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A NOTE ON A CHARACTERIZATION OF LOCAL NULL SEQUENCE

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Abstract

We replace an arbitrary sequence of positive real numbers by a sequence of positive integers in a characterization of local null sequence, and consequently procure an example of a separated locally convex space in which convergence is Mackey.

1. Introduction

Our terminology shall be standard as found, for example, in $[1, 2, 8, \&3]$ signifies the end or absence of a proof.
All topological vector spaces (E, τ) shall be over the field $K = \mathbb{R}$ or $\mathbb{C}$, the reals or the complex numbers; (E, τ) is called locally convex if it has a base of convex neighborhood of zero. We denote the zero of E by θ and that of its scalar field K by $0[1, p.47]$. Of course $\mathbb{R}$ and $\mathbb{C}$ with their usual topologies are locally convex spaces in their own right. By a $lcs(E, \tau)$ we shall mean a separated locally convex space.
If τ_1, τ_2 are topologies on $X \neq \phi$, by $\tau_1 \leq \tau_2$ we shall mean that τ_1 is coarser than τ_2 and for $E \subseteq X$ and τ a topology on X, by $\tau|E$ we shall mean the topology induced on E by τ.
If E is a vector space and $p : E \longrightarrow \mathbb{R}$ a seminorm on E, following Wilansky $[8, p.38]$, we shall denote the pseudometric topology of p by $\sigma p.\sigma p$ is a vector topology [8,**Example** 4.1.8, p.38], indeed, $(E, \sigma p)$ is a locally convex space [8,

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Problem 7.2.1, p.97].

2. SOME ELEMENTARY FACTS

We note some simple facts which we shall employ, at times without citation, in a number of places. Let (E, τ) be a topological vector space. Suppose $\phi \neq A \subseteq E$. A is called balanced if $tA \subseteq A$ for all $t \in K$ with $|t| \leq 1$; called convex if $rA + sA \subseteq A$ for all $r, s \in K, 0 \leq r \leq 1, 0 \leq s \leq 1$ and $r + s = 1$; and called absolutely convex if it is both balanced and convex or equivalently $[5, p.4]$ if $rA + sA \subseteq A$ for all $r, s \in K$ such that $|r| + |s| \leq 1$.

FACT 1:

Let (E, τ) be a topological vector space, and suppose $\phi \neq A \subseteq E$. If A is balanced, then

(i) $|\mu|A \subseteq |s|A$, for $\mu, s \in K, |\mu| \leq |s|$, and

(ii) $\mu A = |\mu|A$, for all scalar μ . By (i) and (ii) therefore,

(ii)$\mu A \subseteq sA$, for $|\mu| \leq |s|$[1(17.2), p.68] If A is a convex set, then

(iii) $\lambda A + \mu A = (\lambda + \mu)A$, for $\lambda > 0, \mu > 0$. If A is absolutely convex, then

(iv) for scalars $\lambda \neq 0, \mu \neq 0, \lambda A + \mu A = |\lambda|A + |\mu|A = (|\lambda| + |\mu|)A$.

Proof:

(i): For $s = 0$, the result is clearly true. Also, if $\mu = 0$, the result is true since A is balanced and so contains the zero 0 of the space. So, suppose $\mu \neq 0, s \neq 0$, and consider $(\frac{|\mu|}{|s|})A$. Since A is balanced and $\frac{|\mu|}{|s|} \leq 1$, it follows that$(|\mu||s|)A \subseteq A$, from which follows that $|\mu|A \subseteq |s|A$.

(ii): [8, **Problem** 1.5.5, p.9].

(iii): [8 ,**Problem** 1.5, 3, p.9][6, **Theorem** 10.1 p.100][4, (v) of **Theorem** 13.6, p.135][1, (25.10), p.101].

(iv): Immediate from (ii) and (iii) [4, (vi) of **Theorem** 13.6, p.135].

Let (E, τ) be a topological vector space,$x \in E$ and $\emptyset \neq A \subseteq E$. A is said to absorb x if there exists $\alpha > 0$ such that $x \in \lambda A$ for all $\lambda \in K$ with $|\lambda| \geq \alpha$; equivalently, if there exists $\epsilon > 0$ such that $\lambda x \in A$ for all $\lambda \in K$ with $0 < |\lambda| \leq \epsilon$ [6, p.95]. A is called an absorbing set if it absorbs every $x \in E$. Similarly, for $\emptyset \neq A, B \subseteq E$, A is said to absorb B provided there exists $\alpha > 0$ such that $B \subseteq \lambda A$ for all $\lambda \in K$ with $|\lambda| \geq K$[2, **Definition** 2.6.1, p.108]. If B is absorbed by every neighborhood of zero of (E, τ), B is called a bounded set [8, p.47][JOR, **Definition** 2.6.2, p.108].

FACT 2:

Let A and B be non-empty subsets of a vector space E such that$A$ is balanced. Then A absorbs B if and only if there exists $\mu \in K$ such that $B \subseteq \mu A$.

Proof :
Paragraph following [2, **Definition** 2.6.1, p.108].
Let B be an absolutely convex absorbing subset of the topological vector space (E,τ). Since B is absorbing, for $x \in E$ there exists $\lambda_x > 0$ such that $x \in \alpha B$ for all $\alpha \in K$ with $|\alpha| \geq \lambda_x$. So, the non-negative function $q_B : E \longrightarrow \mathbb{R}, q_B(x) = inf\{\alpha > 0 : x \in \alpha B\}, x \in E$, is well-defined,$q_B$ is called the gauge or the Minkowski functional of $B.q_B$ is a seminorm [2, p.94]

Example 3 : (E_B, σ_{qB}) By a disc of $lcs(E,\tau)$[3, **Definition** $3.1, p.82$] is meant an absolutely convex bounded subset B of E. We denote by E_B the linear span in E of B. Let $x \in E_B$, and so, .2

$$x = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n$$

for scalars $\alpha_1, \alpha_2, ..., \alpha_n$and $b_1, b_2, ..., b_n \in B$ and $n \in N$. Then

$$x = \alpha_1 B + \alpha_2 B + ... + \alpha_n B \text{ which by } \textbf{Fact } 1\text{(ii)},$$
$$= |\alpha_1|B + |\alpha_2|B + ... + |\alpha_n| \text{ which by } \textbf{Fact } 1\text{(iv)},$$
$$= (|\alpha_1 + |\alpha_2| + ... + |\alpha_n|)B$$
$$= \lambda_x B, \text{where} \lambda_x = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$$

. Since x was an arbitrary element of E_B and B is balanced, it follows from **Fact 2** that b is absorbing in E_B. Denoting by q_B the Minkowski functional of $B(w.r.tE_B), (E_B, q_B)$ is a seminormed space. We have

FACT 4:
With notation as in the preceding example,
(i) (E_B, q_B) is a normed space and has $\{\epsilon B : \epsilon > 0\}$ as a base of neighborhoods of zero of $(E_B, \sigma q_B)$[2, **Proposition** 3.5.6 and its proof, p.$207-208$], and (ii) [3, **Proposition** $3.2.2, p.82]X|E_B \leq \sigma q_B$.

3. LOCAL CONVERGENCE

Let (E,τ) be a lcs. A sequence $\{x_n\}_{l=1}^{\infty}$ in E is said to locally converge or to converge in the Mackey sense to $x \in E$ if $\{x_n\}_{l=1}^{\infty}$ converges to $x \in (E_B, q_B)$ for some disc B[2, **Exercise** $3.7.7, p.225$][3, **Definition** $5.1.1, p.151$], and $\{x_n\}_{l=1}^{\infty}$ called a local null sequence if x is the zero 0 of E.

FACT 1:
Let (E,τ) be a lcs.
(i) [8, **Problem** 4.1.1, p.39] Net $(x_\delta)_{\delta\in(I,\leq)} in(E,\tau)$ converges to $x \in E \Longleftrightarrow$ net $(x_{\delta-}x)$ converges to zero.
(ii) [2, **Exercise** 3.7.7 (a), p.225] A sequence $\{x_n\}_{l=1}^{\infty}$ in (E,τ) locally converges to $x \in E$ if and only if $(x_n x)_{n=1}^{\infty}$is local null.

(iii) [3, **Proposition** 5.1.3(ii), p.151] A sequence $\{x_n\}_{l=1}^{\infty}$ in (E,τ) is local null if and only if there is an increasing unbounded sequence $\{\alpha_n\}_{l=1}^{\infty}$ of positive real numbers such that $(\alpha_n x_n)_{l=1}^{\infty}$ a null sequence in (E,τ).
(iv) A local null sequence is a null sequence [by (ii) of **Fact** 1.4].
(v) $x_n \longrightarrow x$ locally $\Longleftrightarrow x_n \longrightarrow x$ ordinarily [by (ii), (iv) and (i)].

THEOREM 2:
Let (E,τ) be a *lcs* and $\{x_n\}_{l=1}^{\infty}$ a local null sequence in (E,τ). Suppose $\{\alpha_n\}_{l=1}^{\infty}$ is an unbounded increasing sequence of positive real numbers such that $\{\alpha_n x_n\}_{l=1}^{\infty}$ is a null sequence [by (iii) of the preceding **Fact 1**]. Let $\{\beta_n\}_{l=1}^{\infty}$ be an unbounded increasing sequence of positive real numbers such that $\beta_n \leq \alpha_n$, for all n. Then, $(\beta_n x_n)_{l=1}^{\infty}$ is also a null sequence.

Proof Suppose V is an absolutely convex neighborhood of zero in (E,τ). Then, $\alpha_n x_n \longrightarrow 0$ in (E,τ) implies that there exists a positive integer N such that for all $n \geq \natural$, $\alpha_n x_n \in V$. That is,
$x_n \in \frac{1}{\alpha_n}V$, for all $n \geq N$
For all $n \geq \natural, \beta_n \leq \alpha_n$ and so $\frac{1}{\beta_n} \geq \frac{1}{\alpha_n}$ and so by (ii) of **Fact 1.1**, $\frac{1}{\alpha_n}V \subseteq \frac{1}{\beta_n}V$. Hence, from (1) follows that $x_n \in \frac{1}{\beta_n}V$, for all $n \geq N$, and so $\beta_n x_n \in V$, for all $n \geq N$.

COROLLARY 3 :
Let (E,τ) be a lcs and $\{x_n\}_{l=1}^{\infty}$ a sequence in (E,τ). Then, $\{x_n\}_{l=1}^{\infty}$ is local null if and only if there exists an increasing sequence of positive integers $\{\lambda_n\}_{l=1}^{\infty}$ diverging to ∞ such that $\{\lambda_n x_n\}_{l=1}^{\infty}$ is a null sequence.

Proof The implication $\Longleftarrow$ is trivial. So we establish $\Longrightarrow$, i.e., that $\{x_n\}_{l=1}^{\infty}$ is local null implies there exists a sequence of positive integers $\{\lambda_n\}_{l=1}^{\infty}$ such that $\{\lambda_n x_n\}_{l=1}^{\infty}$ is a null sequence. If $\{\alpha_n\}_{l=1}^{\infty}$ is as in **Fact 1**(iii) above, let β_n be the largest integer less than or equal to α_n; if $0 < \alpha_n < 1$ take $\beta_n = 1$. Now evoke **Theorem 2**, by considering a tail of $\{\alpha_n x_n\}_{l=1}^{\infty}$ if necessary.
Now, (iv) of **Fact 1** says that local null sequences are also null. If in $lcs(E,\tau)$ null sequences are also local null, (e.g., if (E,τ) is metrizable [3, **Proposition** $5.1.4, p.152$]) then we say that convergence is Mackey in (E,τ). Indeed, local convergence is also referred to as Mackey convergence. By **Fact 1**(i) and (ii) it is immediate that if convergence is Mackey, then, ordinary convergence implies local convergence. And so we may say that convergence is Mackey if and only if ordinary convergence $\Longrightarrow$ local convergence.
Finally, employing **Corollary 3** we give the promised example of the abstract.

Example 4:
Let $(MF([0,1],\mathbb{R}))$ be the collection of all real-valued Lebesgue measurable functions on the closed bounded interval $[0,1].lcs(MF([0,1],\mathbb{R}))$ is a real vector space

under ordinary addition and scalar multiplication.
Consider $\left(\prod\mathbb{R},\prod_{[0,1]}\right)$ the product space of $[0,1]$ copies of $\mathbb{R}$ with the topology $\prod$ of pointwise convergence (the product topology). Consider,

$$(MF([0,1],\mathbb{R}),\prod)$$

$lcs(MF([0,1],\mathbb{R}))$ with the topology of pointwise convergence $\prod$ restricted to it and still denoted by $\prod$. By COROLLARY 3 AND [7, Theorem $3.5, p.95$], convergence is Mackey in the $lcs(MF([0,1],\mathbb{R}))$.

Proof :

If $\{f_n\}_{l=1}^{\infty}$ is a sequence in $lcs(MF([0,1],\mathbb{R}))$ such that

$$f_n \xrightarrow{\Pi} 0$$

which is $\Longleftrightarrow$

$$|f_n| \xrightarrow{\Pi} 0$$

By [7, Theorem 3.5, p.95] there exists an increasing sequence $\{\lambda_n\}_{l=1}^{\infty}$ of positive integers, $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that.

$$\lambda_n|f_n| \xrightarrow{\Pi} 0$$

which is $\Longleftrightarrow$

$$\lambda_n f_n \xrightarrow{\Pi} 0$$

By our COROLLARY 3 therefore $\{f_n\}_{l=1}^{\infty}$ is local null in $(MF([0,1],\mathbb{R}),\prod)$.

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