Nigerian Journal of Mathematics and Applications
Volume **25**, (2016), 8 − 18. *Printed by Unilorin press*
©*Nig. J. Math. Appl.* *http://www.njmaman.com*

Some generalized k-Fractional companions of Hadamard's inequality

[1]Hussain R., [1]Kehkashan B., [1]Ali A., [1]Latif A. and [2]Rauf K.*

Abstract

The present study deals with the derivation of some new companions of Hadamard's inequality of fractional type by means of generalised convex functions, viz. h-convex functions, s-convex functions and also by co-ordinated h-convex functions on a rectangle from the plane $\mathbb{R}^2$.

1. Introduction

A convex function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, induces the following classical inequality

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)\mathrm{d}x \leq \frac{f(p)+f(q)}{2},$$

known in the literature as Hadamard's inequality, where $p, q \in I$ with $p < q$. Hadamard's inequality is considered to be the first fundamental result for convex functions with a natural geometrical interpretation and has wide range of applications. Over the years this inequality has been refined, extended, generalised, for different classes of functions, as can be seen in a large number of research papers and books devoted to the field. The aim of the present work is to give some generalisations and extensions of this inequality. In order to endow the results, we require the following preliminaries.

Received: 05/01/2016, Accepted: 19/04/2016, Revised: 19/05/2016.
2015 *Mathematics Subject Classification.* 47Jxx & 35A23. * Corresponding author.
Key words and phrases. Hadamard's inequality, h-Convex function, co-ordinated h-Convex function, s- convex function in the second sense and k-Riemann-Liouville fractional integrals.
[1]Department of Mathematics, Mirpur University of Science and Technology, Mirpur, AJK, Pakistan; E-mail: chairperson.maths@must.edu.pk, unigraz2009@yahoo.com
[2]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
E-mail: krauf@unilorin.edu.ng

In [16], Mubeen and Habibullah introduced the following generalization for Riemann-Liouville fractional integrals.

Definition 1.1. Let $f \in L_1[p,q]$, the k-Riemann-Liouville integrals ${}_kJ^{\gamma}_{p+}f$ and ${}_kJ^{\gamma}_{q-}f$ of order $\gamma > 0$ with $p \geq 0, k > 0$ be defined by:

$$(1) \qquad {}_kJ^{\gamma}_{p+}f(x) = \frac{1}{k\Gamma_k(\gamma)}\int_p^x (x-r)^{\frac{\gamma}{k}-1}f(r)\mathrm{d}r, \qquad x > p$$

and

$$(2) \qquad {}_kJ^{\gamma}_{q-}f(x) = \frac{1}{k\Gamma_k(\gamma)}\int_x^q (r-x)^{\frac{\gamma}{k}-1}f(r)\mathrm{d}r, \qquad x < q,$$

respectively, where $\Gamma_k(\gamma)$ is the k-gamma function given by $\Gamma_k(\gamma) = \int_0^\infty r^{\gamma-1}e^{\frac{-r^k}{k}}dr$.

Motivated by [16] and [23], we define double order k-Riemann-Liouville fractional integrals as follows:

Definition 1.2. Let $f \in L_1([p,q]\times[m,n])$, the k-Riemann-Liouville fractional integrals ${}_kJ^{\gamma,\delta}_{p^+,m^+}f$, ${}_kJ^{\gamma,\delta}_{p^+,n^-}f$, ${}_kJ^{\gamma,\delta}_{q^-,m^+}f$ and ${}_kJ^{\gamma,\delta}_{q^-,n^-}f$ of order $\gamma, \delta > 0$ with $p, m \geq 0, k > 0$ be defined by:

$${}_kJ^{\gamma,\delta}_{p^+,m^+}f(x,y) = \frac{1}{k\Gamma_k(\gamma)\ k\Gamma_k(\delta)}\int_p^x\int_m^y (x-r)^{\frac{\gamma}{k}-1}(y-s)^{\frac{\delta}{k}-1}\mathrm{d}s\mathrm{d}r, \quad x > p,\ y > m,$$

$${}_kJ^{\gamma,\delta}_{p^+,n^-}f(x,y) = \frac{1}{k\Gamma_k(\gamma)\ k\Gamma_k(\delta)}\int_p^x\int_y^n (x-r)^{\frac{\gamma}{k}-1}(s-y)^{\frac{\delta}{k}-1}\mathrm{d}s\mathrm{d}r, \quad x > p,\ y < n,$$

$${}_kJ^{\gamma,\delta}_{q^-,m^+}f(x,y) = \frac{1}{k\Gamma_k(\gamma)\ k\Gamma(\delta)}\int_x^q\int_m^y (r-x)^{\frac{\gamma}{k}-1}(y-s)^{\frac{\delta}{k}-1}\mathrm{d}s\mathrm{d}r, \quad x < q,\ y > m,$$

$${}_kJ^{\delta,\delta}_{q^-,n^-}f(x,y) = \frac{1}{k\Gamma_k(\gamma)\ k\Gamma_k(\delta)}\int_x^q\int_y^n (r-x)^{\frac{\gamma}{k}-1}(s-y)^{\frac{\delta}{k}-1}\mathrm{d}s\mathrm{d}r, \quad x < q,\ y < n,$$

where $\Gamma_k(\gamma)$ and $\Gamma_k(\delta)$ are the k-gamma functions given by $\int_0^\infty r^{\gamma-1}e^{\frac{-r^k}{k}}dr$ and $\int_0^\infty r^{\delta-1}e^{\frac{-r^k}{k}}dr$ respectively.

Note that when $k \to 1$, these k-fractional integrals turn out to be the usual Riemann-Liouville fractional integrals. For details on k-gamma function, k-beta function, k-zeta function, k-hypergeometric functions based on Pochhammer k-symbols, see [1], [2], [3], [10], [11], [13], [14]. We also recommend [5], [6], [15] and [20] for detailed study of fractional calculus.

In [21], Varosanec defined h-convex functions, interested reader can see the interaction between h-convex functions and other generalised convex functions in [7], [17], [19], [24] and also the references therein.

Definition 1.3. Let $h : (0,1) \subseteq \mathcal{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be h-convex or that f is said to belong to the class $SX(h, I)$, if f is non-negative and $\forall\ x, y \in I$ and $\gamma \in (0,1)$, we have

$$f(rx + (1-r)y) \leq h(r)f(x) + h(1-r)f(y). \tag{3}$$

In [12] Latif and Alomari described the following class of functions as co-ordinated h-convex functions, which allows a functions to be h-convex on each co-ordinate individually, rather than being simultaneously on the whole plane.

Definition 1.4. A function $f : \triangle \longrightarrow \mathbb{R}$ is said to be h-convex for the co-ordinates on $\triangle$ if the inequality

$$\begin{aligned} f(rx + (1-r)y, su + (1-s)v) &\leq h(r)h(s)f(x,u) + h(r)h(1-s)f(x,v) \\ &+h(1-r)h(s)f(y,u) + h(1-r)h(1-s)f(y,v) \end{aligned} \tag{4}$$

holds $\forall\ r, s \in [0,1]$ and $(x,u), (x,v), (y,u), (y,v) \in \triangle$, where $\triangle$ is the bidimentional interval given by $\triangle := [p,q] \times [m,n]$.

In [18] Orlicz introduced the following classes of s-convex functions. Interested readers are to see [4], [8] and [22], and the references therein.

Definition 1.5. A function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, where $\mathbb{R}^+ = [0,\infty)$, is said to be s-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{5}$$

$\forall\ x, y \in [0,\infty)$, $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ and for some fixed $s \in (0,1]$. We denote this class of functions by K_s^1.

Also, a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, where $\mathbb{R}^+ = [0,\infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{6}$$

$\forall\ x, y \in [0,\infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. We denote this class of functions by K_s^2.

In the next section, we shall discuss the main results of our work. Initially, we generalize Hadamard's inequality by means of right and left k-Riemann-Liouville fractional integrals for generalized convex functions viz. h and s-convex functions. Then, we give k-fractional extension of Hadamard's inequality for co-ordinated h-convex function on a rectangle from the plane. In the future, we intend to generalise these results to substantially generalised convex functions such as $(h-(\alpha,m))$ and (m,h_1,h_2)-convex functions.

2. Main Results

Theorem 2.1. *Let $f:[p,q]\to\mathbb{R}$ be a positive function with $0\le p<q$ and $f\in L_1[p,q]$. If f is a h-convex function on $[p,q]$, then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{p+q}{2}\right)\le h\Big(\frac{1}{2}\Big)\frac{\Gamma_k(\gamma+k)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)]$$

$$\le h\Big(\frac{1}{2}\Big)\Gamma_k(\gamma+k)[f(p)+f(q)][{}_kJ^{\gamma}_{1-}h(0)+{}_kJ^{\gamma}_{0+}h(1)]. \tag{7}$$

Proof. Since f is h-convex function then from (3), we have

$$f(rx+(1-r)y)\le h(r)f(x)+h(1-r)f(y).$$

Let $x=r_1p+(1-r_1)q$, $y=(1-r_1)p+r_1q$ and $r=\frac{1}{2}$, then

$$f\left(\frac{p+q}{2}\right)\le h\Big(\frac{1}{2}\Big)[f(r_1p+(1-r_1)q)+f((1-r_1)p+r_1q)],$$

multiplying both sides by $r_1^{\frac{\gamma}{k}-1}$ and then integrate with respect to r_1 on $[0,1]$, we get

$$\int_0^1 r_1^{\frac{\gamma}{k}-1}f\left(\frac{p+q}{2}\right)\mathrm{d}r_1\le h\Big(\frac{1}{2}\Big)\left[\int_0^1 r_1^{\frac{\gamma}{k}-1}f(r_1p+(1-r_1)q)\mathrm{d}r_1+\int_0^1 r_1^{\frac{\gamma}{k}-1}f((1-r_1)p+r_1q)\mathrm{d}r_1\right],$$

$$\frac{1}{\gamma h(\frac{1}{2})}f\left(\frac{p+q}{2}\right)\le\frac{\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)], \tag{8}$$

By replacing $x=p$ and $y=q$ in (3) yields:

$$f(rp+(1-r)q)\le h(r)f(p)+h(1-r)f(q),$$

$$f((1-r)p+rq)\le h(1-r)f(p)+h(r)f(q).$$

Adding the last two inequalities above, we get

$$f(rp+(1-r)q)+f((1-r)p+rq)\le[h(r)+h(1-r)][f(p)+f(q)],$$

multiplying both sides of the last inequality by $r^{\frac{\gamma}{k}-1}$ and integrate with respect to r over $[0,1]$, we obtain

$$\frac{k\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q-}f(p)]\le[f(p)+f(q)]\int_0^1 r^{\frac{\gamma}{k}-1}[h(r)+h(1-r)]\mathrm{d}r. \tag{9}$$

from (2) and (1), we have

$$\int_0^1 r^{\frac{\gamma}{k}-1}h(r)\mathrm{d}r=k\Gamma_k(\gamma){}_kJ^{\gamma}_{1-}h(0),$$

$$\int_0^1 r^{\frac{\gamma}{k}-1}h(1-r)\mathrm{d}r = k\Gamma_k(\gamma)_k J^{\gamma}_{0+}h(1).$$

Using above results in (9), we get

$$\frac{\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q^-}f(p)] \leq \Gamma_k(\gamma)[f(p)+f(q)][{}_kJ^{\gamma}_{1^-}h(0) + {}_kJ^{\gamma}_{0+}h(1)] \tag{10}$$

from (8) and (10), we have

$$\begin{aligned}\frac{1}{\gamma h(\frac{1}{2})} f\left(\frac{p+q}{2}\right) &\leq \frac{\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q^-}f(p)] \\ &\leq \Gamma_k(\gamma)[f(p)+f(q)][{}_kJ^{\gamma}_{1^-}h(0) + {}_kJ^{\gamma}_{0+}h(1)],\end{aligned}$$

multiplying the above inequality by $\gamma h(\frac{1}{2})$, we obtain the required result (7).

Theorem 2.2. *Let $f : [p,q] \to R$ be a positive function with $0 \leq p < q$ and $f \in L_1[p,q]$. If f is a s-convex function in the second sense on $[p,q]$, then the following inequalities for k-fractional integrals hold:*

$$\begin{aligned}f\left(\frac{p+q}{2}\right) &\leq \frac{\Gamma_k(\gamma+k)}{2^s(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q^-}f(p)] \leq \left(\frac{\frac{\gamma}{k}}{2^s}\right)[f(p)+f(q)] \\ &\left[\frac{1}{\frac{\gamma}{k}+s} + \frac{s(s-1)(s-2)...(s-n)}{\frac{\gamma}{k}(\frac{\gamma}{k}+1)(\frac{\gamma}{k}+2)...(\frac{\gamma}{k}+n)}\int_0^1 r^{\frac{\gamma}{k}+n}(1-r)^{s-(n+1)}\mathrm{d}r\right].\end{aligned} \tag{11}$$

Proof. Since f is s-convex function in the second sense then from (6), we have

$$f(rx+(1-r)y) \leq r^s f(x) + (1-r)^s f(y). \tag{12}$$

Let $x = r_1p + (1-r_1)q$, $y = (1-r_1)p + r_1q$ and $r = \frac{1}{2}$ in (6), we get

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{2^s}[f(r_1p+(1-r_1)q) + f((1-r_1)p + r_1q)],$$

multiplying both sides by $r_1^{\frac{\gamma}{k}-1}$ then integrate with respect to $r_1$over [0,1], we get

$$\frac{2^s}{\gamma} f\left(\frac{p+q}{2}\right) \leq \frac{\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q-}f(p)], \tag{13}$$

$$f\left(\frac{p+q}{2}\right) \leq \frac{\Gamma_k(\gamma+k)}{2^s(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q) + {}_kJ^{\gamma}_{q-}f(p)],$$

with this the first inequality in (11) is proved. To prove the second inequality put $x = p$, $y = q$ in (6), we have

$$f(rp+(1-r)q) \leq r^s f(p) + (1-r)^s f(q),$$

and

$$f((1-r)p + rq) \leq (1-r)^s f(p) + r^s f(q).$$

Adding the above inequalities, we get

$$f(rp+(1-r)q)+f((1-r)p+rq)\le [r^s+(1-r)^s][f(p)+f(q)],$$

multiplying the above inequality by $r^{\frac{\gamma}{k}-1}$, then integrating with respect to r over $[0,1]$, we get

$$\int_0^1 r^{\frac{\gamma}{k}-1}[f(rp+(1-r)q)+f((1-r)p+rq)]\mathrm{d}r \le \int_0^1 r^{\frac{\gamma}{k}-1}[r^s+(1-r)^s][f(p)+f(q)]\mathrm{d}r,$$

$$\frac{k\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q^-}f(p)]\le [f(p)+f(q)]\int_0^1 r^{\frac{\gamma}{k}-1}[r^s+(1-r)^s]\mathrm{d}r, \tag{14}$$

Considering

$$\int_0^1 r^{\frac{\gamma}{k}-1}[r^s+(1-r)^s]\mathrm{d}r=\int_0^1 r^{\frac{\gamma}{k}+s-1}\mathrm{d}r+\int_0^1 r^{\frac{\gamma}{k}-1}(1-r)^s\mathrm{d}r = \frac{1}{\frac{\gamma}{k}+s}+\int_0^1 r^{\frac{\gamma}{k}-1}(1-r)^s\mathrm{d}r,$$

considering the integral term on right hand side of above equation, taking $(1-r)^s$ as first function and applying n-times integration by parts, so that

$$\int_0^1 r^{\frac{\gamma}{k}-1}[r^s+(1-r)^s]\mathrm{d}r=\frac{1}{\frac{\gamma}{k}+s}+\frac{s(s-1)(s-2)...(s-n)}{\frac{\gamma}{k}(\frac{\gamma}{k}+1)(\frac{\gamma}{k}+2)...(\frac{\gamma}{k}+n)}\int_0^1 r^{\frac{\gamma}{k}+n}(1-r)^{s-(n+1)}\mathrm{d}r.$$

By putting this result into (14), we have

$$\frac{k\Gamma_k(\gamma)}{(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q^-}f(p)]\le [f(p)+f(q)] \left[\frac{1}{\frac{\gamma}{k}+s}+\frac{s(s-1)(s-2)...(s-n)}{\frac{\gamma}{k}(\frac{\gamma}{k}+1)(\frac{\gamma}{k}+2)...(\frac{\gamma}{k}+n)}\int_0^1 r^{\frac{\gamma}{k}+n}(1-r)^{s-(n+1)}\mathrm{d}r\right], \tag{15}$$

multiplying the above double inequality by $\left(\frac{\frac{\gamma}{k}}{2^s}\right)$, we get

$$\frac{\Gamma_k(\gamma+k)}{2^s(q-p)^{\frac{\gamma}{k}}}[{}_kJ^{\gamma}_{p+}f(q)+{}_kJ^{\gamma}_{q^-}f(p)]\le \left(\frac{\frac{\gamma}{k}}{2^s}\right)[f(p)+f(q)] \left[\frac{1}{\frac{\gamma}{k}+s}+\frac{s(s-1)(s-2)...(s-n)}{\frac{\gamma}{k}(\frac{\gamma}{k}+1)(\frac{\gamma}{k}+2)...(\frac{\gamma}{k}+n)}\int_0^1 r^{\frac{\gamma}{k}+n}(1-r)^{s-(n+1)}\mathrm{d}r\right]$$

Hence, the second inequality in (11) is also proved.

Theorem 2.3. *Let $f : \triangle \subseteq \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated h-convex on $\triangle := [p,q] \times [m,n]$ in $\mathbb{R}^2$ with $0 \le p < q$, $0 \le m < n$ and $f \in L_1(\triangle)$. Then, following inequalities hold:*

$$
\begin{aligned}
f\Big(\frac{p+q}{2}, \frac{m+n}{2}\Big) &\le \Big[h\Big(\frac{1}{2}\Big)\Big]^2 \frac{\Gamma_k(\gamma+k)\Gamma_k(\delta+k)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}} \\
&\Big[{}_kJ^{\gamma,\delta}_{p+,m+}f(q,n) + {}_kJ^{\gamma,\delta}_{p+,n-}f(q,m) + {}_kJ^{\gamma,\delta}_{q^-,m+}f(p,n) + {}_kJ^{\gamma,\delta}_{q^-,n^-}f(p,m)\Big] \\
&\le \left[h\left(\frac{1}{2}\right)\right]^2 \left[{}_kJ^{\gamma}_{1^-}h(0) + {}_kJ^{\gamma}_{0+}h(1)\right] \left[{}_kJ^{\delta}_{1^-}h(0) + {}_kJ^{\delta}_{0+}h(1)\right] \Gamma_k(\gamma+k)\Gamma_k(\delta+k) \\
&\quad [f(p,m) + f(p,n) + f(q,m) + f(q,n)]. \qquad (16)
\end{aligned}
$$

Proof. In (4), taking
$x = r_1p + (1-r_1)q$, $y = (1-r_1)p + r_1q$, $u = s_1m + (1-s_1)n$, $v = (1-s_1)m + s_1n$ and $r = s = \frac{1}{2}$, so that we have

$$
\begin{aligned}
f\left(\frac{p+q}{2}, \frac{m+n}{2}\right) \le \Big[h\Big(\frac{1}{2}\Big)\Big]^2 &[f(r_1p + (1-r_1)q, s_1m + (1-s_1)n) \\
&+ f(r_1p + (1-r_1)q, (1-s_1)m + s_1n) \\
&+ f((1-r_1)p + r_1q, s_1m + (1-s_1)n) \\
&+ f((1-r_1)p + r_1q, (1-s_1)m + s_1n)].
\end{aligned}
$$

Multiply both side by $r_1^{\frac{\gamma}{k}-1}s_1^{\frac{\delta}{k}-1}$ and then integrate with respect to (r_1, s_1) on $[0,1] \times [0,1]$, we get

$$
\begin{aligned}
\frac{k^2}{\gamma\delta} f\left(\frac{p+q}{2}, \frac{m+n}{2}\right) \le \Big[h\Big(\frac{1}{2}\Big)\Big]^2 &[\int_0^1 \int_0^1 r_1^{\frac{\gamma}{k}-1}s_1^{\frac{\delta}{k}-1}[f(r_1p + (1-r_1)q, s_1m + (1-s_1)n) \\
&+ f(r_1p + (1-r_1)q, (1-s_1)m + s_1n)]\mathrm{d}s_1\mathrm{d}r_1 \\
&+ \int_0^1 \int_0^1 r_1^{\frac{\gamma}{k}-1}s_1^{\frac{\delta}{k}-1}[f((1-r_1)p + r_1q, s_1m + (1-s_1)n) \\
&\quad f((1-r_1)p + r_1q, (1-s_1)m + s_1n)]\mathrm{d}s_1\mathrm{d}r_1],
\end{aligned}
$$

Consider,

$$
\int_0^1 \int_0^1 r_1^{\frac{\gamma}{k}-1}s_1^{\frac{\delta}{k}-1} f(r_1p + (1-r_1)q, s_1m + (1-s_1)n)\mathrm{d}s_1\mathrm{d}r_1.
$$

Let $\quad r_1p + (1-r_1)q = x$, $s_1m + (1-s_1)n = y$, then

$$\int_0^1\int_0^1 r_1^{\frac{\gamma}{k}-1}s_1^{\frac{\delta}{k}-1}f(r_1p+(1-r_1)q,s_1m+(1-s_1)n)\mathrm{d}s_1\mathrm{d}r_1$$
$$=\frac{1}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}\int_p^q\int_m^n(q-x)^{\frac{\gamma}{k}-1}(n-y)^{\frac{\delta}{k}-1}f(x,y)\mathrm{d}y\mathrm{d}x,$$

similarly, by considering the remaining integral in the equality above, solving them the same way and putting them back into the above inequality, we get

$$\frac{k^2}{\gamma\delta}f\left(\frac{p+q}{2},\frac{m+n}{2}\right)\leq\left[h\Big(\frac{1}{2}\Big)\right]^2\frac{1}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}$$
$$\Big[\int_p^q\int_m^n(q-x)^{\frac{\gamma}{k}-1}(n-y)^{\frac{\delta}{k}-1}f(x,y)\mathrm{d}y\mathrm{d}x+\int_p^q\int_m^n(q-x)^{\frac{\gamma}{k}-1}(y-m)^{\frac{\delta}{k}-1}f(x,y)\mathrm{d}y\mathrm{d}x$$
$$+\int_p^q\int_m^n(x-p)^{\frac{\gamma}{k}-1}(n-y)^{\frac{\delta}{k}-1}f(x,y)\mathrm{d}y\mathrm{d}x+\int_p^q\int_m^n(x-p)^{\frac{\gamma}{k}-1}(y-m)^{\frac{\delta}{k}-1}f(x,y)\mathrm{d}y\mathrm{d}x\Big],$$

$$\frac{1}{\gamma\delta h(\frac{1}{2})^2}f\left(\frac{p+q}{2},\frac{m+n}{2}\right)\leq\frac{\Gamma_k(\gamma)\Gamma_k(\delta)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}\Big[{}_kJ^{\gamma,\delta}_{q^+,m^+}f(p,n)+{}_kJ^{\gamma,\delta}_{p^+,n^-}f(p,m)$$
$$+{}_kJ^{\gamma,\delta}_{q^-,m^+}f(p,n)+{}_kJ^{\gamma,\delta}_{q^-,n^-}f(q,m)\Big],\tag{17}$$

now let $x=p$, $y=q$, $u=m$ and $w=n$ in (4), we have

$$f(rp+(1-r)q,sm+(1-s)n)\leq h(r)h(s)f(p,m)+h(r)h(1-s)f(p,n)$$
$$+h(s)h(1-r)f(q,m)+h(1-r)h(1-s)f(q,n),$$

$$f(rp+(1-r)q,(1-s)m+sn)\leq h(r)h(1-s)f(p,m)+h(r)h(s)f(p,n)$$
$$+h(1-s)h(1-r)f(q,m)+h(1-r)h(s)f(q,n),$$

$$f((1-r)p+rq,sm+(1-s)n)\leq h(1-r)h(s)f(p,m)+h(1-r)h(1-s)f(p,n)$$
$$+h(s)h(r)f(q,m)+h(r)h(1-s)f(q,n),$$

$$f((1-r)p+rq,(1-s)m+sn)\leq h(1-r)h(1-s)f(p,m)$$
$$+h(1-r)h(s)f(p,n)+h(1-s)h(r)f(q,m)+h(r)h(s)f(q,n).$$

Adding the above inequalities, we have

$$f(rp+(1-r)q,sm+(1-s)n)+f(rp+(1-r)q,(1-s)m+sn)$$
$$+f((1-r)p+rq,sm+(1-s)n)+f((1-r)p+rq,(1-s)m+sn)$$
$$\leq[h(r)+h(1-r)][h(s)+h(1-s)][f(p,m)+f(p,n)+f(q,m)+f(q,n)],$$

multiplying both side of the inequality by $t^{\frac{\gamma}{k}-1}s^{\frac{\delta}{k}-1}$ and integrating with respect to (r,s) on $[0,1]\times[0,1]$, we have

$$\int_0^1\int_0^1 r^{\frac{\gamma}{k}-1}s^{\frac{\delta}{k}-1}[f(rp+(1-r)q,sm+(1-s)n)$$
$$+f(rp+(1-r)q,(1-s)m+sn)+f((1-r)p+rq,sm+(1-s)n)$$
$$+f((1-r)p+rq,(1-s)m+sn)]\mathrm{d}s\mathrm{d}r$$
$$\leq\int_0^1\int_0^1 r^{\frac{\gamma}{k}-1}s^{\frac{\delta}{k}-1}[h(r)+h(1-r)][h(s)+h(1-s)]$$
$$[f(p,m)+f(p,n)+f(q,m)+f(q,n)]\mathrm{d}s\mathrm{d}r,$$

therefore,

$$\frac{k^2\Gamma_k(\gamma)\Gamma_k(\delta)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}\left[{}_kJ^{\gamma,\delta}_{p^+,m^+}f(q,n)+{}_kJ^{\gamma,\delta}_{p^+,n^-}f(q,m)+{}_kJ^{\gamma,\delta}_{q^-,m^+}f(p,n)+{}_kJ^{\gamma,\delta}_{q^-,n^-}f(p,m)\right]$$
$$\leq[f(p,m)+f(p,n)+f(q,m)+f(q,n)]$$
$$(18)\int_0^1 r^{\frac{\gamma}{k}-1}\left[\int_0^1 s^{\frac{\delta}{k}-1}[h(s)+h(1-s)]\mathrm{d}s][h(r)+h(1-r)\right]\mathrm{d}r.$$

By defination of k-Riemann-Liouville fractional integrals, we have

$$\int_0^1 s^{\frac{\delta}{k}-1}[h(s)+h(1-s)]\mathrm{d}s=k\Gamma_k(\delta)\left[{}_kJ^{\delta}_{1^-}h(0)+{}_kJ^{\delta}_{0^+}h(1)\right],$$

similarly,

$$\int_0^1 r^{\frac{\gamma}{k}-1}[h(r)+h(1-r)]\mathrm{d}r=k\Gamma_k(\gamma)\left[{}_kJ^{\gamma}_{1^-}h(0)+{}_kJ^{\gamma}_{0^+}h(1)\right],$$

then (18) becomes,

$$\frac{\Gamma_k(\gamma)\Gamma_k(\delta)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}\left[{}_kJ^{\gamma,\delta}_{p+,m+}f(q,n)+{}_kJ^{\gamma,\delta}_{p+,n^-}f(q,m)+{}_kJ^{\gamma,\delta}_{q^-,m+}f(p,n)+{}_kJ^{\gamma,\delta}_{q^-,n^-}f(p,m)\right]$$
$$\leq\Gamma_k(\gamma)\Gamma_k(\delta)\left[{}_kJ^{\gamma}_{1^-}h(0)+{}_kJ^{\gamma}_{0+}h(1)\right]\left[{}_kJ^{\delta}_{1^-}h(0)+{}_kJ^{\delta}_{0+}h(1)\right][f(p,m)+f(p,n)+f(q,m)+f(q,n)]$$

combining (17) with the above inequality, we get

$$\frac{1}{\gamma\delta h(\frac{1}{2})^2}f\left(\frac{p+q}{2},\frac{m+n}{2}\right)\leq\frac{\Gamma_k(\gamma)\Gamma_k(\delta)}{(q-p)^{\frac{\gamma}{k}}(n-m)^{\frac{\delta}{k}}}$$
$$\left[{}_kJ^{\gamma,\delta}_{p+,m+}f(q,n)+{}_kJ^{\gamma,\delta}_{p+,n^-}f(q,m)+{}_kJ^{\gamma,\delta}_{q^-,m+}f(p,n)+{}_kJ^{\gamma,\delta}_{q^-,n^-}f(p,m)\right]$$
$$\leq[f(p,m)+f(p,n)+f(q,m)+f(q,n)]\Gamma_k(\gamma)\Gamma_k(\delta)\left[{}_kJ^{\gamma}_{1^-}h(0)+{}_kJ^{\gamma}_{0+}h(1)\right]\left[{}_kJ^{\delta}_{1^-}h(0)+{}_kJ^{\delta}_{0+}h(1)\right],$$

we obtain the required result by multiplying the above double inequality by $\gamma\delta[h(\frac{1}{2})]^2$.

References

[1] Diaz R. and Pariguan E., *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat., 15 (2007), 179-192.

[2] Diaz R. and Teruel C., *q, k Generalized gamma and beta functions*, J. Nonlinear Math. Phys., 12 (2005), 118-134.

[3] Diaz R., Ortiz C. and Pariguan E., *On the k-gamma q-distribution*, Cent. Eur. J. Math., 8 (2010), 448-458.

[4] Dragomir S. S. and Fitzpatrick S., The Hadamard's inequality for s-convex function in the second sense, Demanstratio Math., 32(4), (1999), 687-696.

[5] George A. Anastassiou Fractional Differentiation Inequalities, Springer Springer Dordrecht Heidelberg London New York, 2009.

[6] Gorenflo R. and Mainardi F., Fractional Calculas: Integrals and Differential Equations of Fractional Order, Wien: Springer-Verlag, (1997), 223-276.

[7] Iqbal M., Bahtti M. I. and Muddassar M., *Hadamard-type inequalities for h-Convex functions*, Pakistan Journal of Science (ISSN 1016-2526), 63(3) September 2011 pp. 170-175.

[8] Iscan I., Generalization of different type integral inequalities for s-convex functions via fractional integrals, Applicable Analysis: An Int. J., 93 (9) (2014), 1846-1862.

[9] Kilbas A. A., Srivastava H .M., and Trujillo J. J. , *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. 204, Elsevier, Amsterdam, 2006.

[10] Kokologiannaki C. G., *Properties and inequalities of generalized k-gamma, beta and zeta functions*, Int. J. Contemp. Math. Sciences, 5 (2010), 653-660.

[11] Krasniqi V., *A limit for the k-gamma and k-beta function*, Int. Math. Forum, 5 (2010), 1613-1617.

[12] Latif M. A. and Alomari M. h-convex and co-ordinated h-convex function with some Hadamard-Type Inequalities., Int. J. of Math. Analysis, 3(33), (2009), 1645-1656.

[13] Mansour M., *Determining the k-generalized gamma function $\Gamma_k(x)$ by functional equations*, Int. J. Contemp. Math. Sciences, 4 (2009), 1037-1042.

[14] Merovci F., *Power product inequalities for the Γ_k function*, Int. J. of Math. Analysis, 4 (2010), 1007-1012.

[15] Miller S. and Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., 1993, 384 pages, ISBN 0471588849..

[16] Mubeen S. M. and Habibullah G. M., *k-fractional integrals and applications*, Int. J. Contemp. Math. Sci. **7**(2), 89-94, 2012.

[17] Muhammad Muddassar and Muhammad I. Bhatti, *Some Generalizations of Hermite-Hadamard Type Integral Inequalities and their Applications*, Punjab Univ. Jour. of Math. (ISSN 1016-2526) Vol. 46(1), (2014), pp. 9-18.

[18] Orlicz W., A note on modular spaces, I Bull. Acad. Polon. Sci. Math. Astr. Phys., 9(1961), 157-162.

[19] Ozdemir M. E., Havva Kavurmaci and Merve Avci, *Hermite-Hadamard Type Inequalities for $(h-(\alpha, m))$-convex Functions*, RGMIA Research Report Collection, 14 (2011)Article 31. [ONLINE: http://http://rgmia.org/papers/v14/v14a31.pdf]

[20] Podlubni I., Fractional differential equations, San Diego, CA: Academic Press; (1999).

[21] Sanja Varofsanec, On h-convexity, J. Math. Anal. Appl., 326 (2007) 303-311.

[22] Sarikaya M. Z. and Ertugral F., *Generalized Hermite-Hadamard type integral inequalities for s-convex functions via fractional integrals*, Annals. of the University of Craiova, Math. and Comp. Sci. Series 41(2), 2014, 226-233.

[23] Sarikaya M. Z., On The Hermite-Hadamard-type inequalities for Co-ordinated Convex Function via fractional Integrals, integral transforms and special functions, 25(2), (2013), 134-147.

[24] De-Ping Shi1, Bo-Yan Xi1, Feng Qi, *Hermite-Hadamard Type Inequalities for* (m, h_1, h_2)*-Convex Functions Via Riemann-Liouville Fractional Integrals,* Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 1, 23-28. (http://pubs.sciepub.com/tjant/2/1/6)