-j-k+j}{p-1}$$

$$= \binom{k-1}{p-1}\binom{n+p-k}{p} + \binom{n+p-k}{p-1}\binom{k-1}{p-1}$$

$$= \binom{k-1}{p-1}\binom{n+1+p-k}{p}$$

5.4.4. *THEOREM.* Let $S = ODCI_n$

$F(n; p) = \binom{n+p}{2p}$

Proof

$$F(n;p) = \sum_{m=0}^{p} F(n;p,m)$$
$$= \sum_{m=0}^{p} F(n;p,m) + F(n;p,m_p)$$
$$= \binom{n+p-1}{n-p-1} + \binom{n+p-2}{n-p-1} + \binom{n+p-3}{n-p-1} \cdots + \binom{n}{n-p-1}$$
$$+ \binom{n-1}{n-p-1} + \binom{n-2}{n-p-1} + \cdots + \binom{n-p-1}{n-p-1}$$
$$= \binom{n+p}{n-p} - \binom{n}{n-p}$$

$$\sum_{m=0}^{p-1} \binom{n+p-m-1}{n-p-1} + \binom{n}{n-p}$$
$$= \sum_{m=0}^{2p} \binom{n+p+m-1}{n-p-1} - \sum_{m=p}^{2p} \binom{n+p-m-1}{n-p-1} + \binom{n}{n-p}$$
$$= \left[\binom{n+p}{n-p} - \binom{n}{n-p}\right] + \binom{n}{n-p}$$
$$= \binom{n+p}{2p}$$

5.4.5. *THEOREM.* Let $S = ODCI_n$
Then

$$F(n;m) = F(n-1;m-1) + F(n-1;m)$$

For all$1 \leq m \leq n$.

Proof

$$F(n;m) = \sum_{i=0}^{n-m} \binom{m-1+i}{m-1} F(n-m-i;m_0)$$

$$= \sum_{i=0}^{n-m} \binom{m-2+i}{m-2} F(n-m-i;m_0) + \sum_{i=o}^{n-m} \binom{m-2+i}{m-1} F(n-m-i;m_0)$$

$$= F(n-1;m-1) + \sum_{i=1}^{n-m} \binom{m-1+(i-1)}{m-1} F(n-1-m-(i-1);m_0)$$
$$= F(n-1;m-1) + \sum_{j=0}^{n-1-m} \binom{m-i+j}{m-1} F(n-1-m-j;m_0)$$
$$= F(n-1;m-1) + F(n-1;m)$$

6. CONCLUSION

The combinatorial properties of order preserving and order decreasing semigroup of partial injective contraction mapping were studied. The two and three functions $F(n,p)$, $F(n,m)$, $F(n,k)$ and $F(n,p,k)$ were used to carry out the order of the semigroup. The work can be extended to other subsemigroups of injective contraction mapping.

References

[1] Adeniji, A. O. and Makanjuola, S. O. (2010). Ideals and Elements in Identity Difference Full Transformation Semigroup. *Journal of Mathematical Sciences, 21(3)*, 323-329.

[2] Adeniji, A. O and Makanjuola, S.O (2008). On Some Combinatorial Results of Collapse in Full Transformation Semigroup. *African Journal of Computing and ICTI 2,* 61-62.

[3] Adeniji, A. O (2012): *Identity Difference Transformation Semigroups*, Ph.D Thesis (Unpublished) Submited to the Department of Mathematics, University of Ilorin, Ilorin.

[4] Adeshola, D.A. (2013). *Some Semigroups of Full Contraction Mapping of a Finite Chain*, Ph.D Thesis (Unpublished) Submited to the Department of Mathematics, University of Ilorin, Ilorin.

[5] Garba, G. U. (1990), Idempotent in Partial Transformation Semigroup. *Proc. Roy. Soc. Edinburgh*, 116, 359-366.

[6] Howie, J. M. (1994). *Combinatorial and Probalitistic Results in Transformation Semigroups.* Words, Language and Combinatorics II (kyoto, 1992). World Science Publ., River Edge, NJ, 200-206.

[7] Howie, J. M. (1995). *Fundamentals of Semigroup Theory.* Oxford, Clarendon Press.

[8] Howie, J. M. (1971). *Products of Idempotent in Certain Semigroups of Transformation.* Proc. Ediburgh Math. Soc. 17, 223-236.

[9] Kehinde, R. (2012). *Some Algebraic and Combinatorial Properties of Semigroup of Injective Partial Contraction Mapping and Isometrics of a Finite Chain*, Ph.D Thesis (Unpublished) Submited to the Department of Mathematics, University of Ilorin, Ilorin.

[10] Kehinde, R., Makanjuola, S. O. and Umar, A.(2014). On the Semigroup of Order-Decreasing Partial Isometrics of a Finite Chain. *Communication in Algebra.*

[11] Kehinde, R. and Umar, A. (2014). On the Semigroup of Order-Decreasing Partial Injective Contraction Mapping. *Australiasan Journal.*

[12] Laradji, A. and Umar, A. (2004). *Combinatorics Results for Semigroup of Order Preserving Full Transformation Semigroup Forum.* 72, 51-62.

[13] Laradji, A. and Umar, A. (2004). *Combinatorial Results for Semigroups of Order-Preserving Partial Transformation. Journal of Algebra,* 278,342-359.

[14] Laradji, A. and Umar, A. (2006). Combinatorial Results for Semigroups of Order Preserving Full Transformation Semigroup, *Journal of Algebral*, 278, 342-259.

[15] Sloane, N. J. A. The on-line Ecyclopedia of Integer Sequence, *http://oeis.org/*

[16] Umar, A. (1992). On the Semigroups of Order-decreasing Finite Full Transformations. Proceeding of the Royal Society of Edinburgh, 120A, 129-142.

*Table 4:1:1, Number of Elements in Terms of height F(n,p) in OCI_n

n/p	0	1	2	3	4	5	$\sum$F(n,p)
0	01						01
1	01	01					02
2	01	04	01				06
3	01	09	07	01			18
4	01	16	25	10	01		53
5	01	25	65	49	13	01	154

*Table 4:1:2 Number of Elements in Terms of fixed F(n,m) in OCI_n

n/m	0	1	2	3	4	5	$\sum$F(n,m)
0	01						01
1	01	01					02
2	03	02	01				06
3	09	05	03	01			18
4	26	14	08	04	01		53
5	75	39	22	12	05	01	154

*Table 4:1:3 Number of Elements in Terms of waist elements F(n,w^+) in OCI_n

n/k	0	1	2	3	4	5	$\sum$F(n,w^+)
0	01						01
1	01	01					02
2	01	02	03				06
3	01	03	06	08			18
4	01	04	10	17	21		53
5	01	05	15	31	47	55	154