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CARDINALITY AND IDEMPOTENCY OF SIGNED ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

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Abstract

Let SO_n, SPO_n and SIO_n denotes the sets of signed full order - preserving, signed partial order - preserving and signed partial one - one order preserving transformation semigroups respectively. Let α be a transformation from $X_n \longrightarrow Z^*$. This paper investigate and established the order and idempotency of the set of signed order -preserving transformation semigroups.

1. Introduction

The corresponding object in semigroup theory is full transformation, T_X of all selfmaps of a set (Howie 2006). The full transformation semigroup T_X on a set X, the semigroup analogue of a symmetric group S_X, has been studied over the last fifty years, in both finite and the infinite cases (See, for example Howie(1996), Vorob'ev(1953)). Thus just as the study of syymetric groups, alternating groups and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of T_n, P_n and I_n (Bashar(2010), Fernades et al(2011), Laradji and Umar(2004), Howie(2002)).
A mapping in T_n is called order - preserving if for all i, j in $\{1, 2, 3, \ldots .n\}$, $i \leq y \Rightarrow xi \leq yx$. The semigroup of oder - preserving full transformation X_n will be denoted by O_n. Howie(1971) showed that the order and number of idempotent

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O_n are $|O_n| = \binom{2n-1}{n-1}$ and $|E(O_n)| = F_{2n}$ respectively. He also showed that the *Fibonacci Numbers* F_n defined by $F_1 = F_2 = 1$ and $F_0 = 0$ where $F_n = F_{n-1} + F_{n-2}(n \geq 3)$ and F_{2n} is the alternate Fibonacci numbers. Its integers sequence is $1, 1, 2, 3, 5, \ldots$ (A000045) Sloane(2010). Gomes and Howie (1992), Garba (1992d) study the order of elements of PO_n and IO_n and showed that $|PO_n| = \sum_{r=0}^{n} \binom{n}{r}\binom{n+r-1}{r}$ and $|IO_n| = \binom{2n}{n}$ respectively. An element $\beta \in S$ is an idempotent if if $\beta^2 = \beta$. The set of all idempotents elements in S is denoted by $E(S)$. Richard(2008), James and Kerber(1981) initiate the study of signed semigroup. Here we concerned solely with the case where $\alpha : X_n \longrightarrow Z^*$ where $X_n = \{1, 2, 3, \ldots .n\}$ and $Z^* = \{\pm 1 \pm 2, \pm 3, \ldots, \pm n\}$. The signed (partial) transformation semigroups, ST_n is the set of all mapping from set $X_n \longrightarrow Z^*$ such that $\alpha : dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subset Z^*$, the domain may be empty.The signed partial one - one transformation semigroup is defined in the form of signed partial transformation semigroup as $\alpha : dom(\alpha) \subseteq X_n \longrightarrow Z^*$ but strictly one - one . We shall call a signed partial transformation $\alpha : X_n \longrightarrow Z^*$ signed order-preserving SO_n, if $i \leq j \Rightarrow |i\alpha| \leq |j\alpha|$ for i, j in $dom(\alpha)$.

2. Materials and Methods

2. Some Notation and Fact

Richard(2008) used matrix to lists the elements of signed semigroup. For example, let $\alpha \in SO_2$, this element can be represented by $\alpha(j) = \pm i$ by placing ± 1 in (i, j) entry of $n \times n$ matrix as in the set of elements of SO_2 :

$$|SO_2| = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \right\}$$

Proposition 2.1
Let S be a signed (partial) transformation. Composition of mappings is associative if α, β and δ are partial mappings such that the composition $\delta(\beta\alpha)$ is defined if and only if the composition $(\delta\beta)\ \alpha$ is defined and both are equal.

Proposition 2.2
Both ST_n and SPT_n are semigroup w.r.t. the composition of (partial) transformation.

3.0 RESULTS

3.1 Element in SO_n :
Table 1: Value of Elements in SO_n for each n

$n/\lvert lm(\alpha) = h\rvert$	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$\lvert SO_n\rvert = 2^n\binom{2n-1}{n-1}$
1	2	-	-	-	-	2
2	8	4	-	-	-	12
3	24	48	8	-	-	80
4	64	288	192	16	-	560
5	160	1280	1920	640	32	4032

Theorem 3.1 : Let $S = SO_n$. Then for $n \geq 1$, $|S| = 2^n\binom{2n-1}{n-1}$
***Proof*:** Let $\alpha \in S$ and $Z^* = \{\pm 1 \pm 2, \pm 3, \ldots, \pm n\}$ the base set such that $dom(\alpha) \subseteq X_n \rightarrow Im(\alpha) \subseteq Z^*$. For each n and $\alpha \in SO_n$ if the $|Im(\alpha)| = i$ and $i = 1, 2, 3, \ldots, n$ for $i = 1, 2, 3, \ldots, n$ then $|\alpha S| = 2^n$ and is equivalent to $\sum_{k=0}^{n}\binom{n}{k}$. Since the elements of $dom(\alpha)$ can be chosen from X_n and the remaining $n-1$ elements of $Im(\alpha)$ can be chosen from Z^* and the choices of n is independently from Z^* , it follows by applying the product rule.

Table 2 : Value of Elements in SPO_n for each n

$n/\lvert lm(\alpha)\rvert = h$	h = 0	h = 1	h = 2	h = 3	h = 4	h = 5	$\lvert SPO_n\rvert = \sum_{k=0}^{n}\binom{n}{k}^3 2^k$
1	1	2	-	-	-	-	3
2	1	16	4	-	-	-	21
3	1	78	84	8	-	-	171
4	1	320	832	352	16	-	1521
5	1	1320	6140	55200	1280	32	14283

$|lm(\alpha)| = h =$ number of the of elements in the image of α

Theorem 3.2 : Let $S = SPO_n$, then $|S| = \sum_{k=0}^{n}\binom{n}{k}^3 2^k$
Proof : Let $\alpha \in S$. The semigroup has an empty map since its a partial transformation. Since α is a bijection, k elements of domain can be chosen from X_n in $\binom{n}{k}$ ways. Let $Im(\alpha) \subseteq Z^*$, if $|Im(\alpha)| = 0$, then $|\alpha S| = 1$ and if $|Im(\alpha)| = i$, $i = 1, 2, 3, \ldots$ then $|\alpha S| = 2^n$ for each n and 2^k where $k = 1, 2, 3 \ldots, n$. Then sum the product rule hence the proof. (Note that the number of elements in the image is equal to k i,e $Im(\alpha) = k$

Theorem 3.3 : Let $S = SIO_n$, then $|S| = \sum_{r=0}^{n}\binom{n}{k}\binom{n+k}{k}$
Proof : First observed that $\alpha \in S$. If the $|Im(\alpha)| = 0$, then $|\alpha S| = 1$ for each n. Let $Im(\alpha) =\{\}$ denotes an empty map, and we observed that $|\alpha S| =$

2^n for $|Im(\alpha)| = i$ for $i = 1, 2, 3, \ldots, n$ and $k = 1, 2, 3 \ldots, n$ which is equivalent to $\sum_{k=0}^{n} \binom{n}{k}$ since k elements of the $domain(\alpha)$ can be chosen from Z^* in $\binom{n}{r}$ ways and since the empty map is an element of SIO_n and $n \geq k$ where k is the maximum elements in the image, then k elements of $Im(\alpha)$ can be chosen from Z^* in $\binom{n+k}{k}$ ways.

3.2 Idempotents in SO_n, SPO_n, SIO_n :

Table 3 : Values of Idempotents Elements in Signed Order - preserving transformation semigroup

n/S	$\lvert E(SO_n)\rvert$	$\lvert E(SPO_n)\rvert$	$\lvert E(SIO_n)\rvert$
1	1	2	2
2	4	8	4
3	22	18	8
4	140	200	16
5	969	1010	32

Proposition 3.2.1 : Let $S = SO_n$, and $\alpha \in S$. Then, $|E(S)| = \frac{1}{3n+1}\binom{4n}{n}$

Proof : Let $X_n = \{1, 2, 3, \ldots, n\}$, $X_n \longrightarrow Z^*$, and the $lm(\alpha) = \{i, -i\}$ $i = 1, 2, 3, \ldots, n$ and let α be transformation in S and observed that n elements of $dom(\alpha)$ can be chosen from X_n in $\binom{4n}{n}$ ways and if the $lm(\alpha)$ can be chosen in Z^* in one -one fashions, then the number of idempotent elements in $\alpha \in S$ is $\frac{1}{3n+1}\binom{4n}{n}$

Theorem 3.2.2 : Let $S = SIO_n$. Then $|E(S)| = 2^n$

Proof : Let $\alpha \in S$. Idempotents are special case of binomial theorem which say $\sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x + y)^n = 2^n$ if $x = y = 1$

i.e $\sum_{r=0}^{n} \binom{n}{r} = 2^n \Rightarrow E(IO_n) = 2^n$

4.0 : Summary of the Results

Signed semigroup is a new class of transformation semigroup and much research work has not been done. The following results were obtained of set $X_n \longrightarrow Z^*$;

$|SO_n| = 2^n \binom{2n-1}{n-1}$

$|PSO_n| = \sum_{k=0}^{n} \binom{n}{k}^3 2^k$

$|IO_n| = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$

$|E(SO_n)| = \frac{1}{3n+1}\binom{4n}{n}$

$|E(SIO_n)| = 2^n$

$|E(PSIO_n)| = ?$

4.1 Conclusion

The formula for idempotent of signed partial order - preserving transformation semigroup has not been known.

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