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Review of a Study of Soft Fields

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Abstract

Molodtsov in 1999 introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty about vague concept. In this paper, we present the definition of soft field and soft subfield with illustrative examples. Some properties of soft fields and soft subfield are investigated.

1. Introduction

Soft set theory was first initiated as a novel concept by Molodtsov [1] as a new general Mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. He pointed out that, the important existing theories such as Probability Theory, Fuzzy set Theory, Intuitionistic Fuzzy sets, Rough set Theory, etc, which are considered as mathematical tools for dealing with uncertainties have their own inherent limitations due to the inadequacies of their parameterization tools. These theories cannot be successfully applied to solve complicated problems in the field of Engineering, Social Science, Economics, Medical Science, etc.Maji et al. [2] introduced several operations on soft sets and established their properties.

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Aktas and Cagman [3] define the notion of soft group and presented its properties. Feng et al. [4] initiated the study of soft semi rings, soft ideal and idealistic soft semi rings.
Acar et al. [5] introduce the definition of soft ring and investigated their basic algebraic properties, while, Singh and Onyeozili [6] further discuss on the algebraic properties of soft rings with illustrative examples. Nagarajan and Meenambigar [7] initiated the study of soft lattices and introduced several related properties.
The theory of soft set continues to receive attention and diversification such as in the area of its application, and in the decision making problems in [8], [9], [10],[11], [12].Some soft set operations, relations, derivatives and its properties can be obtained in [13], [14], [15], [16], [17] and [20].
In this paper, we present the definition of soft field and soft subfield with illustrative examples. Some properties of soft fields and soft subfield are stated and proved.

2. Preliminaries and Basic Definitions

In this section, we present the notion of field and some basic definitions in soft set theory.

Definition 2.1 [18]: A field is a nonempty set $\boldsymbol{F}$ together with two binary operations, usually called addition and multiplication, denoted by $+$ and **.**, such that the following axioms hold (note that subtraction and division are defined in terms of the inverse operation of addition and multiplication:)

(i) **F is closed under addition and multiplication:** For all $a, b \in F$, both $a + b$ and $a.b \in F$;
(ii) **Addition and Multiplication are associative:** For all $a, b, c \in F$, the following holds: $a + (b + c) = (a + b)$+c and $a \bullet (b \bullet c) = (a \bullet b)\ \bullet c$;
(iii) **Addition and Multiplication are commutative:** For all $a, b \in F$, the following holds: $a + b = b + a$ and $a \bullet b = b \bullet a$;
(iv) **Additive and multiplicative identity element exists:** There exists an element of F, called the additive identity element and is denoted by 0, such that for every $a \in F$, $a + 0 = a$. Similarly, there is an element, called the multiplicative identity element and denoted by 1, such that for every $a \in F$, $a \bullet 1 = a$;
(v) **Additive and Multiplicative inverses exist:** For every $a \in F$, there exists an element $-a \in F$ (called an additive inverse of $a \in F$), such that $a + (-a) = 0$. Likewise, for every $a \in F$ other than 0, there exists an element $a^{-1} \in F$ (called multiplicative inverse of $a \in F$), such that $a \bullet a^{-1} = 1$;
(vi) **Multiplication is distributive over addition:** For all $a, b, c \in F$, the following equality holds: $a \bullet (b + c) = (a \bullet b) + (a \bullet c)$.

A field is therefore an algebraic structure $(F,\ +,\ \ \bullet)$. For the purpose of convenience we simply denote $(F,\ +,\ \ \bullet)$ by $\boldsymbol{F}$.

Definition 2.2 [18]: If a subset S of a field F satisfies the field axioms with the same operations on F, then S is called a subfield of F.

Theorem 2.1 [18]: Let K and S be subfields of a field F**.** Then,
(i) $K \cap S$ is a subfield of F;
(ii) If $K \subset S$ or $S \subset K$, then $K \cup S$ is a subfield of F.
We recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, P(U) be the power set of U and A⊆E.

Definition 2.3 [1]: A pair *(Γ, A)* is called a **soft set** over *U*, where *Γ* is a mapping given by $\Gamma : A \longrightarrow P(U)$. In other words, a soft set over *U* is a parameterized family of subsets of the universe *U*. For $x \in$A, $\Gamma(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set *(Γ, A).* Thus, (*Γ*, *A*) is defined as:

$$(\Gamma,\ A) = \{\Gamma\,(x) \in P\,(U)\,, x \in A\}$$

The soft set (*Γ*, *A*) can be represented as a set of ordered pairs as follows:
$(\Gamma,\ A) = \{(\ x,\ \Gamma(x)),\ x \in A,\ \Gamma(x) \in P(U)\}$

Example 2.2.: Let $U = \{c_1,\ c_2,\ c_3,\ c_4, c_5\}$ be a universal set consisting of five cars and $E = \{\ e_1,\ e_2,\ e_3,\ e_4\}$ be the set of parameters under consideration, where each parameter e_i, i=1,2,3,4, stands for, Expensive, Fuel economy, Fast, Modern, respectively.
Let $A = \{\ e_1,\ e_3,\ e_4\} \subset E$ such that $\Gamma(e_1)=\{c_1,\ c_3\}$, $\Gamma(e_3)=\{c_1,\ c_2,\ c_4\}$ and $\Gamma(e_4)=\{c_4\}$. Then, the soft set (*Γ*, *A*) over *U* is given by $(\Gamma,\ A) = \{(e_1,\{c_1,\ c_3\}),\ (e_3,\{c_1,\ c_2,\ c_4\}),\ (\ e_4,\{c_4\})\}$.

Definition 2.4 [4]: Let (*Γ*, *A*) and (*G, B*) be two soft sets over *U*. Then,

(1) (*Γ, A*) is said to be a **soft subset** of (*G, B*), denoted by
$(\Gamma,\ A)\,\widetilde{\subseteq}\,(\mathrm{G},\ \mathrm{B})$, if $A \subseteq$B and $\Gamma\,(x) \subseteq \mathrm{G}\,(x)\,, \forall x \in$A
(2) (*Γ, A*) and (*G, B*) are said to be **soft equal**, denoted by
$(\Gamma,\ A)\,\widetilde{=}\,(\mathrm{G},\ \mathrm{B})$ if $(\Gamma,\ A)\,\widetilde{\subseteq}(\mathrm{G},\ \mathrm{B})$ and $(\mathrm{G},\ \mathrm{B})\,\widetilde{\subseteq}\,(\Gamma,\ A)$

Definition 2.5 [2]: Let (*Γ*, *A*) be a soft set over *U*. Then the support of (*Γ*, *A*) written **supp (*Γ*, *A*)** is the set defined as: supp(*Γ*, *A*) = $\{x \in A\colon \Gamma(x) \neq \emptyset\}$ where,
(i) (*Γ, A*) is called a **non-null** soft set if supp $(\Gamma,\ A) \neq \emptyset$;
(ii) (*Γ, A*) is called a **relative null** soft set denoted by $\emptyset_A\ if\ \Gamma\,(x) = \emptyset,\ \forall x \in A$;
(iii) (*Γ, A*) is called a **relative whole** soft set, denoted by
$U_A\ if\ \Gamma\,(x) = U,\ \forall x \in A$.

Definition 2.6 [2]: Let (*Γ*, *A*) and (*G, B*) be two soft sets over *U*. Then the **union** of (*Γ*, *A*) and (*G, B*), denoted by $(\Gamma, A)\,\widetilde{\cup}(G,\ B)$ is a soft set defined as:

$(\mathit{\Gamma}, A)\,\widetilde{\cup}\,(G,\ B) = (H,\ C)\,,\ where\ C = A \cup B\ and\ \forall x \in C$

$$H(x) = \begin{cases} \mathit{\Gamma}(x)\,, & if\ x \in A - B \\ G(x)\,, & if x \in B - A \\ \mathit{\Gamma}(x) \cup G(x)\,, & if\ x \in A \cap B \end{cases}$$

Definition 2.7 [2]: Let $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **restricted union** of $(\mathit{\Gamma},\ A)$ and $(G,\ B)$, denoted by $(\mathit{\Gamma},\ A)\ \widetilde{\cup}_R\ (G,\ B)$ is a soft set defined as: $(\mathit{\Gamma},\ A)\ \widetilde{\cup}_R\ \ (G,\ B) = (H,\ C)$, where $C = A{\cap}B{\neq}\ \emptyset$ and $\forall\ x{\in}C$ $H(x) = \mathit{\Gamma}(x)\ {\cup}G(x)$.

Definition 2.8 [2]: Let $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **extended intersection** of $(\mathit{\Gamma},\ A)$ and $(G,\ B)$, denoted by $(\mathit{\Gamma},\ A)\ \widetilde{\cap}_E\ (G,\ B) = (H,\ C)$, where $C = A\ \cup\ B$ and $\forall x{\in}C$

$$H(x) = \begin{cases} \mathit{\Gamma}(x)\,, & if\ x \in A - B \\ G(x)\,, & if x \in B - A \\ \mathit{\Gamma}(x) \cap G(x)\,, & if\ x \in A \cap B \end{cases}$$

Definition 2.9 [2]: Let $(\mathit{\Gamma}, A)$ and $(G,\ B)$ be two soft sets over U. Then the **restricted intersection** of $(\mathit{\Gamma}, A)$ and $(G,\ B)$ denoted by $(\mathit{\Gamma}, A)\Cap (G,\ B) = (H,\ C)$, where $C = A \cap B \neq \emptyset$ and $\forall\ x{\in}C$, $H(x) = \mathit{\Gamma}(x)\ {\cap}G\ (x)$.

Definition 2.10 [2]: Let $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **AND-product** or **AND-intersection** of $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ denoted by $(\mathit{\Gamma},\ A)\widetilde{\bigwedge}(G,\ B)$ is a soft set defined as:

$$(\mathit{\Gamma},\ A)\widetilde{\bigwedge}(G,\ B) = (H,\ C)\,,\ where\ C = A \times B\ and\ \forall\ (x,\ y) \in A \times B$$

$H(x,\ y) = \mathit{\Gamma}(x)\ {\cap}G(y)$.

Definition 2.11 [2]: Let $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ be two soft sets over U. Then the **OR-product** or **OR-union** of $(\mathit{\Gamma},\ A)$ and $(G,\ B)$, denoted by$(\mathit{\Gamma},\ A)\widetilde{\bigvee}(G,\ B)$ is a soft set defined as $(\mathit{\Gamma},\ A)\widetilde{\bigvee}(G,\ B) = (H, C)\,,\ where\ \ C = A \times B\ ,\ \forall\ (x, y) \in A \times B$ and $H(x,\ y) = \mathit{\Gamma}(x)\ {\cup}G(y)$.

Definition 2.12 [2]: Let $(\mathit{\Gamma},\ A)$ and $(G,\ B)$ be two soft sets over the universes $U_1$and U_2 respectively. Then the **Cartesian product** of $(\mathit{\Gamma},\ A)$ and $(G,\ B)$, denoted by $(\mathit{\Gamma},\ A)\ \widetilde{\times}\ (G,\ B)$ is a soft define as:

$$(\mathit{\Gamma},\ A)\,\widetilde{\times}\,(G,\ B) = (\mathrm{H},\ \mathrm{C})\,,\ \text{where C} = \mathrm{A} \times \mathrm{B}\ \text{and}\ \forall\ \ (x,\ y) \in \mathrm{A} \times \mathrm{B}$$

$$H(x,\ y) =\ \mathit{\Gamma}(x){\times}\ \mathrm{G(y)}.$$

Definition 2.13 [2]: Let $(\Gamma,\ A)$ and $(G,\ B)$ be two nonempty soft sets over U. The **sum** $(\Gamma,\ A)\tilde{+}(G,\ B)$ is define as the soft set $(H,\ C) = (\Gamma,\ A)\tilde{+}(G,\ B)$, where $C = A \times B$ and $H(x, y) = \Gamma(x) + G(y),\ \forall\ (x,\ y) \in C$.

Definition 2.14: Let $(\Gamma_i,\ A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U. The union of these soft sets is defined to be the soft set $(G,\ B)$ such that $B = \cup_{i \in I} A_i$ and for all $x \in B,\ G(x) = \cup_{i \in I(x)} \Gamma_i(x)$ *where* $I(x) = \{i \in I : x \in A_i\}$. In this case we write $\widetilde{\cup}_{\ i \in I}(\Gamma_i,\ \ A_i) = (G,\ B)$.

Definition 2.15 [4]: Let $(\Gamma_i,\ A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U. The AND-soft set $\widetilde{\bigwedge}_{\boldsymbol{i \in I}}(\Gamma_i,\ \ A_i)$ of these soft sets is defined to be the soft set $(H,\ B)$ such that $B = \prod_{i \in I} A_i$ and $H(x) = \cap_{i \in I} \Gamma_i(x_i)$ *for all* $x = (x_i)_{i \in I} \in B$.

Definition 2.16 [20]: Let $(\Gamma_i,\ A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe set U. The OR-soft set $\widetilde{\bigvee}_{\ i \in I}(\Gamma_i,\ \ A_i)$ of these soft sets is defined to be the soft set $(H,\ B)$ such that $B = \prod_{i \in I} A_i$ and $H(x) = \cup_{i \in I} \Gamma_i(x_i)$ for all $x = (x_i)_{i \in I} \in B$.
Note that if $A_i = A$ and $\Gamma_i = \Gamma$ for all $i \in I$, then $\widetilde{\bigwedge}_{\boldsymbol{i \in I}}(\Gamma_i,\ \ A_i)$ (res. $\widetilde{\bigvee}_{\ i \in I}(\Gamma_i,\ \ A_i)$) is denoted by $\widetilde{\bigwedge}_{\boldsymbol{i \in I}}\boldsymbol{(}\Gamma,\ A\boldsymbol{)}$ **(**res. $\widetilde{\bigvee}_{\ i \in I}\boldsymbol{(}\Gamma,\ A\boldsymbol{))}$**.** In this case, $\prod_{i \in I} A_i = \prod_{i \in I} A$ means the direct power A^I.

Definition 2.17 [20]: The restricted union of a nonempty family of soft sets $(\Gamma_i,\ A_i)_{i \in \Delta}$ over a common universe U is defined as the soft set $(H,\ B) = \widetilde{\cup}_{R\ \ i \in \Delta}(\Gamma_i,\ \ A_i)$, where $B = \cap_{i \in \Delta}\ A_i \neq \emptyset$ and $H(x) = \cup_{i \in \Delta} \Gamma_i(x)$ for all $x \in B$.

Definition 2.18 [19]: The extended intersection of a nonempty family of soft sets $(\Gamma_i,\ \ A_i)_{i \in \Delta}$ over a common universe set U is defined as the soft set $(H,\ B) = \widetilde{\cap}_{E\ \ i \in \Delta}(\Gamma_i,\ A_i)$ such that $B = \cup_{i \in \Delta} A_i$ and $H(x) = \cap_{i \in \Delta} \Gamma_i(x)$ where $\Delta(x) = \{i \in \Delta : x \in A_i\}$ for all $x \in B$.

Definition 2.19 [19]: Let $(\Gamma_i,\ A_i)_{i \in \Delta}$ be a nonempty family of soft sets over a common universe set U. The restricted intersection of these soft sets is defined to be the soft set $(G,\ B)$ such that $B = \cap_{i \in \Delta} A_i \neq \emptyset$ and for all $x \in B,\ G(x) = \cap_{i \in \Delta} \Gamma_i(x)$. In this case we write $\Cap_{\boldsymbol{i \in \Delta}}(\Gamma_i,\ \ A_i) = (G,\ B)$.

Definition 2.20 [19]: Let $(\Gamma_i,\ A_i)_{i \in \Delta}$ be a nonempty family of soft sets over $U_i,\ i \in \Delta$. The Cartesian product of $(\Gamma_i,\ A_i)_{i \in \Delta}$ over U_i is defined as the soft set $(H,\ B) = \widetilde{\prod}_{i \in \Delta}(\Gamma_i,\ A_i)$ *where* $B = \prod_{i \in \Delta}\ A_i$ and $H(x) = \prod_{i \in \Delta} \Gamma_i(x_i)$ *for all* $x = (x_i)_{i \in \Delta} \in B$. It is worth noting that if $A_i = A$ and $\Gamma_i = \Gamma$ for all $i \in \Delta$, then $\widetilde{\prod}_{i \in \Delta}(\Gamma_i,\ A_i)$ is denoted by $\prod_{i \in \Delta}(\Gamma,\ A)$. In this case $\prod_{i \in \Delta}(\ A_i) = \prod_{i \in \Delta} A$ means the direct power A^Δ.

Definition 2.21: Let $(\Gamma_i,\ A_i)_{i\in\Delta}$ be a nonempty family of soft sets over $U_i,\ i\in\Delta$. The sum of $(\Gamma_i,\ A_i)_{i\in\Delta}$ over U_i is defined as the soft set
$(H,\ B)=\widetilde{\sum}_{i\in\Delta}(\Gamma_i,\ A_i)$ *where* $B=\prod_{i\in\Delta} A_i$ and $H(x)=\sum_{i\in\Delta}\Gamma_i(x_i)$ *for all* $x=(x_i)_{i\in\Delta}\in B$.

3. Soft Fields

Definition 3.1 [20]: Let F be a field and A be a nonempty set. Let R be an arbitrary binary relation between an elements of A and an elements of F, that is, R is a subset of $A\times F$, mathematically written as $R=\{(x,y)\in A\times F : y\in\Gamma(x)\}$, where Γ is a set-valued function $\Gamma : A\to P(F)$ is defined as:
$\Gamma(x)=\{y\in F : (x,y)\in R\}$ for all $x\in A$. Then the pair $(\Gamma,\ A)$ is a soft set over F, which is derived from the relation R. Let $(\Gamma,\ A)$ be a soft set over F. Then $(\Gamma,\ A)$ is called a soft field over F if and only if $\Gamma(x)$ is a subfield of F denoted by $\Gamma(x)<_F F$ for all $x\in supp(\Gamma,\ A)$.

Example 3.1 [21]: Let $A=\mathbb{Z}^+$and F=$\mathbb{R}$. Consider the function $\Gamma : A\to P(F)$ be defined by $\Gamma(a)=\mathbb{Q}(\sqrt{a})$ for all $a\in A$, where $\mathbb{Q}(\sqrt{a})$ is the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q}$ and $\sqrt{a}$. Then $\Gamma(1)=\mathbb{Q}\left(\sqrt{1}\right)=\mathbb{Q}$, $\Gamma(2)=\mathbb{Q}\left(\sqrt{2}\right)$, $\Gamma(3)=\mathbb{Q}\left(\sqrt{3}\right)$, $\Gamma(4)=\mathbb{Q}\left(\sqrt{4}\right)=2\mathbb{Q}$, $\ldots$, $\Gamma(n)=\mathbb{Q}\left(\sqrt{n}\right)$, $\ldots$ which are all subfields of F. Hence, $(\Gamma,\ A)$ is a soft field over F.

Example 3.2: Let $A=\mathbb{Z}^+$and F=$\mathbb{C}$. Consider the function $\Gamma : A\to P(F)$ be defined by $\Gamma(a)=\mathbb{R}(\sqrt{a^3})$ for all $a\in A$. Then $\Gamma(1)=\mathbb{R}$, $\Gamma(2)=(2\sqrt{2})\mathbb{R}$, $\Gamma(3)=(3\sqrt{3})\mathbb{R}$, $\Gamma(4)=(8)\mathbb{R}$, $\Gamma(5)=(5\sqrt{5})\mathbb{R}$, $\ldots$ which are all subfields of F. Hence, $(\Gamma,\ A)$ is a soft field over F.

Example 3.3: Let $A=\mathbb{Z}^+$and F=$\mathbb{R}(x)$ be a field of fractions of the polynomial with indeterminate x. Consider the function $\Gamma : A\to P(F)$ be defined by $\Gamma(a)=\mathbb{Q}(\sqrt{a})(x)$ for all $a\in A$.Then $\Gamma(1)=\mathbb{Q}(x)$, $\Gamma(2)=\sqrt{2}\mathbb{Q}(x)$, $\Gamma(3)=\sqrt{3}\mathbb{Q}(x)$, $\Gamma(4)=2\mathbb{Q}(x)$, $\Gamma(5)=\sqrt{5}\mathbb{Q}(x)$, $\ldots$ which are all subfield of F. Hence, $(\Gamma,\ A)$ is a soft field over F.

Example 3.4: Let $B=\{p\in\mathbb{Z}^+ : p \textit{ is prime}\}$ and F=$\mathbb{R}$. Consider the function $G : B\to P(F)$ be defined by $G(b)=\mathbb{Q}$ for all $b\in B$. Then $\Gamma(2)=\mathbb{Q}$, $\Gamma(3)=\mathbb{Q}$, $\Gamma(5)=\mathbb{Q}$, $\Gamma(7)=\mathbb{Q}$,... which are all subfields of F. Then (G,B) is a soft field over F.

Example 3.5: Let $F=A=\{0,\ 1,\ a,\ b\}$ be a finite field whose addition and multiplication tables are shown in Table 1 and Table 2

Table 1. Addition table of F

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

Table 2. Multiplication table of F

.	0	1	a	b
0	0	0	0	0
1	0	1	b	a
a	0	a	b	1
b	0	b	1	a

Consider the soft set $(\Gamma,\ A)$ over F, where $\Gamma : A \to P(F)$ is a set valued function defined by $\Gamma(x) = \{y \in F : (x, y) \in R \Longleftrightarrow x + y \bigvee$ x.y=0 for all $x \in A$, where $\bigvee$ refers to Disjunction (OR).
Then $\Gamma(0) = F,\ \Gamma(1) = \{0,\ 1\},\ \Gamma(a) = \{0,\ a\},\ \Gamma(b) = \{0, b\}$, since $\Gamma(a)$ and $\Gamma(b)$ are not subfields of F. Hence, $(\Gamma,\ A)$ is **not** a soft field over F.

Proposition 3.1: Let $(\Gamma,\ A)$ and $(G,\ B)$ be soft fields over F. Then
(i) $(\Gamma,\ A) \Cap (G,\ B)$ is a soft field over F, if it is non-null.
(ii) $(\Gamma,\ A)\widetilde{\cap}_E(G,\ B)$ is a soft field over F, if it is non-null.
(iii) $(\Gamma,\ A)\widetilde{\cup}_R\ (G,\ B)$ is a soft field over F, whenever, it is non-null and if $\Gamma(x)$ and $G(x)$ are ordered by inclusion relation for all $x \in supp((\Gamma,\ A)\widetilde{\cup}_R\ (G,\ B))$.
(iv) $(\Gamma,\ A)\widetilde{\bigwedge}(G,\ B)$ is a soft field over F, if it is non-null.
(v) $(\Gamma,\ A)\widetilde{\cup}(G,\ B)$ is a soft field over F, if A and B are disjoint and if it is non-null.
(vi) $(\Gamma,\ A)\widetilde{\bigvee}(G,\ B) = (N,\ A \times B)$ is a soft field over F, if it is non-null and if $\Gamma(x)$ and $G(y)$ are ordered by inclusion relation for all $(x,\ y) \in supp((N,\ A \times B)$.
(vii) $(\Gamma,\ A)\tilde{+}(G,\ B)$ is a soft field over F, if it is non-null.

Proof: (i) Let $(\Gamma,\ A) \Cap (G,\ B) = (K,\ C)$ where $K(x) = \Gamma(x) \cap G(x)$ for all $x \in C = A \cap B\ \neq \emptyset$. By hypothesis,$(K,\ C)$ is a non-null soft set over F. If $x \in supp(K,\ C)$, then $K(x) = \Gamma(x) \cap G(x) \neq \emptyset$. It follows that $\Gamma(x) \neq \emptyset$ and $G(x) \neq \emptyset$ are both subfields of F. Hence, $K(x)$ is a subfield of F, for all $x \in supp(K,\ C)$.Thus, $(K,\ C)$ is a soft field over F.
(ii) Let $(\Gamma,\ A)\widetilde{\cap}_E(G,\ B) = (M,\ A \cup B)$,

where $$M(x) = \begin{cases} \Gamma(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ \Gamma(x) \cap G(x), & if\ x \in A \cap B \end{cases}$$

For all $x \in A \cup B$. Then by the hypothesis$(M,\ A \cup B)$ is a non-null soft set over F. Let $x \in supp(M,\ A \cup B)$. If $x \in A - B$, then $\emptyset \neq M(x) = \Gamma(x)$. If $x \in B - A$, then $\emptyset \neq M(x) = G(x)$, and if $x \in A \cap B$, then $M(x) = \Gamma(x) \cap G(x) \neq \emptyset$. Since, $\Gamma(x) \neq \emptyset$ and $G(x) \neq \emptyset$, are both subfields of F then $M(x)$ *is a subfield* for all $x \in supp(M, A \cup B)$. Therefore, $(\Gamma, A) \widetilde{\cap}_E (G, B) = (M, A \cup B)$ is a soft field over F.

(iii) Let $(\Gamma, A) \widetilde{\cup}_R (G, B) = (R, A \cap B)$ where $R(x) = \Gamma(x) \cup G(x)$ for all $x \in A \cap B \neq \emptyset$. Then by hypothesis $(R, A \cap B)$ is a non-null soft set over F. if $x \in supp(R, A \cap B)$, $R(x) = \Gamma(x) \cup G(x) \neq \emptyset$. Since, $\Gamma(x)$ and $G(x)$ are ordered by inclusion relation for all $x \in supp(R, A \cap B), \Gamma(x) \cup G(x) = \Gamma(x)$ or $\Gamma(x) \cup G(x) = G(x)$. Since, $\Gamma(x) \neq \emptyset$ and $G(x) \neq \emptyset$ are both subfields of $F, R(x)$ is a subfield of F for all $x \in supp(R, A \cap B)$. Therefore, $(R, A \cap B)$ is a soft field over F.

(iv) Let $(\Gamma, A) \widetilde{\bigwedge} (G, B) = (Q, A \times B)$, where $Q(x, y) = \Gamma(x) \cap G(y)$, for all $(x, y) \in A \times B$. Then by hypothesis, $(Q, A \times B)$ is non-null soft set over F. If $(x, y) \in supp(Q, A \times B)$, then $Q(x, y) = \Gamma(x) \cap G(y) \neq \emptyset$. It follows that $\Gamma(x) \neq \emptyset$ and $G(y) \neq \emptyset$ are both subfields of F. Hence, $Q(x, y)$ is a subfield of F for all $(x, y) \in supp(Q, A \times B)$. Therefore, $(\Gamma, A) \widetilde{\bigwedge} (G, B)$ is a soft field over F.

(v) Let $(\Gamma, A) \widetilde{\cup} (G, B) = (V, A \cup B)$ where

$$V(x) = \begin{cases} \Gamma(x), & if\ x \in A - B, \\ G(x), & if x \in B - A, \\ \Gamma(x) \cup G(x), & if\ x \in A \cap B, \end{cases}$$

For all $x \in A \cup B$,and $A \cap B = \emptyset$, it follows that either $x \in A - B$ or $x \in B - A$, for all $x \in A \cup B$. If $x \in A - B$, then $V(x) = \Gamma(x)$ is a subfield of F and if $x \in B - A$, then $V(x) = G(x)$ is a subfield of F. Therefore, $(\Gamma, A) \widetilde{\cup} (G, B)$ is a soft field over F.

(vi) Let $(\Gamma, A) \widetilde{\bigvee} (G, B) = (N, A \times B)$, where $N(x, y) = \Gamma(x) \cup G(y)$, for all $(x, y) \in A \times B$. Then by hypothesis, $(N, A \times B)$ is a non-null soft set over F. If $(x, y) \in supp(N, A \times B)$, then $N(x, y) = \Gamma(x) \cup G(y) \neq \emptyset$. Since $\Gamma(x)$ and $G(y)$ are ordered by inclusion relation for all $(x, y) \in supp(N, A \times B)$, $\Gamma(x) \cup G(y) = \Gamma(x)$ or $\Gamma(x) \cup G(y) = G(y)$. Since, $\Gamma(x) \neq \emptyset$ and $G(y) \neq \emptyset$ are both subfields of F for all $(x, y) \in supp(N, A \times B)$. Therefore, $(\Gamma, A) \widetilde{\bigvee} (G, B)$ is a soft field over F.

(vii) $(\Gamma, A) \tilde{+} (G, B) = (H, A \times B)$, where $H(x, y) = \Gamma(x) + G(y)$, for all $(x, y) \in A \times B$. Then by the hypothesis, $(H, A \times B)$ is a non-null soft set over F. Suppose $(x, y) \in supp(H, A \times B)$, then $H(x, y) = \Gamma(x) + G(y) \neq \emptyset$. It means that $\Gamma(x) \neq \emptyset$ and $G(y) \neq \emptyset$ are both subfields of F. Hence, $H(x, y)$ is a subfield of F for all $(x, y) \in supp(H, A \times B)$. Therefore, $(\Gamma, A) \tilde{+} (G, B)$ is a soft field over F.

Proposition 3.2: Let $(\Gamma,\ A)$ and $(G,\ B)$ be two soft fields over M and N respectively. If it is non-null, then the product $(\Gamma,\ A)\widetilde{\times}(G,\ B)$ is a soft field over $M\times N$.

Proof: Let $(\Gamma,\ A)\widetilde{\times}(G,\ B)=(J,\ A\times B)$, where $J\,(x,\ y)=\Gamma(x)\times G(y)$, for all $(x,\ y)\in A\times B$. Then by hypothesis, $(J,\ A\times B)$ is non-null soft set over $M\times N$.If $(x,\ y)\in supp(J,\ A\times B)$, then $J\,(x,\ y)=\Gamma(x)\times G(y)\neq\emptyset$. Since, $\Gamma(x)\neq\emptyset$ is a subfield of M and $G(y)\neq\emptyset$ is a subfield of N, it implies that $J\,(x,\ y)$ is a subfield of $M\times N$ for all $(x,\ y)\in supp(J,\ A\times B)$. Therefore, $(\Gamma,\ A)\widetilde{\times}(G,\ B)$ is a soft field over $M\times N$

Proposition 3.3: Let $(\Gamma_i,\ \ A_i)_{i\in I}$ be a nonempty family of soft fields over F. Then the following holds:

(i) $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft field over F, if it is non-null.
(ii) $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft field over F, if it is non-null.
(iii) $\widetilde{\cap}_{\boldsymbol{E\ i\in I}}(\Gamma_i,\ \ A_i)$ is a soft field over F, if it is non-null.
(iv) If $\{\ A_i : i\in I\}$ are pairwise disjoint, that is, $i\neq j$ implies $A_i\cap\ A_j=\emptyset$, then $\widetilde{\cup}_{i\in I}(\Gamma_i,\ \ A_i)$ is a soft field over F.
(v) Let $\Gamma_i\,(x_i)\subseteq\Gamma_j(x_j)$ or $\Gamma_j\,(x_j)\subseteq\Gamma_i(x_i)$, for all $i,\ j\in I$ and $x_i\in A_i$, then $\widetilde{\cup}_{R\ i\in I}(\Gamma_i,\ \ A_i)$ is a soft field over F, whenever it is non-null.
(vi) Let $\Gamma_i\,(x_i)\subseteq\Gamma_j(x_j)$ or $\Gamma_j\,(x_j)\subseteq\Gamma_i(x_i)$, for all $i,\ j\in I$ and $x_i\in A_i$, then $\widetilde{\bigvee}_{\ i\in I}(\Gamma_i,\ \ A_i)$ is a soft field over F, whenever it is non-null.
(vii) $\widetilde{\sum}_{i\in I}(\Gamma_i,\ \ A_i)$ is a soft field over F, whenever it is non-null.

Proof: (i) Let $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$=$(H,\ C)$, where $C=\prod_{i\in I}A_i$, for all $x=(x_i)\in C$, we have $H\,(x)=\cap_{i\in I}\Gamma_i\,(x_i)$. By the hypothesis $(H,\ C)$ is a non-null soft set over F. If $x=(x_i)_{i\in I}\in supp(H,\ C)$. Then, $H(x)$= $\cap_{i\in I}\Gamma_i\,(x_i)\neq\emptyset$, so we have $\Gamma_i\,(x_i)\neq\emptyset$ for all $i\in I$, since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i\in I$, we have $\cap_{i\in I}\Gamma_i\,(x_i)$ are subfields of F, for all $x=(x_i)_{i\in I}\in C$. Therefore, $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft field over F.

(ii) Let $\Cap_{\boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i)=(G,\ B)$, where $B=\ \cap_{i\in\Delta}A_i\neq\emptyset$ and for all $x=(x_i)_{i\in\Delta}\in B$, $G\,(x)=\ \cap_{i\in\Delta}\Gamma_i\,(x)$.

Suppose that $(G,\ B)$ is a non-null soft set over F. If $x\in supp(G,\ B)$, then $G\,(x)=\ \cap_{i\in\Delta}\Gamma_i\,(x)\neq\emptyset$, so we have $\Gamma_i\,(x)\neq\emptyset$ for all $i\in\Delta$, since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i\in\Delta$. It follows that, $\cap_{i\in\Delta}\Gamma_i\,(x_i)$ are subfields of F, for all $x=(x_i)_{i\in I}\in B$. Therefore, $\Cap_{\boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i)$ is a soft field over F.

(iii) Let $\widetilde{\cap}_{\boldsymbol{E\ i\in I}}\,(\Gamma_i,\ \ A_i)=(M,\ C)$, where $C=\cup_{i\in\Delta}A_i$ and $M\,(x)=\cap_{i\in\Delta}\Gamma_i\,(x)$ where $\Delta\,(x)=\{i\in\Delta : x\in A_i\}$ for all $x\in C=\cup_{i\in\Delta}A_i$. Suppose that $(M,\ C)$ is a non-null soft set over F. If $x\in supp(M,\ C)$, then $M\,(x)=\cap_{i\in\Delta}\Gamma_i\,(x)\neq\emptyset$, so, we have $\Gamma_i\,(x)\neq\emptyset$ for all $i\in\Delta$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i\in\Delta$. It follows that, $\cap_{i\in\Delta}\Gamma_i\,(x)$ are subfields of F for all $x\in C$. Therefore, $\widetilde{\cap}_{\boldsymbol{E\ i\in I}}\,(\Gamma_i,\ \ A_i)$ is a soft field over F.

(iv) Let $\widetilde{\cup}_{i\in I}(\Gamma_i,\ A_i)=(G,\ B)$, where $B=\cup_{i\in I}A_i$ and for all $x=(x_i)_{i\in I}\in B$, $G(x)=\ \cup_{i\in I}\Gamma_i(x)$ where $I(x)=\{i\in I: x\in A_i\}$. Then by the hypothesis, $(G,\ B)$ is a non-null soft set over F. If $x\in supp(G,\ B)$, then $G(x)=\cup_{i\in I}\Gamma_i(x)\neq\emptyset$. Since, $\{\ A_i: i\in I\}$ are pairwise disjoint, that is, $i\neq j$ implies $A_i\cap\ A_j=\emptyset$, so we have $\Gamma_i(x)\neq\emptyset$ for all $i\in I(x)$. Since $(\Gamma_i,\ A_i)$ is a family of soft fields over F for all $i\in I(x)$. Therefore, $\widetilde{\cup}_{i\in I}(\Gamma_i,\ A_i)$ is a soft field over F.

(v) Let $\widetilde{\cup}_{R\ i\in I}(\Gamma_i,\ A_i)=(H,\ B)$, where $B=\cap_{i\in I}A_i\neq\emptyset$ and $H(x)=\cup_{i\in I}\Gamma_i(x)$ for all $x=(x_i)_{i\in I}\in B$. By the hypothesis $(H,\ B)$ is a non-null soft set over F. If $x\in supp(H,\ B)$, then $H(x)=\ \cup_{i\in I}\Gamma_i(x)\neq\emptyset$ for all $x\in B$. Since, $\Gamma_i(x_i)\subseteq\Gamma_j(x_j)$ or $\Gamma_j(x_j)\subseteq\Gamma_i(x_i)$, for all $i,\ j\in I$ and $x_i\in A_i$, so we have $\Gamma_i(x)\neq\emptyset$ for all $i\in I$. Since $(\Gamma_i,\ A_i)$ is a family of soft fields over F for all $i\in I$. It follows that $\cup_{i\in I}\Gamma_i(x)$ are subfields of F for all $x\in B$. Therefore, $\widetilde{\cup}_{R\ i\in I}(\Gamma_i,\ A_i)$ is a soft field over F.

(vi) Let $\widetilde{\bigvee}_{i\in I}(\Gamma_i,\ A_i)$=$(M,\ C)$, where $C=\prod_{i\in I}A_i\neq\emptyset$ and $M(x)=\cup_{i\in I}\Gamma_i(x_i)$ for all $x=(x_i)_{i\in I}\in C$. By the hypothesis, $(M,\ C)$ is a non-null soft set over F. If $x\in supp(M,\ C)$, then $M(x)=\ \cup_{i\in I}\Gamma_i(x_i)\neq\emptyset$ for all $x=(x_i)_{i\in I}\in C$. Since, $\Gamma_i(x_i)\subseteq\Gamma_j(x_j)$ or $\Gamma_j(x_j)\subseteq\Gamma_i(x_i)$, for all $i,\ j\in I$ and $x_i\in A_i$, so we have $\Gamma_i(x_i)\neq\emptyset$ for all $i\in I$. Since $(\Gamma_i,\ A_i)$ is a family of soft fields over F for all $i\in I$, it follows that $\cup_{i\in I}\Gamma_i(x_i)$ are subfields of F for all $x\in C$. Therefore, $\widetilde{\bigvee}_{i\in I}(\Gamma_i,\ A_i)$ is a soft field over F.

(vii) Let $\widetilde{\sum}_{i\in I}(\Gamma_i,\ A_i)=(G,\ B)$, where $B=\prod_{i\in I}A_i$ and $G(x)=\sum_{i\in I}\Gamma_i(\ x_i)$ for all $x=(x_i)_{i\in I}\in B$. By the hypothesis $(G,\ B)$ is a non-null soft set over F. If $x\in supp(G,\ B)$, then $G(x)=\sum_{i\in I}\Gamma_i(\ x_i)\neq\emptyset$ for all $x\in B$. So, we have $\Gamma_i(\ x_i)\neq\emptyset$ for all $i\in I$. Since $(\Gamma_i,\ A_i)$ is a family of soft fields over F for all $i\in I$, it follows that $\sum_{i\in I}\Gamma_i(\ x_i)$ are subfields of F for all $x\in B$. Therefore, $\widetilde{\sum}_{i\in I}(\Gamma_i,\ A_i)$ is a soft field over F.

Proposition 3.4: Let $(\Gamma_i,\ A_i)_{i\in I}$ be a nonempty family of soft field over F_i. Then $\widetilde{\prod}_{i\in I}(\Gamma_i,\ A_i)$ is a soft field over $\prod_{i\in I}\ F_i$.

Proof: We can write $(H,\ B)=\ \widetilde{\prod}_{i\in I}(\Gamma_i,\ A_i)$, where $B=\ \prod_{i\in I}A_i$ and $H(x)=\prod_{i\in I}\Gamma_i(x_i)$ for all $x=\ (x_i)_{i\in I}\in B$. Let $x=(x_i)_{i\in I}\in supp(H,\ B)$. Then, $H(x)=\prod_{i\in I}\Gamma_i(x_i)\neq\emptyset$, so we have $\Gamma_i(x_i)\neq\emptyset$ for all $i\in I$. Since $(\Gamma_i,\ A_i)$ is a family of soft fields over F_i, for all $i\in I$ we have $\prod_{i\in I}\Gamma_i(x_i)$ is a subfield of $\prod_{i\in I}\ F_i$, for all $x=\ (x_i)_{i\in I}\in B$. That is, the Cartesian product $\widetilde{\prod}_{i\in I}(\Gamma_i,\ A_i)$ is a soft field over $\prod_{i\in I}\ F_i$.

Definition 3.4 [20]: Let $(\Gamma,\ A)$ and $(G,\ B)$ be two soft fields over F. Then $(\Gamma,\ A)$ is a soft subfield of $(G,\ B)$ written $(\Gamma,\ A)\tilde{<}_F(G,\ B)$ if the following conditions hold:

(i) $A\subseteq B$,

(ii) $\Gamma(x) <_F G(x)$, for all $x \in supp(\Gamma,\ A)$. (The notation $\tilde{<}_F$ refers to soft subfield).

Definition 3.5 [20]: Let $(\Gamma,\ A)$ be a soft field over F. Then,
(i) $(\Gamma,\ A)$ is called trivial if $\Gamma(x) = \{0_F\}$ (*the zero element of* F) for all $x \in supp(\Gamma,\ A)$.
(ii) $(\Gamma,\ A)$ is said to be whole if $\Gamma(x) = F$ for all $x \in supp(\Gamma,\ A)$.

Example 3.7: Let $F = B = \{0,\ 1,\ a,\ b\}$ and $A = \{0, 1\} \subset B$and $\Gamma : A \to P(F)$, if we define the set-valued function

$\Gamma(x) = \{y \in F : (x,\ y) \in R \Longleftrightarrow x + y \bigwedge x \bullet y = 0\}$ *for all* $x \in A$, Then $\Gamma(x)$ is a subfield of F, for all $x \in supp(\Gamma,\ A)$.(see Tables 5 and 6) Thus, conditions (i) and (ii) in definition 3.5 are satisfied. Hence, $(\Gamma,\ A)$ is a soft subfield of $(G,\ B)$.

Table 6. Addition table of subfield

+	0	1
0	0	1
1	1	0

Table 7.Multiplication table of subfield

•	0	1
0	0	0
1	0	1

Proposition 3.5: Let $(\Gamma,\ A)$,$(G,\ A)$ and $(H,\ B)$ be soft fields over F. Then the following hold:

(1) If $\Gamma(x) \subseteq G(x)$ for all $x \in A$, then $(\Gamma,\ A)$ is a soft subfield of $(G,\ A)$.
(2) $(\Gamma,\ A) \Cap (H,\ B)$ is a soft subfield of both $(\Gamma,\ A)$ *and* $(H,\ B)$, if it is non- null.
(3) $(\Gamma,\ A) \widetilde{\cap}_E (G,\ A)$ is a soft subfield of both $(\Gamma,\ A)$ *and* $(G,\ A)$ if it is non-null.

Proof: (1) Since $\Gamma(x) \subseteq G(x)$ for all $x \in A$ and $A \cap A = A \subseteq A$ it is obvious that $\Gamma(x)$ is a subfield of $G(x)$, denoted by $\Gamma(x) <_F G(x)$. Therefore, $(\Gamma,\ A)$ is a soft subfield of $(G,\ A)$. (the notation $<_F$ refers to subfield)

(2) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, the first condition of definition 3.5 is satisfied. Let $(\Gamma,\ A) \Cap (H,\ B) = (K,\ C)$, where $C = A \cap B$ and $K(x) = \Gamma(x) \cap H(x)$ for all $x \in C$. Since $K(x) = \Gamma(x) \cap H(x) \subseteq \Gamma(x)$ and $K(x) = \Gamma(x) \cap H(x) \subseteq H(x)$ for all $x \in C$. Therefore, $(\Gamma,\ A) \Cap (H,\ B)$ is a soft subfield of both $(\Gamma,\ A)$ *and* $(H,\ B)$ (see definition 3.5).

(3) Since $A \cap A = A \subseteq A$.*Let* $(\Gamma,\ A) \widetilde{\cap}_E (G,\ A) = (Q,\ A)$ where $Q(x) = \Gamma(x) \cap G(x)$ for all $x \in A$. Since $Q(x) = \Gamma(x) \cap G(x) \subseteq \Gamma(x)$ and $Q(x) = \Gamma(x) \cap G(x) \subseteq G(x)$ for all $x \in A$. Therefore, $(\Gamma,\ A) \widetilde{\cap}_E (G,\ A)$ is a soft subfield of both $(\Gamma,\ A)$ *and* $(G,\ A)$ (see definition 3.5).

Proposition 3.6: Let $(\Gamma,\ A)$ be a soft field over F and $(\Gamma_i,\ A_i)_{i\in I}$ be a nonempty family of soft subfields of $(\Gamma,\ A)$. Then the following results hold:

(i) $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, if it is non-null;

(ii) $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, if it is non-null;

(iii) $\widetilde{\cap}_{\boldsymbol{E}\ \boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, if it is non-null;

(iv) If $\{\ A_i : i \in I\}$ are pairwise disjoint, that is, $i \neq j$ implies $\ A_i \cap\ A_j = \emptyset$, then $\widetilde{\cup}_{i\in I}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$;

(v) Let $\Gamma_i\,(x_i) \subseteq \Gamma_j(x_j)$ or $\Gamma_j\,(x_j) \subseteq \Gamma_i(x_i)$, for all $i,\ j \in I$ and $x_i \in A_i$, then $\widetilde{\cup}_{R\ i\in I}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, whenever it is non-null;

(vi) Let $\Gamma_i\,(x_i) \subseteq \Gamma_j(x_j)$ or $\Gamma_j\,(x_j) \subseteq \Gamma_i(x_i)$, for all $i,\ j \in I$ and $x_i \in A_i$, then $\widetilde{\bigvee}_{\ i\in I}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, whenever it is non-null;

(vii) $\widetilde{\sum}_{i\in I}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$, whenever it is non-null;

(viii) $\widetilde{\prod}_{i\in I}\ (\Gamma_i,\ A_i)$ is a soft subfield $of\,(\Gamma,\ A)$, whenever it is non-null.

Proof: (i) Let $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i) =\ (K,\ C)$, where $C = \prod_{i\in I} A_i$ and $K\,(x) = \cap_{i\in I}\Gamma_i\,(x_i)$. for all $x = (x_i)_{i\in I} \in C$. Suppose that $(K,\ C)$ is a non-null soft set over F. If $x \in supp(K,\ C)$, then $K\,(x) = \cap_{i\in I}\Gamma_i\,(x) \neq \emptyset$. Thus, we have $\Gamma_i\,(x) \neq \emptyset$ for all $i \in I$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in I$. It follows that, $C = \prod_{i\in I} A_i \neq \emptyset$, It means that C is nonempty and $C \subseteq A$, meaning every $x \in C$ implies $x \in A$. Also, $K\,(x) = \cap_{i\in I}\Gamma_i\,(x_i) <_F \Gamma(x)$, which means $K\,(x)$ is a subfield of $\Gamma(x)$ .Therefore, $\widetilde{\bigwedge}_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$.

(ii) Let $\Cap_{\boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i) = (H,\ C)$, where $C = \cap_{i\in\Delta} A_i \neq \emptyset$ for all $x = (x_i)_{i\in\Delta} \in C$, $H\,(x) = \cap_{i\in\Delta}\Gamma_i\,(x)$. By the hypothesis $(H,\ C)$ is a non-null soft set over F. If $x \in supp(H,\ C)$, then $H\,(x) = \cap_{i\in\Delta}\Gamma_i\,(x) \neq \emptyset$, so we have $\Gamma_i\,(x) \neq \emptyset$ for all $i \in \Delta$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in \Delta$. It follows that, $C = \cap_{i\in\Delta} A_i \neq \emptyset$, it implies that C is nonempty and $C \subseteq A$, means every $x \in C$ implies $x \in A$. Also $H\,(x) = \cap_{i\in\Delta}\Gamma_i\,(x) <_F \Gamma(x)$,which means $H\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(H,\ C)$. Therefore, $\Cap_{\boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma,\ A)$.

(iii) Let $\widetilde{\cap}_{\boldsymbol{E}\ \boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i) = (G,\ B)$, where $B = \cup_{i\in\Delta} A_i$ and $G\,(x) = \cap_{i\in\Delta}\Gamma_i(x)$ where $\Delta\,(x) = \{i \in \Delta : x \in A_i\}$ for all $x \in B = \cup_{i\in\Delta} A_i$. By the hypothesis $(G,\ B)$ is a non-null soft set over F. If $x \in supp(G,\ B)$, then $G\,(x) = \cap_{i\in\Delta}\Gamma_i(x) \neq \emptyset$. Since, $\{\ A_i : i \in I\}$ are pairwise disjoint, that is, $i \neq j$ implies $\ A_i \cap\ A_j = \emptyset$, so we have $\Gamma_i(x) \neq \emptyset$ for all $i \in \Delta$, since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in \Delta$. It follows that, $B = \cup_{i\in\Delta} A_i \neq \emptyset$, it means B is nonempty and $B \subseteq A$, it means $x \in B \to x \in A$ for all $x \in B$. Also, $G\,(x) = \cap_{i\in\Delta}\Gamma_i\,(x) <_F \Gamma(x)$, which means $G\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(G,\ B)$. Therefore, $\widetilde{\cap}_{\boldsymbol{E}\ \boldsymbol{i\in I}}\,(\Gamma_i,\ \ A_i)$ is a soft subfield of (Γ, A).

(iv) Let $\widetilde{\cup}_{i\in I}\,(\Gamma_i,\ \ A_i) = (K,\ C)$, where $C = \cup_{i\in I} A_i$ and for all $x \in C$, $K\,(x) = \cup_{i\in I(x)}\Gamma_i(x)$ where $I\,(x) = \{i \in I : x \in A_i\}$. By the hypothesis, $(K,\ C)$

is non-null soft set over F. If $x \in supp(K,\ C)$, then $K\,(x) = \cup_{i\in I(x)}\Gamma_i(x) \neq \emptyset$, so we have $\Gamma_i(x) \neq \emptyset$ for all $i \in I(x)$. It follows that, $C = \cup_{i\in I}A_i \neq \emptyset$. It means C is nonempty and $C \subseteq A$, this means $x \in B \to x \in A$ for all $x \in C$. Also, $K\,(x) = \cup_{i\in I(x)}\Gamma_i\,(x) <_F \Gamma(x)$, which means $K\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(K,\ C)$. Therefore, $\widetilde{\cup}_{i\in I}\,(\Gamma_i,\ \ A_i)$ is a soft subfield of (Γ, A).

(v) Let $\widetilde{\cup}_{R\ i\in I}\,(\Gamma_i,\ \ A_i) = (H,\ C)$, where $C = \cap_{i\in I}A_i$ and $H\,(x) = \cup_{i\in I(x)}\Gamma_i(x)$ for all $x \in C$. Then by the hypothesis $(H,\ C)$ is a non-null soft set over F. If $x \in supp(H,\ C)$, then $H\,(x) = \cup_{i\in I(x)}\Gamma_i(x) \neq \emptyset$. Since, $\Gamma_i\,(x_i) \subseteq \Gamma_j(x_j)$ or $\Gamma_j\,(x_j) \subseteq \Gamma_i(x_i)$, for all $i,\ j \in I$ and $x_i \in A_i$, so we have $\Gamma_i(x) \neq \emptyset$ for all $i \in I$. It follows that $C = \cap_{i\in I}A_i \neq \emptyset$, this means C is nonempty and $C \subseteq A$, it means that $x \in C \to x \in A$ for all $x \in C$. Also, $H\,(x) = \cup_{i\in I(x)}\Gamma_i(x) <_F \Gamma(x)$, it means that $H\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(H,\ C)$. Therefore, $\widetilde{\cup}_{R\ i\in I}\,(\Gamma_i,\ \ A_i)$ is a soft subfield of (Γ, A).

(vi) Let $\widetilde{\bigvee}_{i\in I}(\Gamma_i,\ \ A_i) = (M,\ D)$, where $D = \prod_{i\in I} A_i$ and $M\,(x) = \cup_{i\in I}\Gamma_i(x_i)$ for all $x = (x_i) \in D$. Then by the hypothesis $(M,\ D)$ is a non-null soft set over F. If $x \in supp(M,\ D)$, then $M\,(x) = \cup_{i\in I}\Gamma_i(x_i) \neq \emptyset$. Since, $\Gamma_i\,(x_i) \subseteq \Gamma_j(x_j)$ or $\Gamma_j\,(x_j) \subseteq \Gamma_i(x_i)$, for all $i,\ j \in I$ and $x_i \in A_i$, so we have $\Gamma_i(x_i) \neq \emptyset$ for all $i \in I$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in I$. It follows that, $D = \prod_{i\in I} A_i \neq \emptyset$, it means D is nonempty and $D \subseteq A$, this implies that for $x \in D \to x \in A$ for all $x \in D$. Also, $M\,(x) = \cup_{i\in I}\Gamma_i(x_i) <_F \Gamma(x)$, which means that $M\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(M,\ D)$. Therefore, $\widetilde{\bigvee}_{i\in I}(\Gamma_i,\ \ A_i)$ is a soft subfield of (Γ, A).

(vii) Let $\widetilde{\sum}_{i\in I}\,(\Gamma_i, \{\ A_i) = (W,\ C)$, where $C = \prod_{i\in I} A_i$ and $W\,(x) = \sum_{i\in I}\Gamma_i(x_i)$ for all $x = (x_i)_{i\in I} \in C$. Then by the hypothesis $(W,\ C)$ is a non-null soft set over F. If $x \in supp(W,\ C)$, then $W\,(x) = \sum_{i\in I}\Gamma_i(x_i) \neq \emptyset$ for all $i \in I$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in I$. It follows that, $C = \prod_{i\in I} A_i \neq \emptyset$. It means C is nonempty and $C \subseteq A$. It means $x \in C \to x \in A$ for all $x \in C$. Also, $W\,(x) = \sum_{i\in I}\Gamma_i(x_i) <_F \Gamma(x)$, which means that $W\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(W,\ C)$. Therefore, $\widetilde{\sum}_{i\in I}\,(\Gamma_i,\ \ A_i)$ is a soft subfield of (Γ, A).

(viii) Let $\widetilde{\prod}_{i\in I}\ (\Gamma_i,\ A_i) = (H,\ B)$, where $B = \prod_{i\in\Delta} A_i$ and $H\,(x) = \prod_{i\in\Delta}\Gamma_i(x_i)$ for all $x = (x_i)_{i\in\Delta} \in B$. Then by the hypothesis $(H,\ B)$ is a non-null soft set over F. If $x \in supp(H,\ B)$, then $H\,(x) = \prod_{i\in\Delta}\Gamma_i(x_i) \neq \emptyset$, so we have $\Gamma_i(x_i) \neq \emptyset$ for all $i \in \Delta$. Since $(\Gamma_i,\ \ A_i)$ is a family of soft fields over F for all $i \in \Delta$. It follows that, $B = \prod_{i\in\Delta} A_i \neq \emptyset$, it means that B is nonempty and $B \subseteq A$. Also, $H\,(x) = \prod_{i\in\Delta}\Gamma_i(x_i) <_F \Gamma(x)$, which means $H\,(x)$ is a subfield of $\Gamma(x)$ for all $x \in supp(H,\ B)$. Therefore, $\widetilde{\prod}_{i\in I}\ (\Gamma_i,\ A_i)$ is a soft subfield of (Γ, A).

Proposition 3.7: Let $(\Gamma,\ A)$ be a soft field over F and $(\Gamma_i,\ \ A_i)_{i\in I}$ be a nonempty family of soft subfield of $(\Gamma,\ A)$. Then $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ \ A_i)$ is a soft subfield of $(\Gamma_i,\ \ A_i)$ for each $i \in I$, if it is non-null.

Proof: Let $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ A_i)=(H,\ C)$, where $C=\cap_{\boldsymbol{i\in I}} A_i\neq\emptyset$ and $H(x)=\cap_{\boldsymbol{i\in I}}\ \Gamma_i(x)$ for all $x\in C$. The parameter set of the soft set $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ A_i)$, that is $\cap_{\boldsymbol{i\in I}} A_i$ is a subset of the parameter set of the soft set $(\Gamma_i,\ A_i)_{i\in I}$ for all $i\in I$. Then by the hypothesis $(H,\ C)$ is a non-null soft set over F. If $x\in supp(H,\ C)$, then $H(x)=\cap_{\boldsymbol{i\in I}}\ \Gamma_i(x)\neq\emptyset$. Therefore, $\Gamma_i(x)\neq\emptyset$ are subfields of F for all $i\in I$. Thus, $H(x)=\cap_{\boldsymbol{i\in I}}\ \Gamma_i(x)$ is a subfield of F. Moreover, since $\cap_{\boldsymbol{i\in I}}\ \Gamma_i(x)\subseteq\ \Gamma_i(x)$, for all $i\in I$ and for all $x\in\cap_{\boldsymbol{i\in I}} A_i$. Hence, $\cap_{\boldsymbol{i\in I}}\ \Gamma_i(x)<_F\ \Gamma_i(x)$, for all $x\in\cap_{\boldsymbol{i\in I}} A_i$. Therefore, $\Cap_{\boldsymbol{i\in I}}(\Gamma_i,\ A_i)$ is a soft subfield of $(\Gamma_i,\ A_i)$ for each $i\in I$.

Proposition 3.8: Let $(\Gamma,\ A)$ and $(G,\ B)$ be two soft fields over F. Then,

(i) If $(\Gamma,\ A)$ and $(G,\ B)$ are trivial soft fields over F, then $(\Gamma,\ A)\Cap\ (G,\ B)$ is a trivial soft field over F.

(ii) If $(\Gamma,\ A)$ and $(G,\ B)$ are whole soft field over F, then $(\Gamma,\ A)\Cap\ (G,\ B)$ is a whole soft field over F.

(iii) If $(\Gamma,\ A)$ is a trivial soft field over F and $(G,\ A)$ is a whole soft field over F, then $(\Gamma,\ A)\Cap\ (G,\ B)$ is a trivial soft field over F.

(iv) If $(\Gamma,\ A)$ and $(G,\ B)$ are trivial soft fields over F, then $(\Gamma,\ A)\tilde{+}\ (G,\ B)$ is a trivial soft field over F.

(v) If $(\Gamma,\ A)$ and $(G,\ B)$ are whole soft field over F, then $(\Gamma,\ A)\tilde{+}\ (G,\ B)$ is a whole soft field over F.

(vi) If $(\Gamma,\ A)$ is a trivial soft field over F and $(G,\ B)$ is a whole soft soft field over F, then $(\Gamma,\ A)\tilde{+}\ (G,\ B)$ is a whole soft field over F.

Proof: (i) Suppose that, $(\Gamma,\ A)\Cap\ (G,\ B)=(H,\ C)$, where $C=A\cap B$ and for all $x\in C$, $H(x)=\Gamma(x)\cap G(x)$. Since $(\Gamma,\ A)$ and $(G,\ B)$ are trivial soft fields over F, it implies that $\Gamma(x)=\{0_F\}$ for all $x\in supp(\Gamma,\ A)$and $G(x)=\{0_F\}$ for all $x\in supp(G,\ B)$. Thus, $H(x)=\Gamma(x)\cap G(x)=\{0_F\}\cap\{0_F\}=\{0_F\}$ is trivial, $\forall x\in supp(H,\ A\cap B)$. Therefore, $(\Gamma,\ A)\Cap\ (G,\ B)$ is a trivial soft field over F.

($\{0_F\}$ refers to set of zero elements of the field F)

(ii) Suppose, $(\Gamma,\ A)\Cap\ (G,\ B)=(K,\ C)$, where$C=A\cap B$, for all $x\in C$, $K(x)=\Gamma(x)\cap G(x)$. Since $(\Gamma,\ A)$and $(G,\ B)$ are whole soft fields over F, it follows that $\Gamma(x)=F$, for all $x\in supp(\Gamma,\ A)$ and $G(x)=F$, for all $x\in supp(G,\ B)$. This implies that $K(x)=\Gamma(x)\cap G(x)=F\cap F=F$, for all $\forall x\in supp(K,\ A\cap B)$. Therefore, $(\Gamma,\ A)\Cap\ (G,\ B)$ is a whole soft field over F.

(iii) Let $(\Gamma,\ A)\Cap\ (G,\ B)=(T,\ N)$, where $C=A\cap B$, for all $x\in C$, $T(x)=\Gamma(x)\cap G(x)$. Since $(\Gamma,\ A)$ is a trivial soft field over F, it implies that $\Gamma(x)=\{0_f\}$, $\forall x\in supp(\Gamma,\ A)$ and $(G,\ B)$ is a whole soft field over F, then $G(x)=F$, for all $\forall x\in supp(G,\ B)$. It means that $T(x)=\Gamma(x)\cap G(x)=\{0_F\}\cap F=\{0_F\}$. Therefore, $(\Gamma,\ A)\Cap\ (G,\ B)$ is a trivial soft field over F.

(iv) Let $(\Gamma,\ A)\tilde{+}\ (G,\ B)=(W,\ A\times B)$, where $W(x,\ y)=\Gamma(x)+G(y)$, for all $(x,\ y)\in A\times B$. Let $(W,\ A\times B)$ be a non-null soft set over F. If $(x,\ y)\in$

$supp(W,\ A\times B)$, then $W(x,\ y)=\varGamma(x)+G(y)\neq\emptyset$. It means that, $\varGamma(x)=\{0_F\}$ is a trivial soft field over F, for all $x\in supp(\varGamma,\ A)$ and $G(y)=\{0_F\}$ is a trivial soft field over F, then $W(x,\ y)=\varGamma(x)+G(y)=\{0_F\}+\{0_F\}=\{0_F\}$, for all $x\in supp(W,\ A\times B)$. Therefore, $(\varGamma,\ A)\tilde{+}\ (G,\ B)$ is a trivial soft field over F.

(v) Let $(\varGamma,\ A)\tilde{+}\ (G,\ B)=(\ M,\ A\times B)$, where $M(x,\ y)=\varGamma(x)+G(y)$, for all $(x,\ y)\in A\times B$. Let, $(M,\ \ A\times B)$ be a non-null soft set over F. If $(x,\ y)\in supp(M,\ A\times B)$, then $M(x,\ y)=\varGamma(x)+G(y)\neq\emptyset$. Since, $(\varGamma,\ A)$and $(G,\ B)$ are whole soft fields, it means that $\varGamma(x)=F$ and $G(y)=F$. It implies that, $M(x,\ y)=\varGamma(x)+G(y)=F+F=F$ is a trivial soft field over F. Therefore, $(\varGamma,\ A)\tilde{+}\ (G,\ B)$ is a whole soft field over F.

(vi) Let $(\varGamma,\ A)\tilde{+}\ (G,\ B)=(\ T,\ \ A\times B)$, where $T(x,\ y)=\varGamma(x)+G(y)$, for all $(x,\ y)\in A\times B$. Let, $(T,\ \ A\times B)$ be a non-null soft set over F. If $(x,\ y)\in supp(T,\ A\times B)$, then $T(x,\ y)=\varGamma(x)+G(y)\neq\emptyset$. Since, $(\varGamma,\ A)$ is a trivial soft field over F, it implies that for all $x\in supp(\Gamma,\ A)$, $\Gamma(x)=\ \{0_F\}$ similarly$(G,\ B)$ is whole soft fields, it means for all $x\in supp(G,\ B)$, and $G(y)=F$.

It implies that, $T(x,\ y)=\Gamma(x)+G(y)=\{0_F\}+F=F$ is a whole soft field over F. Therefore, $(\Gamma,\ A)\overset{\sim}{+}\ (G,\ B)$ is a whole soft field over F.

4. Conclusion

In this paper, basic concept of soft set theory and algebraic fields were reviewed. The definition of soft fields and soft subfields with illustrative examples were presented. Important propositions on operations on family of soft fields were stated and proved. We also defined soft subfield and proved some important results.

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