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Analysis of a Steady Magnetohydrodynamic (MHD) Flow of a Convective Third Grade Fluid in a Constrained Geometry

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Abstract

This paper presents magnetohydrodynamic (MHD) flow of steady convective third grade fluid in a constrained geometry. The non-linear governing equations were solved analytically using homotopy perturbation method (HPM). The influences of dimensionless parameters on magnetohydrodynamic (MHD) flow of a convective third grade fluid in a constrained geometry were investigated. Simulation results revealed that temperature and the velocity depend on the values of combination of magnetic field and porosity. The obtained results are presented graphically and discussed. It is observed that velocity decreases and increases with increasing magnetic field and porosity, temperature increases as magnetic field increases.

1. Introduction

In recent years, investigation and problem dealing with the flow of non-Newtonian fluid in a cylindrical system were carried out by researchers. This interest is due

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to several important applications in engineering and industry such as reactive polymer flows in heterogeneous porous media, extraction of crude oil from the petroleum production, synthetic fibres and paper production (Schowaller 1978). Many practical situations we have deals with the natural convection of heat which play significant role in the behaviour of the flow.

Convection problems associate will heat sources within fluid saturated porous media are of great practical signification, such as in geophysics and energy related problems (petroleum resources, geophysical flows, cooling of underground electric cable e.t.c).In generally terms, the difference between Non-Newtonian fluids and the single component Newtonian fluid in that, in the latter case the mathematical formulation is known but the macroscopic physical processes are complex and often not well understood, especially for turbulent flow conditions are for Non - Newtonian fluids even the appropriation governing equations and conditions at the boundaries are still not well understood. However, the flow of Non - Newtonian fluids plays an important role in many practical applications.

Sajid et al (2008) discussed the effect of variable viscosity on the flow and heat transfer in a thin film flow for a third grade fluid. The thin film was considered on the outer side of an infinitely long vertical cylinder. The governing non-linear differential equation of momentum and energy were solved analytically by using homotopy analysis method (HAM). The study of MHD convection in a vertical channel, Aiyesimi et al. (2013) considered the MHD heat transfer in the ?ow between 2 concentric cylinders.

Siddiqui et al (2012) carried out studies on two phase flow of a third grade fluid between parallel plates in three different cases. Makinde et al. (2010) studied MHD mixed convection from a vertical plate embedded in a porous medium with a convective boundary condition. Chamkha and Ahmad (2012) investigated unsteady MHD heat and mass transfer by mixed convection flow in the forward stagnation region of a rotating sphere at different wall conditions. Unsteady MHD convective heat and mass transfer in a boundary layer slip flow part a vertical permeable with thermal radiation and chemical reaction was examined by Dulal and Talukdar (2000). Singh and Pathak (2010) studied effect of slip condition on rotating vertical channel. Kandasamy et al. (2011) group theory transformation for Soret and Dufour effects on free convective heat and mass transfer with thermophoresis and chemical reaction over a porous stretching surface in the presence of heat source/sink. Rao *et al.* (2012) have found out the chemical effects on an unsteady MHD free convection fluid past a semi-infinite vertical plate embedded in a porous medium with heat absorption.

In this paper, magnetohydrodynamics (MHD) flows of a steady convective third grade fluid in a constrained geometry were investigated. The governing equations arising from the steady flow are solved using homotopy perturbation method (HPM).

2. Model Formulation

The one-dimensional momentum and energy equations describing MHD flows of a steady convective third grade fluid in a constrained geometry are:
Momentum equation

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)= \tag{1}$$

$$\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-p+2\alpha_1\left(\frac{\partial u}{\partial r}\right)\right)+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+6\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)+2\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)\right)$$

$$+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial r}+\alpha_1\frac{\partial^2 u}{\partial t\partial r}+\beta_1\frac{\partial^3 u}{\partial t^2\partial r}+2\beta_2\left(\frac{\partial u}{\partial r}\right)^3+2\beta_3\left(\frac{\partial u}{\partial r}\right)^3\right)-\sigma\beta_0^2u-\mu\frac{\emptyset}{k}u$$

Energy equation

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}\right)= \tag{2}$$

$$k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+2\mu\left(-p+2\alpha_1\left(\frac{\partial u}{\partial r}\right)^2+\alpha_2\left(\frac{\partial u}{\partial r}\right)^2+6\beta_1\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial t\partial r}\right)+2\beta_2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial t\partial r}\right)^2$$

$$+\sigma\beta_0^2\ u^2$$

together with initial and boundary conditions

$$u(r,\ 0)=U\left(1-\frac{r}{R}\right),\quad u(0,\ t)=U,\quad \left.\frac{\partial u}{\partial r}\right|_{r=R}=0$$

$$T(r,\ 0)=(T_1-T_0)\left(1-\frac{r}{R}\right)+T_0,\quad T(0,\ t)=T_0,\quad T(R,\ t)=T_1 \tag{3}$$

where u, p, k, T, ρ, c_p, μ, B_0, t and σ are velocity, pressure, thermal conductivity, temperature, fluid, specific heat, fluid viscosity, applied magnetic field, non-dimensional time and electrical conductivity respectively.
Here, equations (1) – (3) are transformed using the following coordinate transformations:

$$\frac{\partial}{\partial r}\longrightarrow\frac{\partial}{\partial\eta}\bullet\frac{\partial\eta}{\partial r}=\rho\frac{\partial}{\partial\eta} \tag{4}$$

$$\frac{\partial}{\partial t}\longrightarrow\frac{\partial}{\partial\eta}\bullet\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial\eta}=-\rho u\frac{\partial}{\partial\eta}+\frac{\partial}{\partial t} \tag{5}$$

$$r=\frac{\eta}{\rho} \tag{6}$$

and we obtain

(7)

$$\rho\frac{\partial u}{\partial t}=\rho\frac{\partial p}{\partial ?\bar{a}}+\frac{\rho}{\eta}\left(-p+2\alpha_1\rho^2\frac{\partial^2 u}{\partial\eta^2}+2\alpha_2\rho^3\frac{\partial^2 u}{\partial\eta^2}\frac{\partial u}{\partial\eta}+(6\beta_1+2\beta_2)\rho^3\left(\frac{\partial^2 u}{\partial t\partial\eta}\bullet\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial u}{\partial\eta}\bullet\frac{\partial^3 u}{\partial t\partial\eta^2}\right)\right)+\frac{\rho}{\eta}$$
$$\left(\mu\rho^2\frac{\partial^2 u}{\partial\eta^2}+\alpha_1\rho^2\frac{\partial^3 u}{\partial t\partial\eta^2}+\beta_1\rho^2\frac{\partial^4 u}{\partial t^2\partial\eta^2}+6(\beta_2+\beta_3)\rho^2\frac{\partial^2 u}{\partial\eta^2}\bullet\left(\rho\frac{\partial u}{\partial\eta}\right)^2\right)-\sigma B_0^2 u-\mu\frac{\emptyset}{k}u$$

(8)

$$\rho c_p\frac{\partial T}{\partial t}=k\rho^2\frac{\partial^2 T}{\partial\eta^2}+k\frac{\rho^2}{\eta}\frac{\partial T}{\partial\eta}+2\mu\left(-p+(2\alpha_1+\alpha_2)\rho^2\left(\frac{\partial u}{\partial\eta}\right)^2+(6\beta_1+2\beta_2)\rho^2\frac{\partial u}{\partial\eta}\frac{\partial^3 u}{\partial t\partial\eta^2}\right)^2+\sigma B_0^2u^2$$

together with initial and boundary conditions;

$$u(\eta,0)=U\left(1-\frac{\eta}{R\rho}\right),\quad u(0,t)=U,\quad \left.\frac{\partial u}{\partial\eta}\right|_{\eta=R}=0$$

(9) $$T(\eta,0)=(T_1-T_0)\left(1-\frac{\eta}{R\rho}\right)+T_0,\quad T(0,t)=T_0,\quad T(R,t)=T_1$$

3. Method of solution

3.1 Non-dimensionalization

Here, we let pressure p be constant and non-dimensionlised equations (7) – (9) using the following dimensionless variables,

$$\eta'=\frac{\eta}{R},\qquad u'=\frac{u}{U},\qquad t'=\frac{tU}{R},\qquad \theta=\frac{T-T_0}{T_1-T_0}\quad(10)$$

and obtain

(11) $$\left(\frac{\beta+\sigma_3}{\eta}\right)\frac{d^2u}{d\eta^2}+\frac{\sigma_1}{\eta}\frac{d^2u}{d\eta^2}\left(\frac{du}{d\eta}\right)+\frac{\sigma_6}{\eta}\frac{d^2u}{d\eta^2}\left(\frac{du}{d\eta}\right)^2-\sigma_7u+\frac{\alpha}{\eta}=0$$

(12) $$\frac{d^2\theta}{d\eta^2}+\frac{Re}{Pr}\frac{1}{\eta}\frac{d\theta}{d\eta}+\frac{1}{Pr}\left(-2Br+2Gr\gamma\left(\frac{du}{d\eta}\right)^2\right)^2+\frac{Mu^2}{Pr}=0$$

together with initial and boundary conditions:

$$u(0,t)=1,\qquad \left.\frac{\partial u}{\partial\eta}\right|_{\eta=R}=0$$
$$\theta(0,\ t)=0,\qquad \theta(1,\ t)=1$$

where

$$\alpha=\frac{-p}{U^2},\qquad \beta=\frac{2\alpha_1\ \rho}{U},\qquad \sigma_1=\frac{2\alpha_2\rho^2}{R},\qquad \sigma_2=(6\beta_1+2\beta_2)\frac{\rho^3U}{R^2}$$
$$\sigma_3=\frac{\rho^2\mu}{UR},\qquad \sigma_4=\frac{\alpha_1\rho^2}{R},\qquad \sigma_5=\frac{\beta_1\rho^2}{R},\qquad \sigma_6=6(\beta_2+\beta_3)\frac{\rho^4}{R^2}U$$

$$\sigma_7 = \frac{R}{\rho U}\left(\sigma B_0^2 - \mu\frac{\emptyset}{k}\right) \ , \ Ec = \frac{\mu\rho U^2}{R^2 c_p\left(T_1 - T_0\right)} \ , \ Br = \frac{\mu p U}{U\rho c_p\left(T_1 - T_0\right)} \ , \ Re = \frac{K\rho}{UR},$$

$$Gr = \frac{\mu\rho U}{Rc_p\left(T_1 - T_0\right)} \ , \quad Pr = \frac{K\rho}{Uc_p} \ , \quad \gamma = \left(2\alpha_1 + \alpha_2\right) \ , \quad M = \frac{\sigma U B_0^2}{\rho c_p\left(T_1 - T_0\right)}$$

3.2 Existence and uniqueness of solution

Theorem 3.1: Let $Pr = \beta + \sigma_3$, $Re = Br = Gr = Ec = 0$, $\alpha = \sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$. Then, the equations (11) and (12) with initial and boundary conditions (13) has a unique solution for all $t = 0$.

Proof; Let $Pr = \beta + \sigma_3$, $Re = Br = Gr = Ec = 0$, $\alpha = \sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$.in equations (11) and (12), we obtain

(13)
$$\frac{\partial u}{\partial t} = \frac{Pr}{\eta}\frac{\partial^2 u}{\partial \eta^2} - \sigma_7 u$$

$$u\left(\eta, \ 0\right) = \left(1 - \frac{\eta}{\rho}\right) \ , \quad u\left(0 \ ,t\right) = 1, \quad u_\eta\left(1, \ t\right) = 0$$

(14)
$$\frac{\partial \theta}{\partial t} = Pr\frac{\partial^2 \theta}{\partial \eta^2} + Mu^2$$

$$(\ ,0) = \left(1 - \ \right), \quad (0,\mathrm{t}) = 0, \quad (1,\mathrm{t}) = 1$$

Using frobenius method, we obtain the solution of problem (14) as

(15)
$$u\left(\eta, \ t\right) = \left(\left(1 + \eta + \frac{\sigma_7}{6}\eta^3 + \frac{\sigma_7}{12}\eta^4 + \cdots\right) + \left(\frac{1 + \frac{\sigma_7}{2}\eta^2 + \frac{\sigma_7}{3}\eta^3 + \cdots}{1 + \frac{\sigma_7}{3}\eta^3 + \cdots}\right)\left(\eta + \frac{\sigma_7}{12}\eta^4 + \cdots\right)\right)$$

where

$$u_n\left(t\right) = \frac{4}{\left(2n-1\right)\pi}\left(1 + \frac{\left(-1\right)^n}{\left(2n-1\right)\pi}\right)e^{-\left(\sigma_7 + Pr\left(\frac{2n-1}{2}\pi\right)^2\right)t}$$

and using eigenfunction expansion method, the solution of problem as

(16)
$$\theta\left(\eta, \ t\right) = \eta + \sum_{n=1}^{8} V_n\left(t\right)\sin n\pi\eta \ ,$$

where

$$V_n\left(t\right) = q_n\left(t\right) + b_n e^{-Pr\left(n^2\pi^2\right)t}$$

$$q_n\left(t\right) = 2M\int_0^t T_n\left(\tau\right)e^{-Prn^2\pi^2\left(t-\tau\right)}$$

$$T_n\left(\tau\right) = \int_0^1\left(\sum_{n=1}^{8} u_n\left(t\right)\sin\left(\frac{2n-1}{2}\right)\eta\pi\right)^2 \sin n\pi\eta d\eta$$

Hence, there exists a unique solution of problems (11) and (12). This completes the proof.

3.3 Analytical solution

Applying homotopy perturbation method (HPM), we have

$$(1-p)\frac{d^2u}{d\eta^2}+p\left(\frac{d^2u}{d\eta^2}+\frac{d^2u}{d\eta^2}+\frac{\sigma_1}{(\beta+\sigma_3)}\frac{d^2u}{d\eta^2}\left(\frac{du}{d\eta}\right)+\frac{\sigma_6}{(\beta+\sigma_3)}\frac{d^2u}{d\eta^2}\left(\frac{du}{d\eta}\right)^2\right)$$

$$-p\left(\frac{\sigma_7\eta}{(\beta+\sigma_3)}u+\frac{\alpha}{(\beta+\sigma_3)}\right)=0$$

and

$$(1-p)\frac{d^2\theta}{d\eta^2}+p\left(\frac{d^2\theta}{d\eta^2}+\frac{Re}{Pr}\frac{1}{\eta}\frac{d\theta}{d\eta}+\frac{1}{Pr}\left(-2Br+2Gr\gamma\left(\frac{du}{d\eta}\right)^2\right)^2+\frac{Mu^2}{Pr}\right)=0$$

with initial and boundary condition

$$u(0,t)=1,\qquad \left.\frac{\partial u}{\partial\eta}\right|_{\eta=R}=0$$

$$\theta(0,\ t)=0,\qquad \theta(1,\ t)=1$$

Therefore, we obtain the following at zero order:
Momentum equation

$$u_0(\eta\ ,t)=a\eta+1$$

Energy equation

$$\theta_0(\eta\ ,\ t)=?\eta$$

at first-order Problem, we have
Momentum equation

$$\frac{d^2u_1}{d\eta^2}+\frac{\sigma_1}{(\beta+\sigma_3)}\frac{d^2u_0}{d\eta^2}\left(\frac{du_0}{d\eta}\right)+\frac{\sigma_6}{(\beta+\sigma_3)}\frac{d^2u_0}{d\eta^2}\left(\frac{du_0}{d\eta}\right)^2-\frac{\sigma_7u_0\eta}{(\beta+\sigma_3)}+\frac{\alpha}{(\beta+\sigma_3)}=0$$

$$u_1(\eta\ ,t)=\frac{1}{(\beta+\sigma_3)}\left(\sigma_7\left(\frac{a\eta^4}{12}+\frac{\eta^3}{6}\right)-\frac{a\eta^2}{2}\right)+\eta+1$$

Energy equation:

$$\theta_1(\eta,\ t)=-\frac{ZRe}{Pr}(\eta\ln\eta-\eta\ )-\frac{1}{Pr}\left(-2Br+2Gr\gamma(a)^2\right)^2\eta^2-\frac{M}{Pr}\left(\frac{a^2\eta^4}{12}+\frac{a\eta^3}{3}+\frac{\eta^2}{2}\right)+\eta$$

Let

$$u=u_0+pu_1+\cdots\cdots$$

$$\theta=\theta_0+p\theta_1+\cdots\cdots$$

Collecting like powers of p, we have

$$u(\eta\ ,t)=(a\eta+1)+p\left(\frac{1}{(\beta+\sigma_3)}\left(\sigma_7\left(\frac{a\eta^4}{12}+\frac{\eta^3}{6}\right)-\frac{a\eta^2}{2}\right)+\eta+1\right)+\cdots$$

and

$$\theta\left(\eta\ ,t\right)=$$

$$Z\eta+p\left(-\frac{ZRe}{Pr}\left(\eta\ln\eta-\eta\ \right)-\frac{1}{Pr}\left(-2Br+2Gr\gamma(a)^{2}\right)^{2}\eta^{2}-\frac{M}{Pr}\left(\frac{a^{2}\eta^{4}}{12}+\frac{a\eta^{3}}{3}+\frac{\eta^{2}}{2}\right)+\eta\right)+\cdots$$

The computations were done using algebraic package MAPLE.

4. Results and Discussion

The analytical results obtained through several graphs which demonstrate the various parameter on the velocity and temperature of magnetohyrodynamic (MHD) flow of a convective third grade fluid in a constrained geometry using homotopy perturbation method (HPM). We prove the existence and unigueness of solution by the actural method.

Fig. 1 **Fig. 2**

Graph of velocity u(η) against the position η for $\sigma_3 = 0.5, \quad \alpha = 0.1, \quad a = 1, \quad \beta = 6, \quad t = 5$, while we vary the combination of magnetic field and porosity (σ_7).

Fig.3 **Fig.4**

Graph of temperature against the position for Z = 3, $Pr = 5, \quad Br = 1, \quad Gr = 1, \quad \gamma = 1, \quad Re = 1, \quad t = 5$, while we vary the magnetic field M.

Discussion

Fig. 1 depicts the effect of σ_7 (combination of magnetic field and porosity) on velocity (u) for $\sigma_3 = 0.5, \quad \alpha = 0.1, \quad a = 1, \quad \beta = 6, \quad t = 5$ velocity (u) decreases with magnetic field and porosity increases. It is observed that, velocity (u) and magnetic field and porosity both increases for $\sigma_3 = 0.5, \quad \alpha = 0.1, \quad a = 1, \quad \beta = 6, \quad \eta = 0.5$ in fig. 2.

The effect of magnetic field (M) on temperature (T) for Z = 3, $Pr = 5, \quad Br = 1, \quad Gr = 1, \quad \gamma = 1, \quad Re = 1, \quad t = 5$ is found that, in fig. 3. Temperature (T) increases then decreases with magnetic field increases. In Fig. 4 both increases for Z = 3, $Pr = 5, \quad Br = 1, \quad Gr = 1, \quad \gamma = 1, \quad Re = 1, \quad \eta = 0.5$

5. Conclusion

The MHD flows of steady convective third grade fluid in cylindrical systems were examined. The resulting equations were solved using HPM and graphical results were obtained. The steady flow study were analysed through the effects of physical parameters such as Magnetic field on velocity and temperature distribution.

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