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On the Stability Analysis and Homotopy-Based Solution of three species Lotka-Volterra Model

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Abstract

The Lotka-Volterra model is a description of the dynamic behaviour of a multispecies number of competitors and/or predator-prey interactions which have found usefulness in non-linear control and simultaneous chemical engineering and other engineering fields. In this paper, we present the stability analysis and homotopy based solution of three species Lotka-Volterra model. The three species Lotka-Volterra model equations have been considered and analysed. We performed the stability analysis of the equilibrium points and gave some qualitative information about the solution of the model equations through phase plane analysis. The Homotopy Analysis Method (HAM) has been used to solve the model and the results have been compared with other numerical solution and are found to be in good agreement. Finally, various simulations are done to discuss the solution.

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1. Introduction

The use and applications of mathematical models in various fields cannot be overemphasised. It has also found usefulness in population ecology in order to understand the interaction between species and their environment. Issues of the dynamic process such as predator-prey, competition and symbiotic interactions, renewable resource management, ecological control of the pest, fish harvesting and others can be well understood by exploring mathematical models. There are interactions between species living in an ecosystem and their interaction is of three types, namely, predator-prey interaction, competition interaction and mutualism or symbiotic interaction.In 1926, Volterra [18] proposed a mathematical model which tried to explain then oscillatory levels of some fish species in the Adriatic. In [12], he assumed that q species live in an ecosystem with density P_i for species $i = 1, 2, 3...n$ then the ecological equations is given by equation (1) below. The accurate solution of the Lotka-Volterra equations may become a difficult task when the equations are stiff (even with a small number of species), or when the number of species is large [17]. Variational iteration method(VIM) and Adomian decomposition method(ADM) have successfully been applied to solve Lotka-Volterra model in [4]. In [4], the author compared the VIM solution with ADM and fourth order Runge-Kutta methods. The HAM results in this paper agree well with the solutions given by [4]. HAM was first proposed in 1992 by Liao Shijun [13] in his PhD dissertation and further modified in 1997 to introduce a non-zero auxiliary parameter, referred to as the convergence control parameter to construct a homotopy on a differential system. Homotopy Analysis Method employs the concept of the homotopy from topology to generate a convergent series solution for nonlinear systems which is enabled by utilising a homotopy-McLaurin series to deal with the nonlinearity in the system. The strength of the HAM, to naturally exhibit convergence of the series solution, is strange in most analytic and semi-analytic approaches to nonlinear PDEs [19].[2] solved a problem of stagnation point flow of a viscoelastic fluid towards a stretching surface analytically using the Homotopy Analysis Method (HAM). They obtained the results for velocity and temperature profiles and found out that the behaviour of the HAM solution for the velocity and temperature profiles is in good agreement with the numerical solution given in their stated reference. In [3], the time evolution of the multispecies Lotka-Volterra system is investigated by the HAM. They found that the HAM continuous solution generated by the polynomial base functions is of comparable accuracy to the purely numerical fouth-order Runge-Kutta method. [5], presented a mathematical model for the control of the spread of an infectious disease in a predator-prey ecosystem. Analysis of the model were discussed and they found that to eradicate the intensity of disease spread in the prey-predator ecosystem vaccination strategy with herd immunity. In [7], the problem of the spread of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic with ordinary differential equation is considered. Adomian Decomposition Method(ADM) is employed to solve the system and its convergence was discussed. In [8], systems of linear and nonlinear integral equations were solved using the HAM. In their work, they presented conditions to ensure the convergence of this series and the estimation of the error of approximate solution which is obtained when the partial sum of the series is used. [9] considered the problem of the boundary layer flow of an incompressible viscous fluid over a non-linear stretching sheet. The Homotopy Analysis Method (HAM) was applied in order to obtain analytical solution analytical solution of the governing nonlinear differential equations. Their result were finally compared using the illustrative graphs with the exact solution and an approximate method. [10] considered the problem of magnetohydrodynamic(MHD) boundary layer flow of an upper-convected Maxwell(UCM) fluid for the analytical solution using HAM. They transformed the non-linear partial differential equations to an ordinary differential equation by taking boundary layer approximations into account and then using the similarity transformations. They presented the analytical solution in the form of an infinite series and the recurrence formulated for finding the coefficients were presented and the convergence is established. The effects of the Deborah number and MHD parameter is discussed on the velocity profiles and the skin friction coefficients. It is found that the results are in excellent agreement with the existing results in the literature for the case of hydrodynamic flow. In [16], the authors, applied homotopy analysis method(HAM) to solve the Lotka-Volterra problem and obtained approximate solutions.They studied a different model, which is a typical prey-predator, model. The authors in [17] investigated the numerical solution of fuzzy arbitrary order predator-prey equations using the Homotopy Perturbation Method (HPM). In their work, they took fuzziness in the initial conditions to mean convex normalised fuzzy sets viz triangular fuzzy number.They found that their solution was exactly equal to the crisp solution obtained in other literature.The work presented in this paper is different from the existing work done. The main objective of this study is to understand the dynamics of three species Lotka-Volterra model, to present a semi-analytic solution and provide some qualitative analysis with qualitative information about the solution of the model. In general, we seek a detailed numerical, with semi-analytic solution of the model and characterise some aspects of the system behaviour such as equilibrium points, stability, and phase plane analysis of the three species Lotka-Volterra model. In order to solve our three species Lotka-Volterra model, the Homotopy Analysis Method(HAM) has been applied. The model is analysed by investigating the qualitative features and carried out the stability analysis of the equilibrium points. The phase plane analysis of the model is determined and numerical results in ode45 were compared with the semi-analytic solution of the homotopy analysis method. This paper is organised as follows: in section 2, the model formulation and analysis has been studied. In section 3, we present the homotopy analysis approach to nonlinear system, while, in section 4, we include the solution of the Lotka-Volterra model by HAM and in section 5, numerical results and discussion are carried out. In section 6, we present the conclusion and finally, the references are given.

2. Model Formulation and Analysis

The multi-species Lotka-Volterra equations model explains the dynamic nature of an arbitrary number of competitors.We present below the general Lotka-Volterra model for m-species as

$$\frac{dP_i}{dt} = P_i(\beta_i + \sum_{j=1}^{n} \alpha_{ij} P_j) \tag{1}$$

where $i = 1, 2, ..., m$ and P_i is the density or biomass of species i. We describe the associated model parameters in Table 1

2.1. Single Specie Case.

For the case of single specie, equation (1) becomes single specie competing for a given finite source of food

$$\frac{dP_1}{dt} = P_1(\beta_1 + \alpha_{11} P_1) \tag{2}$$

such that $\beta_1 > 0$, $\alpha_{11} < 0$, and $P_1 > 0$, where α_{11} represent the intraspecific interaction. Equation(2) above has a closed form solution

$$P_1(t) = \frac{\beta_1 e^{\beta_1 t}}{\frac{\beta_1 + \alpha_{11} P(0)}{P(0) - \alpha_{11} e^{\beta_1 t}}} \; if \; \beta_1 \neq 0 \tag{3}$$

$$P_1(t) = \frac{P_1(0)}{1 - \alpha_{11} P_1(0) t} \; if \; \beta_1 = 0 \tag{4}$$

where $P_1(0)$ is the initial condition.

2.2. Two Species Case.

Here we construct the Lokta-Volterra equations for two species competing for a common ecological niche as

$$\begin{aligned} \frac{dP_1}{dt} &= P_1(\beta_1 + \alpha_{11} P_1 + \alpha_{12} P_2) \\ \frac{dP_2}{dt} &= P_2(\beta_2 + \alpha_{21} P_1 + \alpha_{22} P_2) \end{aligned} \tag{5}$$

subject to the initial conditions

$$P_1(0) = P_{1,0} > 0, \; P_2(0) = P_{2,0} > 0$$

where α_{11}, α_{12},α_{21}, α_{22}, β_1 andβ_2 are constants.

2.3. **Three Species Case.** Here we construct and analyse the Lotka-Volterra equations for three species competing for a common ecological niche as

$$\begin{aligned} \frac{dP_1}{dt} &= P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2 + \alpha_{13}P_3) \\ \frac{dP_2}{dt} &= P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2 + \alpha_{23}P_3) \\ \frac{dP_3}{dt} &= P_3(\beta_3 + \alpha_{31}P_1 + \alpha_{32}P_2 + \alpha_{33}P_3) \end{aligned} \tag{6}$$

subject to the initial conditions

$$P_1(0) = P_{1,0} > 0,\ P_2(0) = P_{2,0} > 0,\ P_3(0) = P_{3,0} > 0$$

where α_{11}, α_{12},α_{21}, α_{22},α_{13},α_{23},α_{31},α_{32}, α_{33} β_1,β_2 and β_3 are constants.

We describe the associated model parameters in Table 2

2.4. **Positivity of the solution. Theorem 1.**:Suppose the initial data $P_1 \geq 0$, $P_2 \geq 0$,$P_3 \geq 0 ... P_n \geq 0$ then the solutions $(P_1(t), P_2(t), P_3(t), ..., P_n(t)) \in \mathbb{R}^n_+$ of the Lotka-Volterra model (1) are non-negative for all $t > 0$ and will remain positive for all time.(See [1])

Theorem 2.:The region $\Delta \subset \mathbb{R}_+$ is positively-invariant for the Predator-prey model (1) with non-negative initial conditions in $\mathbb{R}^n_+$.
(See [1])

2.5. **Model Analysis.** We analyse the model equation (6) by finding the equilibrium solution of the system when we set the Left Hand Side (L.H.S) of the equation to zero

$$\begin{aligned} P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2 + \alpha_{13}P_3) &= 0 \\ P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2 + \alpha_{23}P_3) &= 0 \\ P_3(\beta_3 + \alpha_{31}P_1 + \alpha_{32}P_2 + \alpha_{33}P_3) &= 0 \end{aligned} \tag{7}$$

2.6. **Equilibrium solution and Stability Analysis.** The system of Equation (9) has eight equilibrium solutions and to determine the stability of this equilibrium point we find the linearization of the system.

Theorem 3.[6]: Suppose λ_1 and λ_2 are the eigenvalues of the 2×2 matrix of the linear system $\dot{x} = Ax$, then the equilibrium point $(0, 0)$ is

(1) asymptotically stable if both eigenvalues λ_1 and λ_2 of A are real and negative, or have negative real parts.
(2) stable but not asymptotically stable if λ_1 and λ_2 are pure imaginary, that is, $\lambda_1, \lambda_2 = \pm bi$.
(3) unstable if either λ_1 or λ_2 is real and positive, or has a positive real part.
(4) saddle point if either λ_1 or λ_2 is positive and one is negative(are real and of opposite signs.)The saddle is always unstable.

We hereby present the eight equilibrium solution of equation (9) given by
$E_1 = [0, 0, 0]$,
$E_2 = [0, 0, -\frac{\beta_3}{\alpha_{33}}]$
$E_3 = [0, -\frac{\beta_2}{\alpha_{22}}, 0]$
$E_4 = [-\frac{\beta_1}{\alpha_{11}}, 0, 0]$
$E_5 = [0, \frac{\beta_2\alpha_{33}-\alpha_{23}\beta_3}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}, -\frac{\beta_3\alpha_{22}+\alpha_{32}\beta_2}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}]$
$E_6 = [\frac{\beta_1\alpha_{33}-\alpha_{13}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}, 0, \frac{\beta_1\alpha_{31}-\alpha_{11}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}]$
$E_7 = [\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, -\frac{\beta_2\alpha_{11}+\alpha_{21}\beta_1}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, 0]$
$E_8 = [-\frac{-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}+\alpha_{12}\alpha 23\beta 3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\alpha 13\beta_3\alpha_{22}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}},$
$\frac{\alpha_{11}\alpha_{23}\beta_3-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha 13-\alpha_{23}\beta_1\alpha_{31}+\alpha_{21}\alpha_{33}\beta_1-\alpha_{21}\beta_3\alpha_{13}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}},$
$-\frac{\beta_2\alpha_{31}\alpha_{12}-\beta_2\alpha_{32}\alpha_{11}+\alpha_{32}\alpha 21\beta_1-\alpha_{21}\beta_3\alpha_{12}+\alpha_{11}\alpha_{22}\beta_3-\alpha 31\beta_1\alpha_{22}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}]$

Case(i) When $E_1 = (0, 0, 0)$

It represents the trivial equilibrium state. Ecologically, it means the simultaneous extinction of both species.Hence for stability of this equilibrium point we find Jacobian matrix of the system which is given by

(8)
$$J(P_1^*, P_2^*, P_3^*) = \left[\begin{array}{ccc} \beta_1 + 2\alpha_{11}P_1^* + \alpha_{12}P_2^* + \alpha_{13}P_3^* & \alpha_{12}P_1^* & \alpha_{13}P_1^* \\ \alpha_{21}P_2^* & \beta_2 + \alpha_{21}P_2^* + 2\alpha_{22}P_2^* + \alpha_{23}P_3^* & \alpha_{23}P_2^* \\ \alpha_{31}P_3^* & \alpha_{32}P_3^* & \beta_3 + \alpha_{31}P_1^* + \alpha_{32}P_2^* + \end{array} \right.$$

At $(0, 0, 0)$,

$$J(0,0,0) = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} \quad (9)$$

with eigenvalues $\lambda_1 = \beta_1 > 0$, $\lambda_2 = \beta_2 > 0$ and $\lambda_3 = \beta_3 > 0$, with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.003$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = 0.1 > 0$, $\lambda_2 = 0.08 > 0$ and $\lambda_3 = 0.06 > 0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation (6)

Case(ii) When $E_2 = [0, 0, -\frac{\beta_3}{\alpha_{33}}]$

This stands for the non-trivial equilibrium state. Ecologically, it depicts that P_1 and P_2 eventually disappears due to predation or competition while P_3 specie

persist. For the stability of this equilibrium point we find Jacobian matrix of the system which is given by
At $E_2 = [0, 0, -\frac{\beta_3}{\alpha_{33}}]$,

$$J\left(0, 0, -\frac{\beta_3}{\alpha_{33}}\right) = \begin{bmatrix} \beta_1 - \frac{\alpha_{13}\beta_3}{\alpha_{33}} & 0 & 0 \\ 0 & \beta_2 - \frac{\alpha_{23}\beta_3}{\alpha_{33}} & 0 \\ -\frac{\alpha_{31}\beta_3}{\alpha_{33}} & -\frac{\alpha_{32}\beta_3}{\alpha_{33}} & -\beta_3 \\ \end{bmatrix} \tag{10}$$

with eigenvalues $\lambda_1 = \beta_3$, $\lambda_2 = \frac{\beta_2\alpha_{33} - \alpha_{23}\beta_3}{\alpha_{33}}$ and $\lambda_3 = \frac{\beta_1\alpha_{33} - \alpha_{13}\beta_3}{\alpha_{33}}$, with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.003$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = -0.06 < 0$, $\lambda_2 = 0.02 > 0$ and $\lambda_3 = 0.058 > 0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation (6)

Case(iii) When $E_3 = [0, -\frac{\beta_2}{\alpha_{22}}, 0]$

This stands for the non-trivial equilibrium state. Ecologically, it means that the P_1 and P_3 species eventually disappears due to predation or competition while P_2 specie persist.Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system which is given by
At $E_3 = [0, -\frac{\beta_2}{\alpha_{22}}, 0]$,

$$J\left(0, -\frac{\beta_2}{\alpha_{22}}, 0\right) = \begin{bmatrix} \beta_1 - \frac{\alpha_{12}\beta_2}{\alpha_{22}} & 0 & 0 \\ -\frac{\alpha_{12}\beta_2}{\alpha_{22}} & \beta_2 & -\frac{\alpha_{23}\beta_2}{\alpha_{22}} \\ -0 & 0 & \beta_3 - -\frac{\alpha_{32}\beta_2}{\alpha_{22}} \end{bmatrix} \tag{11}$$

with eigenvalues $\lambda_1 = -\beta_2$, $\lambda_2 = \frac{\beta_3\alpha_{22} - \alpha_{32}\beta_2}{\alpha_{22}}$ and $\lambda_3 = \frac{\beta_1\alpha_{22} - \alpha_{12}\beta_2}{\alpha_{22}}$, with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.003$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = -0.08 < 0$, $\lambda_2 = -0.156 > 0$ and $\lambda_3 = 0.004 > 0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation (6)

Case(iv) When $E_4 = [-\frac{\beta_1}{\alpha_{11}}, 0, 0]$

This also denotes the non-trivial equilibrium state. Ecologically, this means that the P_2 and P_3 species eventually disappears due to predation or competition while P_1 specie persist.Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system At $E_4 = [-\frac{\beta_1}{\alpha_{11}}, 0, 0]$,

$$J\left(-\frac{\beta_1}{\alpha_{11}},0,0\right)=\begin{bmatrix}\beta_1 & -\frac{\alpha_{12}\beta_1}{\alpha_{11}} & -\frac{\alpha_{13}\beta_1}{\alpha_{11}}\\ 0 & \beta_2-\frac{\alpha_{21}\beta_1}{\alpha_{11}} & 0\\ 0 & 0 & \beta_3-\frac{\alpha_{31}\beta_1}{\alpha_{11}}\end{bmatrix} \quad (12)$$

with eigenvalues $\lambda_1=-\beta_1$, $\lambda_2=\frac{\beta_3\alpha_{11}-\alpha_{31}\beta_1}{\alpha_{11}}$ and $\lambda_3=\frac{\beta_2\alpha_{11}-\alpha_{21}\beta_1}{\alpha_{11}}$. The parameter values: $\alpha_{11}=-0.0014$,$\alpha_{12}=-0.0012$,$\alpha_{21}=-0.00009$, $\alpha_{22}=-0.001$,$\alpha_{13}=-0.0021$,$\alpha_{23}=-0.003$,$\alpha_{33}=-0.003$, $\alpha_{31}=-0.06$,$\alpha_{32}=-0.0027$ $\beta_1=0.1$,$\beta_2=0.08$ and $\beta_3=0.06$ the eigenvalues become $\lambda_1=-0.1<0$, $\lambda_2=-4.225714286<0$ and $\lambda_3=0.07357142857>0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation (6)

Case(v) When $E_5=[0,\frac{\beta_2\alpha_{33}-\alpha_{23}\beta_3}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}},-\frac{\beta_3\alpha_{22}+\alpha_{32}\beta_2}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}]$

This symbolise the non-trivial equilibrium state. Ecologically, this means that the P_1 specie eventually disappears due to predation or competition while P_2 and P_3 species persist. Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system which is given by

At $E_5=[0,\frac{\beta_2\alpha_{33}-\alpha_{23}\beta_3}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}},-\frac{\beta_3\alpha_{22}+\alpha_{32}\beta_2}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}]$,

$$J\left(0,\frac{\beta_2\alpha_{33}-\alpha_{23}\beta_3}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}},-\frac{\beta_3\alpha_{22}+\alpha_{32}\beta_2}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}\right)=\begin{bmatrix}a_{11} & 0 & 0\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \quad (13)$$

Let $a_{11}=\beta_1-\frac{\alpha_{12}(\beta_2\alpha_{33}-\alpha_{23}\beta_3)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}+\frac{\alpha_{13}(-\beta_3\alpha_{22}+\alpha_{32}\beta_2)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{21}=-\frac{\alpha_{21}(\beta_2\alpha_{33}-\alpha_{23}\beta_3)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{22}=\beta_2-\frac{2\alpha_{22}(\beta_2\alpha_{33}-\alpha_{23}\beta_3)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}+\frac{\alpha_{23}(-\beta_3\alpha_{22}+\alpha_{32}\beta_2)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{23}=-\frac{\alpha_{23}(\beta_2\alpha_{33}-\alpha_{23}\beta_3)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{31}=\frac{\alpha_{31}(-\beta_3\alpha_{22}+\alpha_{32}\beta_2)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{32}=\frac{\alpha_{32}(-\beta_3\alpha_{22}+\alpha_{32}\beta_2)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
$a_{33}=\beta_3-\frac{\alpha_{32}(\beta_2\alpha_{33}-\alpha_{23}\beta_3)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}+\frac{2\alpha_{33}(-\beta_3\alpha_{22}+\alpha_{32}\beta_2)}{-\alpha_{32}\alpha_{23}+\alpha_{33}\alpha_{22}}$
with eigenvalues $\lambda_1=-\beta_1$, $\lambda_2=\frac{\beta_3\alpha_{11}-\alpha_{31}\beta_1}{\alpha_{11}}$ and $\lambda_3=\frac{\beta_2\alpha_{11}-\alpha_{21}\beta_1}{\alpha_{11}}$, with parameter values: $\alpha_{11}=-0.0014$,$\alpha_{12}=-0.0012$,$\alpha_{21}=-0.00009$, $\alpha_{22}=-0.001$,$\alpha_{13}=-0.0021$,$\alpha_{23}=-0.003$,$\alpha_{33}=-0.003$, $\alpha_{31}=-0.06$,$\alpha_{32}=-0.0027$ $\beta_1=0.1$,$\beta_2=0.08$ and $\beta_3=0.06$ the eigenvalues become $\lambda_1=0.0498824<0$, $\lambda_2=-0.0399+0.01534i$ and $\lambda_3=-0.0399-0.01534i$ hence, the equilibrium point is stable, of the Lotka-Volterra equation (6)

Case(vi) When $E_6 = [\frac{\beta_1\alpha_{33}-\alpha_{13}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}, 0, \frac{\beta_1\alpha_{31}-\alpha_{11}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}]$

This as well represents the non-trivial equilibrium state. Ecologically, this means that the P_2 specie eventually disappears due to predation or competition while P_1 and P_3 species persist.Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system which is given by
At $E_6 = [\frac{\beta_1\alpha_{33}-\alpha_{13}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}, 0, \frac{\beta_1\alpha_{31}-\alpha_{11}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}]$,

$$(14) \qquad J\left(\frac{\beta_1\alpha_{33}-\alpha_{13}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}, 0, \frac{\beta_1\alpha_{31}-\alpha_{11}\beta_3}{\alpha_{33}\alpha_{11}-\alpha_{31}\alpha_{13}}\right) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Let $b_{11} = \beta_1 - \frac{2\alpha_{11}(\beta_1\alpha_{33}-\alpha_{13}\beta_3)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}} + \frac{\alpha_{13}(-\beta_3\alpha_{11}+\alpha_{31}\beta_1)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{12} = -\frac{\alpha_{12}(\beta_1\alpha_{33}-\alpha_{13}\beta_3)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{13} = -\frac{\alpha_{13}(\beta_1\alpha_{33}-\alpha_{13}\beta_3)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{22} = \beta_2 - \frac{\alpha_{21}(\beta_1\alpha_{33}-\alpha_{13}\beta_3)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}} + \frac{\alpha_{13}(-\beta_3\alpha_{11}+\alpha_{31}\beta_1)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{31} = \frac{\alpha_{31}(-\beta_3\alpha_{11}+\alpha_{31}\beta_1)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{32} = \frac{\alpha_{32}(-\beta_3\alpha_{11}+\alpha_{31}\beta_1)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
$b_{33} = \beta_3 - \frac{\alpha_{31}(\beta_1\alpha_{33}-\alpha_{13}\beta_3)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}} + \frac{2\alpha_{33}(-\beta_3\alpha_{11}+\alpha_{31}\beta_1)}{-\alpha_{31}\alpha_{13}+\alpha_{33}\alpha_{11}}$
with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.002$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = -0.04759902598 - 0.03588557209i$, $\lambda_2 = 0.04759902598 - 0.03588557209i$ and $\lambda_3 = -0.06400438312 < 0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation (6)

Case(vii) When $E_7 = [\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}, \frac{-\beta_2\alpha_{11}+\alpha_{21}\beta_1}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}, 0]$

This indicates the non-trivial equilibrium state. Ecologically, this means that the P_3 specie eventually disappears due to predation or competition while P_1 and P_2 species persist.Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system which is given by

At $E_7 = [\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, -\frac{\beta_2\alpha_{11}+\alpha_{21}\beta_1}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, 0]$,

$$(15) \qquad J\left(\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, -\frac{\beta_2\alpha_{11}+\alpha_{21}\beta_1}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}, 0\right) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{bmatrix}$$

Let $c_{11} = \beta_1 - \frac{2\alpha_{11}(\beta_1\alpha_{22}-\alpha_{12}\beta_2)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}} + \frac{\alpha_{12}(-\beta_2\alpha_{11}+\alpha_{21}\beta_1)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{12} = -\frac{\alpha_{12}(\beta_1\alpha_{22}-\alpha_{12}\beta_2)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{13} = -\frac{\alpha_{13}(\beta_1\alpha_{22}-\alpha_{12}\beta_2)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{21} = \frac{\alpha_{21}(-\beta_2\alpha_{11}+\alpha_{21}\beta_1)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{22} = \beta_2 - \frac{\alpha_{21}(\beta_1\alpha_{22}-\alpha_{12}\beta_2)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}} + \frac{2\alpha_{22}(-\beta_2\alpha_{11}+\alpha_{21}\beta_1)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{23} = \frac{\alpha_{23}(-\beta_2\alpha_{11}+\alpha_{21}\beta_1)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

$c_{33} = \beta_3 - \frac{\alpha_{31}(\beta_1\alpha_{22}-\alpha_{12}\beta_2)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}} + \frac{\alpha_{32}(-\beta_2\alpha_{11}+\alpha_{21}\beta_1)}{-\alpha_{21}\alpha_{12}+\alpha_{22}\alpha_{11}}$

with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.002$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = -0.00398241877872786 < 0$, $\lambda_2 = -0.0800733088112722 < 0$ and $\lambda_3 = -0.3410061919 < 0$ hence, the equilibrium point is stable, of the Lotka-Volterra equation (6)

Case(viii) When$E_8 = [m_1, m_2, m_3]$

where
$m_1 = -\frac{-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}+\alpha_{12}\alpha 23\beta 3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\alpha 13\beta_3\alpha_{22}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$

$m_2 = \frac{\alpha_{11}\alpha_{23}\beta_3-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha 13-\alpha_{23}\beta_1\alpha_{31}+\alpha_{21}\alpha_{33}\beta_1-\alpha_{21}\beta_3\alpha_{13}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$

$m_3 = -\frac{\beta_2\alpha_{31}\alpha_{12}-\beta_2\alpha_{32}\alpha_{11}+\alpha_{32}\alpha 21\beta_1-\alpha_{21}\beta_3\alpha_{12}+\alpha_{11}\alpha_{22}\beta_3-\alpha 31\beta_1\alpha_{22}}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$

This also represents the non-zero constant predator-prey or three competing species population that can co-exist(Non-zero coexistence equilibrium solution). Hence, we present the stability of this equilibrium point using the Jacobian matrix which is given by

At $E_8 = [m_1, m_2, m_3]$,

$$J(m_1, m_2, m_3) = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \tag{16}$$

Let $d_{11} = \beta_1 - \frac{2\alpha_{11}(-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}-\alpha_{12}\alpha_{23}\beta_3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\beta_3\alpha_{22}\alpha_{13})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$
$+ \frac{\alpha_{12}(\beta_3\alpha_{11}\alpha_{23}-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha_{13}-\alpha_{23}\alpha_{13}\beta_1+\alpha_{33}\alpha_{21}\alpha_{21}+\alpha_{33}\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$
$- \frac{\alpha_{13}(\alpha_{31}\alpha_{12}\beta_2-\alpha_{32}\alpha_{11}\beta_2+\beta_1\alpha_{32}\alpha_{21}-\alpha_{21}\alpha_{12}\beta_3+\alpha_{22}\alpha_{11}\beta_3-\alpha_{31}\beta_1\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$
$d_{12} = -\frac{\alpha_{12}(-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}-\alpha_{12}\alpha_{23}\beta_3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\beta_3\alpha_{22}\alpha_{13})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$

$$d_{13} = -\frac{\alpha_{13}(-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}-\alpha_{12}\alpha_{23}\beta_3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\beta_3\alpha_{22}\alpha_{13})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{21} = \frac{\alpha_{21}(-\beta_3\alpha_{11}\alpha_{23}-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha_{13}-\alpha_{23}\alpha_{13}\beta_1+\alpha_{33}\alpha_{21}\alpha_{21}+\alpha_{33}\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{22} = \beta_2 - \frac{\alpha_{21}(-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}-\alpha_{12}\alpha_{23}\beta_3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\beta_3\alpha_{22}\alpha_{13})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$
$$+ \frac{2\alpha_{22}(-\beta_3\alpha_{11}\alpha_{23}-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha_{13}-\alpha_{23}\alpha_{13}\beta_1+\alpha_{33}\alpha_{21}\alpha_{21}+\alpha_{33}\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$
$$- \frac{\alpha_{23}(\alpha_{31}\alpha_{12}\beta_2-\alpha_{32}\alpha_{11}\beta_2+\beta_1\alpha_{32}\alpha_{21}-\alpha_{21}\alpha_{12}\beta_3+\alpha_{22}\alpha_{11}\beta_3-\alpha_{31}\beta_1\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{23} = \frac{\alpha_{23}(\beta_3\alpha_{11}\alpha_{23}-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha_{13}-\alpha_{23}\alpha_{13}\beta_1+\alpha_{33}\alpha_{21}\alpha_{21}+\alpha_{33}\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{31} = \frac{\alpha_{31}(\beta_2\alpha_{31}\alpha_{12}-\alpha_{11}\beta_2\alpha_{32}+\beta_1\alpha_{32}\alpha_{21}-\alpha_{21}\alpha_{12}\beta_3+\alpha_{22}\alpha_{11}\beta_3-\beta_1\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{32} = -\frac{\alpha_{32}(\beta_2\alpha_{31}\alpha_{12}-\alpha_{11}\beta_2\alpha_{32}+\beta_1\alpha_{32}\alpha_{21}-\alpha_{21}\alpha_{12}\beta_3+\alpha_{22}\alpha_{11}\beta_3-\beta_1\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

$$d_{33} = \beta_3 - \frac{\alpha_{31}(-\beta_1\alpha_{32}\alpha_{23}+\beta_1\alpha_{33}\alpha_{22}-\alpha_{12}\alpha_{23}\beta_3-\alpha_{12}\beta_2\alpha_{33}+\alpha_{13}\alpha_{32}\beta_2-\beta_3\alpha_{22}\alpha_{13})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$
$$+ \frac{\alpha_{32}(\beta_3\alpha_{11}\alpha_{23}-\alpha_{11}\beta_2\alpha_{33}+\beta_2\alpha_{31}\alpha_{13}-\alpha_{23}\alpha_{13}\beta_1+\alpha_{33}\alpha_{21}\alpha_{21}+\alpha_{33}\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$
$$- \frac{2\alpha_{33}(\beta_2\alpha_{31}\alpha_{12}-\alpha_{11}\beta_2\alpha_{32}+\beta_1\alpha_{32}\alpha_{21}-\alpha_{21}\alpha_{12}\beta_3+\alpha_{22}\alpha_{11}\beta_3-\beta_1\alpha_{31}\alpha_{22})}{\alpha_{31}\alpha_{12}\alpha_{23}+\alpha_{32}\alpha_{21}\alpha_{13}-\alpha_{32}\alpha_{11}\alpha_{23}-\alpha_{31}\alpha_{13}\alpha_{22}-\alpha_{33}\alpha_{21}\alpha_{12}+\alpha_{33}\alpha_{11}\alpha_{22}}$$

with parameter values: $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$,$\alpha_{13} = -0.0021$,$\alpha_{23} = -0.003$,$\alpha_{33} = -0.002$, $\alpha_{31} = -0.06$,$\alpha_{32} = -0.0027$ $\beta_1 = 0.1$,$\beta_2 = 0.08$ and $\beta_3 = 0.06$ the eigenvalues become $\lambda_1 = 0.0044899384974119 + 0.0383199816094255i$, $\lambda_2 = 0.0044899384974119 - 0.0383199816094255i$ and $\lambda_3 = -0.0902285584584826 < 0$ hence, the equilibrium point is unstable, of the Lotka-Volterra equation(6)

2.7. **Three species Lotka-Volterra Model(6) Phase Plane Analysis.** A phase plane analysis can be used to predict the long-term behaviour of a system. The solutions to the differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field or by 3d-plot. Vectors representing the derivatives of the points with respect to a parameter (say time t), that is $(\frac{dP_1}{dt}, \frac{dP_2}{dt}, \frac{dP_3}{dt})$, at representative points are drawn. Here, we consider the evolution of the three population $P_1(t)$, $P_2(t)$ and $P_3(t)$ living in the same community, competing for finite source of resources. Thus, we present in figures (1) to (4) the direction field and its phase portraits to obtain some qualitative information about the solution of the system(6) without solving it.

Figure (1), shows the direction field for the three species Lotka-Volterra model (6), figure (2) shows the phase portrait for the two species (P_1 and P_3) Lotka-Volterra model (6), figure (3) shows the phase portrait for the two species (P_2 and P_3) Lotka-Volterra model (6) while the figure (4) shows the phase portrait for the two species (P_1 and P_2) Lotka-Volterra model (6).Figures (1) to (4), shows the qualitative information of the behaviour of the solution without solving the equation. Figure (1), shows a 3-d plot of the direction field for P_1, P_2 and P_3. It is observed that the species P_1, P_2 and P_3 increases and finally remains at a steady state.Figures (2) and (3), presents the phase portrait for two species P_1 and P_3. we observed that as P_1 decreases P_3 increases with time while in the figure (4), the phase portrait shows that P_1 increases with increase in P_2.

3. Homotopy Analysis Method

Homotopy Analysis Method is a very powerful analytical method for solving nonlinear problems. It has a convenient means to control and adjust the convergence region and rate of approximation series, when necessary. We consider the following nonlinear differential equation

$$A[v(t)] = 0 \tag{17}$$

where A is a nonlinear operator, t denotes the time, and $v(t)$ is an unknown function. By applying an embedding parameter $q \in [0,1]$ we formulate the zero-order deformation equations (Liao[13]) and [17]

$$(1-q)L[\varphi(t;q) - v_0(t)] = qh_1H(t)A[\varphi(t;q)] \tag{18}$$

where $h_1 \neq 0$ is an auxilliary parameter, $H(t)$ is an auxilliary function, L is an auxilliary linear operator, $v_0(t)$ is an initial guess of $v(t)$, $\varphi(t;q)$ is an unknown function. In the HAM based solution we have freedom to select h_1, $H(t)$, and L. It can easily be shown that when $q = 0$, we obtain

$$\varphi(t;0) = v_0(t) \tag{19}$$

and when $q = 1$, we obtain

$$\varphi(t;1) = v(t) \tag{20}$$

Hence, as q increases from 0 to 1, the solution $\varphi(t;q)$ varies continuously from the initial approximation $v_0(t)$ to the exact solution $v(t)$.Such a kind of continuous variation is called deformation in topology.Expanding $\varphi(t;p)$ by Taylor series with respect to q, we have

$$\varphi(t;q) = \varphi(t;0) + \Sigma_{m=1}^{\infty} v_m q^m \tag{21}$$

where

$$v_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;q)}{\partial q^m} \tag{22}$$

is the deformation derivative. If the auxilliary nonlinear operator A, the initial approximation $v_0(t)$, the auxilliary parameter h_1 and the auxilliary function $H(t)$ are properly chosen so that

(1) the solution $\varphi(t;q)$ of the zero-order deformation equation (18) exists for all $q \in [0,1]$
(2) the deformation derivative (17) exists for all $m = 1, 2, ..$
(3) the series (23) converges at $q = 1$.

Then, we have the series solution

$$v(t) = v_0(t) + \sum_{m=1}^{\infty} v_m(t) \tag{23}$$

which must be one of the solution of the original nonlinear equation (17) as proved by (Liao[13]). In short, we define the vector

$$\overrightarrow{v}_n(t) = \{v_0(t), v_1(t), ..., v_n(t)\} \tag{24}$$

By differentiating the zero-order equations (18) m-times with respect to the embedding parameter q, by setting $q = 0$ and finally dividing the resulting equation by $m!$, we have the m^{th} order deformation equation as

$$L[v_m(t) - \lambda_m v_{m-1}(t)] = h_1 H(t) P_m(\overrightarrow{v}_{m-1}(t)) \tag{25}$$

subject to initial condition

$$v_m(0) = 0 \tag{26}$$

where

$$P_m(\overrightarrow{v}_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} A(t;q)}{\partial q^{m-1}} \mid q = 0 \tag{27}$$

and

$$\lambda_m = \begin{cases} 0 \; if \; m \leq 1, \\ 1 \; if \; m > 1, \end{cases} \tag{28}$$

Note that $v_m(t)$ for $m \geq 1$ can be obtained from equation (25) from uncoupled linear first order differential equation when solved subject to the initial condition (26). The solution of the system can be obtained easily by using the well known symbolic computation software such as MAPLE, MATLAB or MATHEMATICA.

4. Solution of the Lotka-Volterra model by Homotopy Analysis Method

To solve the model equation (6) by HAM, we consider the first equation in the model equation (6) and choose the linear operator

$$\begin{aligned} L_1[P_1(t;q)] &= \tfrac{dP_1(t;q)}{dt} \\ L_2[P_2(t;q)] &= \tfrac{dP_2(t;q)}{dt} \\ L_3[P_3(t;q)] &= \tfrac{dP_3(t;q)}{dt} \end{aligned} \tag{29}$$

with the property that

$$\begin{array}{c} L_1[k_1] = 0 \\ L_2[k_2] = 0 \\ L_3[k_3] = 0 \end{array} \tag{30}$$

where k_1, k_2 and k_3 are constants of integration.

Suppose $q \in [0,1]$ is an embedding parameter,h_1, h_2 and h_3 are auxilliary nonzero parameters and H_1 , H_2 and H_3 are auxilliary functions, then the zeroth-order deformation equations are of the following form

$$\begin{array}{rcl} (1-q)L_1[P_1(t;q) - P_{1,0}(t)] &=& qh_1H_1(t)A_1[P_1(t;q), P_2(t;q), P_3(t;q)] \\ (1-q)L_2[P_2(t;q) - P_{2,0}(t)] &=& qh_2H_2(t)A_2[P_1(t;q), P_2(t;q), P_3(t;q)] \\ (1-q)L_3[P_3(t;q) - P_{3,0}(t)] &=& qh_3H_3(t)A_3[P_1(t;q), P_2(t;q), P_3(t;q)] \end{array} \tag{31}$$

subject to the initial conditions $P_1(0;q) = P_{1,0}$, $P_2(0;q) = P_{2,0}$, $P_3(0;q) = P_{3,0}$

From the Lotka-Volterra equation (6) we define the following HAM nonlinear operators A_1, A_2 and A_3 as

$$\begin{array}{rcl} A_1[P_1(t;q)] &=& \frac{\partial P_1}{dt} - P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2 + \alpha_{13}P_3) \\ A_2[P_2(t;q)] &=& \frac{\partial P_2}{dt} - P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2 + \alpha_{23}P_3) \\ A_3[P_3(t;q)] &=& \frac{\partial P_3}{dt} - P_3(\beta_3 + \alpha_{31}P_1 + \alpha_{32}P_2 + \alpha_{33}P_3) \end{array} \tag{32}$$

For $q = 0$ and $q = 1$ we have
$P_1(t;0) = P_{1,0}(t)$, $P_2(t;0) = P_{2,0}(t)$ $P_3(t;1) = P_{3,0}(t)$, $P_1(t;1) = P_{1,1}(t)$, $P_2(t;1) = P_{2,1}(t)$, $P_3(t;1) = P_{3,1}(t)$

As q increases from 0 to 1, $P_1(t;q)$, $P_2(t;q)$ and $P_3(t;q)$ vary from $P_{1,0}(t)$ to $P_{1,1}(t)$, $P_{2,1}(t)$ and $P_{3,1}(t)$.Using Taylor theorem, and $P_1(0;q) = P_{1,0}$, $P_2(0;q) = P_{2,0}$ and $P_3(0;q) = P_{3,0}$ we obtain

$$\begin{array}{rcl} P_1(t;q) &=& P_1(t;0) + \sum_{m=1}^{\infty} P_{1,m}(t)q^m \\ P_2(t;q) &=& P_2(t;0) + \sum_{m=1}^{\infty} P_{2,m}(t)q^m \\ P_3(t;q) &=& P_3(t;0) + \sum_{m=1}^{\infty} P_{3,m}(t)q^m \end{array} \tag{33}$$

where

$$\begin{array}{rcl} P_{1,m}(t) &=& \frac{1}{(m)!}\frac{\partial^m P_1(t;q)}{\partial q^m}|q=0 \\ P_{2,m}(t) &=& \frac{1}{(m)!}\frac{\partial^m P_2(t;q)}{\partial q^m}|q=0 \\ P_{3,m}(t) &=& \frac{1}{(m)!}\frac{\partial^m P_3(t;q)}{\partial q^m}|q=0 \end{array} \tag{34}$$

By Liao[13], the convergence of the series (33) strongly depend upon auxillary parameters h_1 and h_2. Assume that h_1 and h_2 are selected such that the series

(33) are convergent at $q = 1$ then due to $P_1(t;0) = P_{1,0}(t)$, $P_2(t;0) = P_{2,0}(t)$ $P_3(t;0) = P_{3,0}(t)$, $P_1(t;1) = P_{1,1}(t)$, $P_2(t;1) = P_{2,1}(t)$, $P_3(t;1) = P_{3,1}(t)$ we have

(35)
$$\begin{array}{rcl} P_1(t) & = & P_{1,0}(t) + \sum_{m=1}^{\infty} P_{1,m}(t) \\ P_2(t) & = & P_{2,0}(t) + \sum_{m=1}^{\infty} P_{2,m}(t) \\ P_3(t) & = & P_{3,0}(t) + \sum_{m=1}^{\infty} P_{3,m}(t) \end{array}$$

From the so-called mth-order deformation equations (25) and (27), we differentiate equation (32) m times with respect to q divide by $m!$ and then set $q = 0$. The resulting deformation equation at the mth-order are

(36)
$$\begin{array}{rcl} L_1[P_{1,m}(t) - \lambda_m P_{1,m-1}(t)] & = & h_1 H(t) B_{1,m}(\overrightarrow{P}_{1,m-1}(t)) \\ L_2[P_{2,m}(t) - \lambda_m P_{2,m-1}(t)] & = & h_2 H(t) B_{2,m}(\overrightarrow{P}_{2,m-1}(t)) \\ L_3[P_{3,m}(t) - \lambda_m P_{3,m-1}(t)] & = & h_3 H(t) B_{3,m}(\overrightarrow{P}_{3,m-1}(t)) \end{array}$$

subject to the initial conditions
$P_{1,m}(0) = 0$, $P_{2,m}(0) = 0$, $P_{3,m}(0) = 0$

where

(37)
$$B_{i,m}(\overrightarrow{P}_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} A_i(t;q)}{\partial q^{m-1}} \mid q = 0 \ for \ all \ i = 1,2,3$$

subject to initial condition

(38)
$$P_{i,m}(0) = 0 \ for \ all \ i = 1,2,3$$

where

(39)
$$\begin{array}{rcl} B_{1,m} & = & [\frac{dP_{1,(m-1)}(t)}{dt} - P_{1,(m-1)}(t)\beta_1 - \sum_{k=0}^{m-1} \alpha_{11} P_{1,k}(t) P_{1,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{12} P_{1,k}(t) P_{2,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{13} P_{3,k} P_{1,m-1} \\ B_{2,m} & = & [\frac{dP_{2,(m-1)}(t)}{dt} - P_{2,(m-1)}(t)\beta_2 - \sum_{k=0}^{m-1} \alpha_{21} P_{2,k}(t) P_{1,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{22} P_{2,k}(t) P_{2,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{23} P_{3,k}(t) P_{2,m} \\ B_{3,m} & = & [\frac{dP_{3,(m-1)}(t)}{dt} - P_{3,(m-1)}(t)\beta_3 - \sum_{k=0}^{m-1} \alpha_{31} P_{1,k}(t) P_{3,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{32} P_{2,k}(t) P_{3,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{33} P_{3,k}(t) P_{3,m} \end{array}$$

Here, we take $h_i = -1$ and $H_i(t) = 1$ since HAM allows one to choose the control-convergence function $H_i(t)$ and the parameter h_i for all $i = 1,2,3$. Finally,the HAM formulation for the mth- order deformation equation (35) for $m \geq 1$ yields

(40)
$$P_{1,m}(t) = \lambda_m P_{1,(m-1)}(t) - \int_{\infty}^{t} [\frac{dP_{1,(m-1)}(t)}{dt} - P_{1,(m-1)}(t)\beta_1 - \sum_{k=0}^{m-1} \alpha_{11} P_{1,k}(t) P_{1,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{12} P_{1,k}(t) P_{2,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{13} P_{1,k} P_{3,m-1-}$$

(41)
$$P_{2,m}(t) = \lambda_m P_{2,(m-1)}(t) - \int_{\infty}^{t} [\frac{dP_{2,(m-1)}(t)}{dt} - P_{2,(m-1)}(t)\beta_2 - \sum_{k=0}^{m-1} \alpha_{21} P_{2,k}(t) P_{1,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{22} P_{2,k}(t) P_{2,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{23} P_{2,k}(t) P_{3,m-}$$

(42)
$$P_{3,m}(t) = \lambda_m P_{3,(m-1)}(t) - \int_{\infty}^{t} [\frac{dP_{3,(m-1)}(t)}{dt} - P_{3,(m-1)}(t)\beta_3 - \sum_{k=0}^{m-1} \alpha_{31} P_{3,k}(t) P_{1,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{32} P_{3,k}(t) P_{2,m-1-k}(t) - \sum_{k=0}^{m-1} \alpha_{33} P_{3,k}(t) P_{3,m-}$$

and

$$\lambda_m = \begin{cases} 0 \; if \;\; m \leq 1, \\ 1 \; if \;\; m > 1, \end{cases} \tag{43}$$

5. Numerical Results and Discussions

We authenticated the model using the parameters obtained from different kinds of literature. Tables 3 and 4 below shows the details of the parameters and their values.The result of the numerical experiment carried out in this paper is similar to that of the ode-45.

The 7^{th} to 9^{th} approximations for P_1, P_2, P_3 are obtained and presented below:

7^{th} approximations:

$P_{1,7}(t) = 4 + 0.1568t + 0.0182952t^2 + 0.0005995175273t^3 + 0.00002166239834t^4 + 0.0000005459583614t^5 + 0.0000000124695351t^6 + 0.0000000002406026199t^7$

$P_{2,7}(t) = 10 + 0.064t + 0.0414488t^2 + 0.0001261403467t^3 + 0.00005365303345t^4 + 0.000001403932621t^5 + 0.00000003381304318t^6 + 0.0000000006709814987t^7$

$P_{3,7}(t) = 20 - 2.6216t - 0.034400336t^2 - 0.004361831453t^3 - 0.0001152367228t^4 - 0.000003655237768t^5 - 0.0000008172281415t^6 - 0.000000001670015513t^7$

8^{th} approximations:

$P_{1,8}(t) = 4 + 0.1568t + 0.0182952t^2 + 0.0005995175273t^3 + 0.00002166239834t^4 + 0.0000005459583614t^5 + 0.0000000124695351t^6 + 0.0000000002406026199t^7 + 0.000000000004147000416t^8$

$P_{2,8}(t) = 10 + 0.064t + 0.0414488t^2 + 0.0001261403467t^3 + 0.00005365303345t^4 + 0.000001403932621t^5 + 0.00000003381304318t^6 + 0.0000000006709814987t^7 + 0.00000000001183170461t^8$

$P_{3,8}(t) = 20 - 2.6216t - 0.034400336t^2 - 0.004361831453t^3 - 0.0001152367228t^4 - 0.000003655237768t^5 - 0.0000008172281415t^6 - 0.000000001670015513t^7 - 0.00000000002911139289t^8$

9^{th} approximations:

$P_{1,9}(t) = 4 + 0.1568t + 0.0182952t^2 + 0.0005995175273t^3 + 0.00002166239834t^4 + 0.0000005459583614t^5 + 0.0000000124695351t^6 + 0.0000000002406026199t^7 + 0.000000000004147000416t^8 + 0.00000000000006367473269t^9$

$P_{2,9}(t) = 10 + 0.064t + 0.0414488t^2 + 0.0001261403467t^3 + 0.00005365303345t^4 + 0.000001403932621t^5 + 0.00000003381304318t^6 + 0.0000000006709814987t^7 + 0.00000000001183170461t^8 + 0.0000000000001843296637t^9$

$P_{3,9}(t) = 20 - 2.6216t - 0.034400336t^2 - 0.004361831453t^3 - 0.0001152367228t^4 - 0.000003655237768t^5 - 0.0000008172281415t^6 - 0.000000001670015513t^7 - 0.00000000002911139289t^8 - 0.000000000000455887707t^9$

5.1. **Homotopy Analysis Method for the solution of the three species Lotka-Volterra Model.**

5.2. **Comparison of HAM and Ode45 on the dynamics of the three species Lotka-Volterra Model.** In this section,we have compared the solutions of HAM with the result of ode45. Figure (5), presents the plot of 9^{th} approximation for P_1 against time and figure (6),shows the plot of 9^{th} approximation for P_2 against time with HAM and ode45 while figure (7) shows the plot of 9^{th} approximation for P_3 against time with HAM and ode45. In figure (3) above, it is observed that P_1 (first specie population) is increasing with time as well as in fig (4), the population of the second specie P_2 is also increasing with time which agrees with the plot obtain by using ode45(for fourth order Runge-Kutta method) for P_1 , P_2 and P_3. Figure (3), figure (4) and figure (5) shows that HAM is in good agreement with RK4. Hence,it is obvious that the analytical approximations to the solutions are reliable method and confirm the power and ability of the Homotopy Analysis Method.

5.3. **Variation of different values of β_1, β_2 and β_3 on the dynamics of the three species Lotka-Volterra Model.** In figures (8) and (9), the variations of P_1 for different high and low values of β_1 are shown and figures (10) and (11) gave the variations of P_2 for different low and high values of β_2 while figures (12) and (13) the results show the variations of P_3 for different high and low values of β_3. In figure(8), it is observed that for different high values of β_1, the population of P_1 decreases to zero except at $\beta_1 = 2.0$ where the population increases and remain at steady state. Figure(9) shows that the population of P_1 increases at $\beta_1 = 0.1$ while at $\beta_1 = 0.0001$ and $\beta_1 = 0.0005$ P_1 decreases.For low values of β_2, the population of P_2 also increases at $\beta_2 = 0.08$ but the population decreases at $\beta_1 = 0.0001$ and $\beta_1 = 0.0005$ in figure(10).Figure (11) presents variation of P_2 for different high values of β_2,at $\beta_2 = 2.0$ the population P_2 increases rapidly to maintain a steady state while at $\beta_2 = 0.5$, P_2 increases and stopped at $P_2 = 399$ but goes to zero at $\beta_2 = 0.08$. Finally, figures (12) and (13), it is observed that at $\beta_3 = 2.0$ the population P_3 increases rapidly to maintain a steady state while at $\beta_3 = 0.5$, P_3 increases and stopped at $P_2 = 50$ but goes to zero at $\beta_3 = 0.06$. Figure (12) shows the variation of P_3 for different low values of β_3. Here, it is observed that the lower the values of β_3, the faster the population declines to zero.

6. Conclusion

We present in this paper, a mathematical model that describes the dynamics of three species Lotka-Volterra model and its qualitative analysis. We further gave some qualitative information about the solution of the model by phase-plane analysis. Approximate analytical solution for the three species Lotka-Volterra model is obtained by HAM and comparisons with Runge-Kutta of fourth order and that of [4] reveals that HAM is a more powerful, easy-to-use analytic method for nonlinear problems. β_1, β_2 and β_3 which represent the intrinsic rate of change of specie 1, 2 and 3 is the rate at which the population increases in size if there are no density-dependent forces regulating the population. This implies that the population of specie 1, 2 and 3 increases as β_1, β_2 and β_3. The advantage of the homotopy analysis method over other perturbative and non-perturbative techniques is that it gives a simple way to adjust the convergence region and rate of given approximate series. Variations of the intrinsic rate of change were also presented with graphical illustrations.

7. Acknowledgement

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References

[1] Agusto, F. B.; Gumel, A. B.; Theoretical assessment of avian influenza vaccine. DCDS Series B. 13 (1) pp 1-25 (2010)

[2] M. Ayub, H. Zaman, M. Sajid, T. Hayat. Analytical solution of a stagnation-point flow of viscoelastic fluid towards a stretching surface, Commun. Nonlinear. Sci. Numer. Simulat., 2008, 13: 1822-1835.

[3] A.S.Bataineh,M.S.M. Noorani, I. Hashim. Series solution of the multispecies Lotka-Voltera equations by means of the homotopy analysis method. Diff. Eq. and Nonlin. Mech.,doi:10.1155/2008/816787.

[4] B. Batiha, M.S.M. Noorani, I.Hashim, Variational iteration for solving multispecies Lotka-Volterra equations, "Computer and Mathematics Applications 54(2007) 903-909.

[5] Bakare E. A., Adekunle Y. and Nwagwo A., Mathematical analysis of the control of the spread of infectious disease in a prey-predator ecosystem, International Journal of Computer & Organization Trends, 2(1), 27-32, 2012.

[6] F. Benyah, Introduction to mathematical modelling II,8^{th} Regional college on modelling, simuation and optimization, University of capecoast, Ghana, 24th April - 5th May, 2006.

[7] J. Biazar, olution of the epidemic model by Adomian decomposition method,pplied Mathematics and Computation,vol. 173, no. 2, pp. 1101106, 2006.

[8] R. Brociek,E. Hetmaniok, J. Matlak, D. Slota, Application of the Homotopy Analysis Method for Solving the Systems of Linear and Nonlinear Integral Equations. Mathematical Modelling and Analysis, Volume 21, 2016 - issue 3, Taylor and Francis, Pg. 350-370.

[9] A.R. Ghotbi. Homotopy analysis method for solving the MHD flow over a non-linear stretching sheet, Commun. Nonlinear Sci. Numer. Simulat., 2008, 14:2653-2663.

[10] T. Hayat, M. Sajid. Homotopy analysis of MHD boundary layer flow of an upper-conveceted Maxwell fluid. Int. Jour. of Eng. Sc., 2007, 45: 393-401.

[11] W.O. Kermack, A.G. McKendrick. Contribution to the mathematical theory of epidemics, Proc. Roy. Soc., 1927, A115: 700-721 .

[12] H.Khan, R.N. Mohapatra, K.Vajravelu, S.J. Liao. Explicit series solution of SIR and SIS epidemic models, Appl. Math.Comput doi:10.1016/j.amc.2009.05.051(2009).
[13] S.J. Liao. Beyond perturbation: Introduction to homotopy analysis method. Chapman and Hall/CRC Press, 2003.
[14] A.A. Momoh, M.O. Ibrahim, Tahir A., Ibrahim I.A., Application of Homotopy Analysis Method for Solving the SEIR Models of Epidemics, Nonlinear Analysis and Differential Equations, Vol. 3, 2015, no. 2, 53 - 68 HIKARI Ltd, www.m-hikari.com, http://dx.doi.org/10.12988/nade.2015.3818.
[15] S. Olek, An accurate solution to the multispecies Lotkaolterra equations, SIAM Rev. 36 (3) (1994) 48088.
[16] D. Rostany, F. zabihi, K. Karimi, The application of homotopy analysis method for solving the prey and predator problem, "Applied Mathematical Sciences Vol5, 2011, no.13, 639-650.
[17] S. Tapaswini, S. Chakarverty, Numerical Solution of Fuzzy Arbitrary order Predator-Prey Equations, Applications and Applied Mathematics: An International Journal(AAM), Vol. 8, issue 2, (December 2013), pp 647-672.
[18] V. Volterra. Variazionie fluttuazioni del numero dndividui in specie animali conviventi. Mem. Acad. Lincei., 2:3113, 1926. Variations and fluctuations of a number of individuals in animal species living together. Translation by R.N. Chapman. In: Animal Ecology. pp. 409-448. McGraw Hill, New York, 1931.
[19] http://en.wikipedia.org/wiki/Homotopy analysis method

Table 1. Parameters in the model Equation 1

Parameters	Description
α_{ij}	the interaction coefficients between species i and j
β_i	the intrinsic rate of change of specie i
n	the number of species

Table 2. Parameters in the model Equation 6

Parameters	Description
α_{11}	the intraspecific interaction between members of the same species 1 and 1
α_{12}	the interaction coefficients between species 1 and 2
α_{21}	the interaction coefficients between species 2 and 1
α_{22}	the intraspecific interaction between members of the same species 2 and 2
α_{13}	the interaction coefficients between species 1 and 3
α_{23}	the interaction coefficients between species 2 and 3
α_{31}	the interaction coefficients between species 3 and 1
α_{32}	the interaction coefficients between species 3 and 2
α_{33}	the intraspecific interaction between members of the same species 3 and 3
β_1	the intrinsic rate of change of specie 1
β_2	the intrinsic rate of change of specie 2
β_3	the intrinsic rate of change of specie 3

Table 3. Parameters of the model Equation 6

Parameter	Symbol	Value	Source
the intraspecific interaction between members of the same species 1 and 1	α_{11}	-0.0014	[4]
the interaction coefficients between species 1 and 2	α_{12}	-0.0012	[4]
the interaction coefficients between species 1 and 2	α_{21}	-0.0009	[4]
the intraspecific interaction between members of the same species 2 and 2	α_{22}	-0.001	[4]
the interaction coefficients between species 1 and 3	α_{13}	-0.0021	[4]
the interaction coefficients between species 2 and 3	α_{23}	-0.003	[4]
the interaction coefficients between species 3 and 2	α_{32}	-0.0027	[4]
the interaction coefficients between species 1 and 3	α_{31}	-0.06	[4]
the intraspecific interaction between members of the same species 3 and 3	α_{33}	-0.002	[4]
the intrinsic rate of change of specie 1	β_1	0.1	[4]
the intrinsic rate of change of specie 2	β_2	0.08	[4]
the intrinsic rate of change of specie 3	β_3	0.06	[4]

Table 4. Initial values of variables used in the simulations.

Variables	Symbol	Value	Source
Initial population of first specie	$P_{1,0}$	4	[4]
Initial population of second specie	$P_{2,0}$	10	[4]
Initial population of third specie	$P_{3,0}$	20	[4]

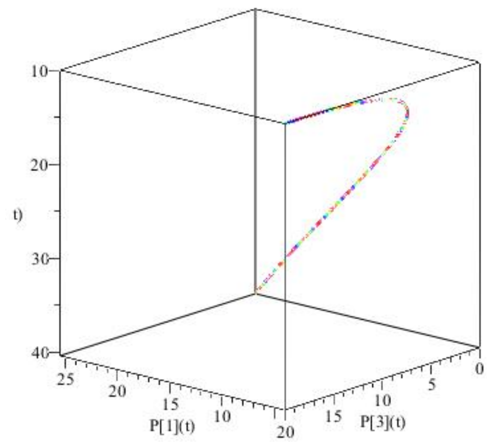


FIGURE 1. The direction field for the two species Lotka-Volterra model (6).

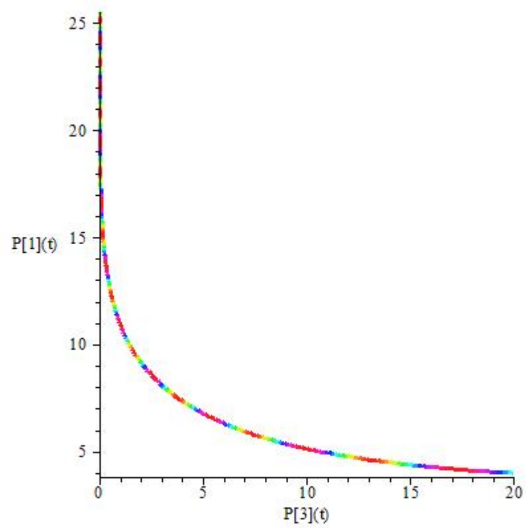
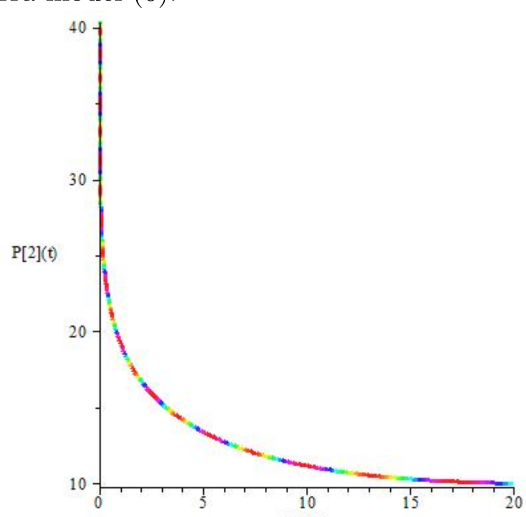


FIGURE 2. The phase portrait for the two(P_1 and P_3) species Lotka-Volterra model (6).



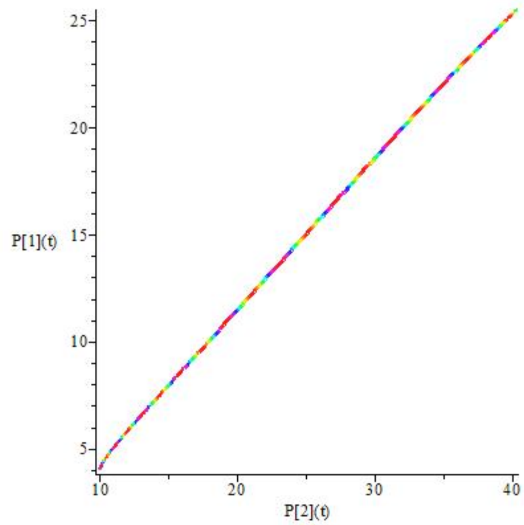


FIGURE 4. The phase portrait for the two(P_1 and P_2) species Lotka-Volterra model (6).

TABLE 5. Result of homotopy analysis method when t=0.2

3*order		t=0.2	
	P_1	P_2	P_3
7th	4.032096639	10.01446813	19.47426892
8th	4.032096639	10.01446813	19.47426892
9th	4.032096639	10.01446813	19.47426892

TABLE 6. Result of homotopy analysis method when t=0.4

3*order		t=0.4	
	P_1	P_2	P_3
7th	4.065686162	10.03231392	18.94557380
8th	4.065686162	10.03231392	18.94557380
9th	4.065686162	10.03231392	18.94557380

TABLE 7. Result of homotopy analysis method when t=0.6

3*order		t=0.6	
	P_1	P_2	P_3
7th	4.100798618	10.05360109	18.41369851
8th	4.100798618	10.05360109	18.41369851
9th	4.100798618	10.05360109	18.41369851

Table 8. Result of homotopy analysis method when t=0.8

3*order		t=0.8	
	P_1	P_2	P_3
7th	4.137464936	10.07839552	17.8784221
8th	4.137464936	10.07839552	17.8784221
9th	4.137464936	10.07839552	17.8784221

Table 9. Result of homotopy analysis method when t=1.0

3*order		t=1.0	
	P_1	P_2	P_3
7th	4.175716938	10.10676528	17.33951885
8th	4.175716938	10.10676528	17.33951885
9th	4.175716938	10.10676528	17.33951885

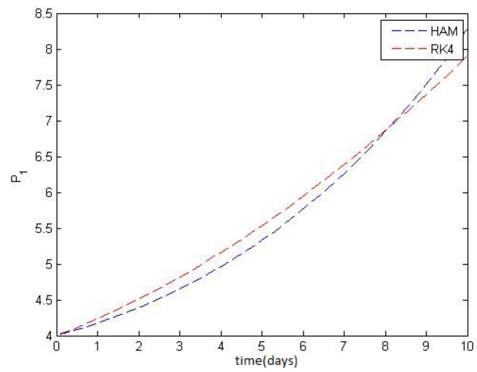


FIGURE 5. Plot of 9^{th} approximation for P_1 against time with HAM and ode45.

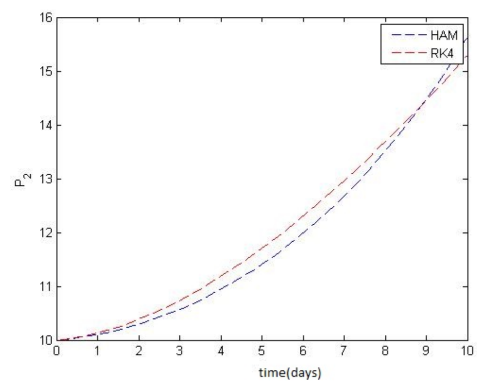


FIGURE 6. Plot of 9^{th} approximation for P_2 against time with HAM and ode45.

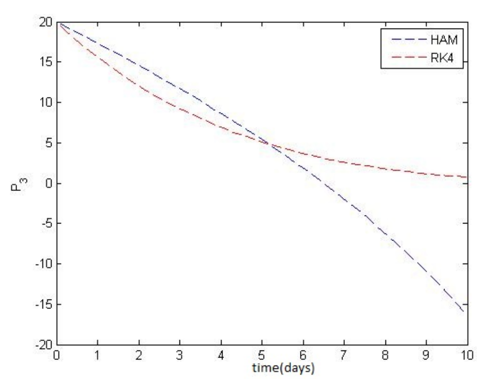


FIGURE 7. Plot of 9^{th} approximation for P_3 against time with HAM and ode45.

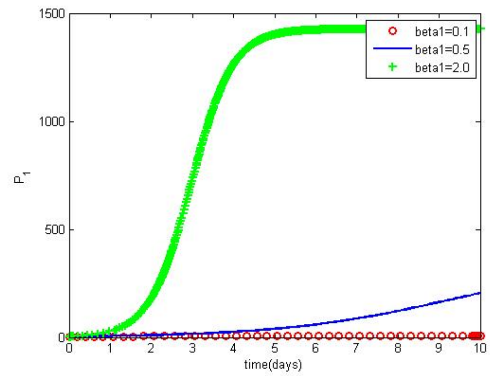


FIGURE 8. Variation of P_1 for different high values of β_1.

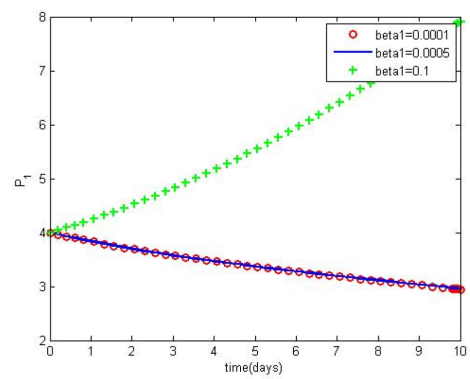


FIGURE 9. Variation of P_1 for different low values of β_1.

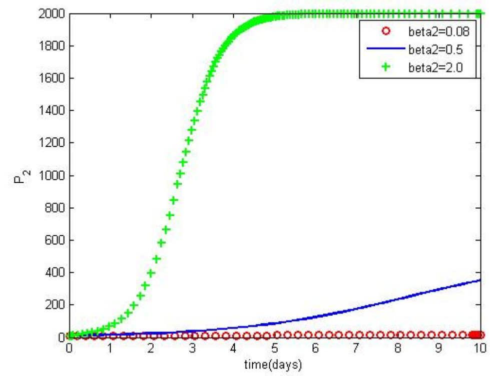


FIGURE 10. Variation of P_2 for different low values of β_2.

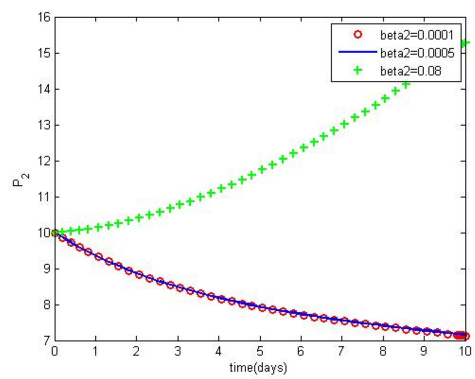


FIGURE 11. Variation of P_2 for different high values of β_2.

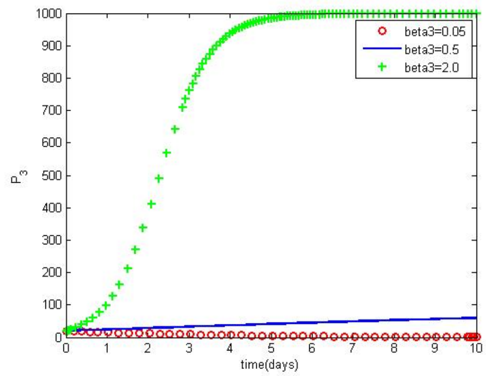


FIGURE 12. Variation of P_3 for different high values of β_3.

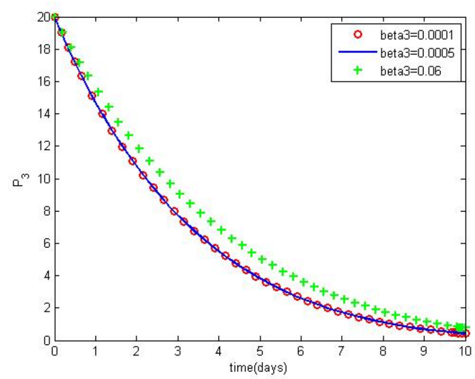


FIGURE 13. Variation of P_3 for different low values of β_3.