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Finite geometries and Finite quantum systems: Duality between lines in phase space and mutually unbiased bases in Hilbert space

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Abstract

In this work, we focus on finite quantum systems with variables in $\mathcal{Z}_d$ and finite geometry $\mathcal{G}_d$ where d is a prime integer. In it, we confirmed the existence of duality between mutually unbiased bases in Hilbert space of a finite quantum system and lines in phase space.

1. Introduction

Over the years, finite quantum systems with variables in $\mathcal{Z}_d$ had received a lot of attention with particular interest on mutually unbiased bases in finite Hilbert space [1, 2, 4-8, 9, 10, 11-19, 21, 22, 24-26, 28 & 30]. In the recent times attention is focused on finite geometry especially on near-linear finite geometry [3, 20 & 27]. It was discussed in [23] there exists a duality between lines non-linear finite geometry and weak mutually unbiased bases in finite quantum systems.
In this work, we discuss similar concept from the perspective of near-linear finite geometry and mutually unbiased bases in a finite dimensional Hilbert space of a finite quantum systems. Our finding is that, as this feature exists between lines in non-near linear finite geometry and weak mutually unbiased bases in finite Hilbert space $\mathcal{H}_d$, there exists such a reflection between lines in phase space of a near-linear finite geometry and mutually unbiased bases in Hilbert space of a finite quantum system. We divide the whole work into the following parts;

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preliminaries of this work is discussed in section II. Here, we define the notation we used in the discourse. Section III covers finite geometry $\mathcal{G}_d$. We discuss Symplectic on $\mathcal{G}_d$ with numerical example was in section IV, also in section V, we discuss quantum systems with variables in $\mathcal{Z}_d$ with examples using Symplectic group on $\mathcal{H}_d$ was discussed. Thereafter, Section VII covers mutually unbiased bases, the duality between lines in finite geometry and mutually unbiased bases is discussed in section. Finally section IX conclude our work.

2. Preliminaries

(i) $\mathcal{Z}_d$ represents the field of integer modulo d.

(ii) $\varphi(d)$ it is called Euler Phi function. It represents the number of invertible elements in $\mathcal{Z}_d$.

(iii) The notation, $\psi(d)$ is called the Dedekind psi function, where;

$$\psi(d) = d\prod\left(1+\frac{1}{d}\right); d = \text{prime} \tag{1}$$

(iv) We present the greatest common divisor of two element θ and λ in $\mathcal{Z}_d$ as $\mathcal{GCD}(\theta, \lambda)$.

(v) The notation $a \cong b$ means a maps b.

3. Finite Geometry $\mathcal{G}_d$

We define a finite geometry as the combination

$$\mathcal{G}_d = (\mathcal{P}_d, L_d) \tag{2}$$

where $\mathcal{P}_d$ represents set of points, L_d represents set of lines in $\mathcal{G}_d$

$$\mathcal{P}_d = \{(g, h)|g, h \in \mathcal{Z}_d\} \tag{3}$$

and a line through $(0, 0)$ is defined as:

$$\mathcal{L}(\theta_i, \lambda_i) = \{(s\theta_i, s(\theta_i + h_i)mod(d))|\ \theta_i = d|d_i, s, \lambda_i = \theta_i + h_i \in \mathcal{Z}_d, h_i \in \mathcal{Z}_d\}, \tag{4}$$

denotes set of lines.

This work focuses on non-linear geometry. That is, in this case two lines for example intersects in one points that is the origin. It is related to the fact that $\mathcal{Z}_d$ is a field of integer modulo d and all the lines in this work are through the origin. From the results of [23] we confirm the following propositions:

(i) If

$$b \in \mathcal{Z}_d^* \text{ then } \mathcal{L}(\theta, \lambda) = \mathcal{L}(b\theta, b\lambda). \tag{5}$$

also, if we can express $\mathcal{L}(b\theta, b\lambda)$ as a subset of $\mathcal{L}(\theta, \lambda)$ we represent it as $\mathcal{L}(b\theta, b\lambda) \prec \mathcal{L}(\theta, \lambda)$.

We confirmed that $\mathcal{L}(\theta, \lambda)$ is a maximal line in $\mathcal{G}_d$ if $\mathcal{GCD}(\theta, \lambda) \in \mathcal{Z}_d^*$ and $\mathcal{L}(\theta, \lambda)$ is a subline in $\mathcal{G}_d$ if $\mathcal{GCD}(\theta, \lambda) \in \mathcal{Z}_d - \mathcal{Z}_d^*$

(ii) There are $\psi(d)$ maximal lines in finite geometry $\mathcal{G}_d$ with exactly d points each.

(iii) Suppose we define a line in finite geometry $\mathcal{G}_d$ as

$$\mathcal{L}(s\theta, s\lambda) = \{(s\xi\theta, s\xi\lambda)|\xi \in \mathcal{Z}_d\}, \tag{6}$$

$\mathcal{L}(\theta,\lambda)$ is also at the same time the line $\mathcal{L}(\xi\theta,\xi\lambda)$ in $\mathcal{G}_{\xi d}$ is a subline of

$$\mathcal{L}(\theta,\lambda) = \{(s'\theta, s'\lambda)|\ s' = 0, ..., \xi d - 1\}, \tag{7}$$

(iv) If two maximal lines have q points in common where $q|d$. The q points gives a subline $\mathcal{L}(\phi,\eta)$ where $\phi,\eta \in \frac{d}{q}\mathcal{Z}_q$.
If we consider the subgeometry $\mathcal{G}_q$, the subline $\mathcal{L}(\phi,\eta)$ in $\mathcal{G}_d$ is a maximal line in $\mathcal{G}_q$. There exists $\psi(q)$ maximal lines in subgeometry $\mathcal{G}_q$ of finite geometry $\mathcal{G}_q$.

(v) There exists a duality between maximal lines in $\mathcal{G}_d$ and weak mutually unbiased bases in $\mathcal{H}_d$.

3.1. **Example.** From (5) above, Lines in geometry $\mathcal{G}_5 \equiv \mathcal{Z}_5 \times \mathcal{Z}_5$ is obtained as follows:

$$\mathcal{L}(0,1) = \{(0,0)(0,1)(0,2)(0,3)(0,4)\} \tag{8}$$

$$\mathcal{L}(1,0) = \{(0,0)(1,0)(2,0)(3,0)(4,0)\} \tag{9}$$

$$\mathcal{L}(1,1) = \{(0,0)(1,1)(2,2)(3,3)(4,4)\} \tag{10}$$

$$\mathcal{L}(1,2) = \{(0,0)(1,2)(2,4)(3,1)(4,3)\} \tag{11}$$

$$\mathcal{L}(1,3) = \{(0,0)(1,3)(2,1)(3,4)(4,2)\} \tag{12}$$

$$\mathcal{L}(1,4) = \{(0,0)(1,4)(2,3)(3,2)(4,1)\} \tag{13}$$

Here; $$\mathcal{L}(1,3) \cong \mathcal{L}(4,2)$$

4. Symplectic Group on $\mathcal{G}_d$

We define the matrices

$$\mathcal{M}(\phi,\eta|\lambda,\theta) \tag{14}$$

where $\mathcal{M}(\phi,\eta|\lambda,\theta) \equiv \begin{pmatrix} \phi & \eta \\ \lambda & \theta \end{pmatrix}$

$\det \mathcal{M} = (\phi\theta - \eta\lambda) = 1(mod\ d)$; where $\phi,\eta,\lambda,\theta \in \mathcal{Z}_d$, $\mathcal{M}$ form a group called symplectic group $Sp(2,\mathcal{Z}_d)$ group.

Suppose we act $\mathcal{M}$ on all points of line $\mathcal{L}(x,y)$ in $\mathcal{Z}_d \times \mathcal{Z}_d$. This produces all the points of the line $\mathcal{L}(\phi x + \eta y, \lambda x + \theta y)$. We write it as $\mathcal{M}(\phi,\eta|\lambda,\theta)\mathcal{L}(x,y)$. This is illustrated below using example for $d = 5$.

If we substitute $\phi = 2, \eta = 1, \lambda = 1, \theta = 1$ into $\mathcal{M}$ then act it on $\mathcal{L}(x,y)$ where

$x = 0, y \in \mathcal{Z}_5$ this yields all the points in line $\mathcal{L}(1,1)$ in (10). Suppose d is a prime, acting $\mathcal{M}(0,1|-1,\chi)$ on the line $\mathcal{L}(0,1)$, we obtain all the lines (maximal lines) through the origin. For

$$\chi = 0, ..., d-1 \rightarrow \Gamma(\chi) = \mathcal{M}(0,1|-1,\chi)\mathcal{L}(0,1) = \mathcal{L}(1,\chi). \tag{15}$$

In this work, we fix a rule that if $\chi = -1, \mathcal{M}(0,1|-1,\chi)$ is replaced by $\mathcal{M}(1,0|0,1)$. If we substitute the value of $\chi = -1, ..., d-1$, we obtain all the lines in $\mathcal{G}_d$ we obtain all the points in the line as shown in eqs.(8) $-$ (13) for $a = 0, b \in \mathcal{Z}_d$.

5. Quantum systems with variables in $\mathcal{Z}_d$

We consider a quantum system with positions and momenta in $\mathcal{Z}_d$ with prime d. Let $|X_d; m\rangle$ and $|P_d; m\rangle$ be positions and momentum states respectively. Here the X_d, P_d are not variables but they simply indicates position and momentum, respectively in a finite quantum system. The variable m belongs to $\mathcal{Z}_d$. The Fourier transform is given by:

$$\mathcal{F}_d = d^{-\frac{1}{2}} \sum_{m,\mathcal{N}} \omega(m\mathcal{N})|X_d; m\rangle\langle X_d; \mathcal{N}|; \omega(m) = \exp\left(i\frac{2\pi m}{d}\right) \tag{16}$$

$|P_d; m\rangle = \mathcal{F}_d|X_d; m\rangle$.
We define displacement operators as

$$\mathcal{D}(\gamma, \delta) = Z^\delta X^\gamma \omega(-2^{-1}\delta\gamma) \tag{17}$$

where

$$Z^\gamma = \sum_{\mathcal{N} \in \mathcal{Z}_d} \omega(\mathcal{N}\gamma)|X; \mathcal{N}\rangle\langle X; \mathcal{N}|; \tag{18}$$

and

$$X^\delta = \sum_{\mathcal{N} \in \mathcal{Z}_d} \omega(-\mathcal{N}\delta)|P; \mathcal{N}\rangle\langle P; \mathcal{N}| \tag{19}$$

where eqs.(18) and (19) satisfy the condition:

$$X^\delta Z^\gamma = Z^\gamma X^\delta \omega(-\gamma\delta); X^d = Z^d = \mathbf{1} \tag{20}$$

The $\mathcal{D}(\gamma, \delta)\omega(\lambda)$ where $\gamma, \delta, \lambda \in \mathcal{Z}_d$ form a representation of Heisenberg-Weyl group.

6. Symplectic Group on $\mathcal{H}_d$

We define the matrices

$$\mathbf{M}(\phi, \eta|\lambda, \theta) \tag{21}$$

Symplectic transformation has been studied in [12&13]. It satisfies the relations

$$X' = [\mathbf{M}(\phi, \eta|\lambda, \theta)]X[\mathbf{M}(\phi, \eta|\lambda, \theta)^\dagger] = \mathcal{D}(\eta, \phi) \tag{22}$$

$$Z' = [\mathbf{M}(\phi,\eta|\lambda,\theta)]Z[\mathbf{M}(\phi,\eta|\lambda,\theta)]^{\dagger} = \mathcal{D}(\theta,\lambda)) \tag{23}$$

$$\mathbf{M}(\phi\theta - \eta\lambda) = 1(mod\ d), \phi,\eta,\lambda,\theta \in \mathcal{Z}_d \tag{24}$$

6.1. **Example.** We define

$$Z' = \langle X_d; m|Z'|X_d; n\rangle = \omega(2^{-1}\eta\lambda + n\eta)\delta(m, n+\lambda) \tag{25}$$

and

$$X' = \omega(2^{-1}\phi\eta + n\eta)\delta(m, n+\lambda) \tag{26}$$

Hence in this work, we define the fourier transform

$$\mathcal{F}_d = \mathbf{M}(0,1|-1,0). \tag{27}$$

$$\mathcal{F}_d = \mathbf{M}(0,1|-1,\chi),\ \phi = 0, \eta = 1, \lambda = -1, \chi = -1, ..., d-1 \tag{28}$$

substituting (27) into eqs.(24) and (25), we obtain the following for $d = 5$,

$$Z' = \begin{pmatrix} 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \omega^1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 \\ \omega^3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An eigenvector of Z' corresponding to eigenvalue 1 is,

$$|X'; 0\rangle = \begin{pmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{pmatrix}$$

Also, from (26),

$$X' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega^1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 \end{pmatrix}$$

If $(X')^m|X'; 0\rangle = |X'; m\rangle$ then $X'|X'; 0\rangle = |X'; 1\rangle$
And

$$\mathcal{S}(n, m) = \langle X; m|X'; n\rangle \tag{29}$$

As a check, if we substitute various values of m and n, into (29) it satisfies the relation $\mathcal{S}\mathcal{S}^{\dagger} = \mathbf{1}$.
Acting $\mathcal{S}$ on the standard bases $|X; m\rangle$, where $m \in \mathcal{Z}_d$ we obtain the remaining bases for $\chi = 0, ..., d-1$. Varying our value of χ_j produces similar results for correspondingly.

7. Mutually Unbiased Bases

We define the standard bases labelled here as

$$|\mathcal{B}(-1); v\rangle = \{\mathcal{S}(1, 0|0, 1)|\mathcal{X}; v\rangle\}, v = 0, ..., d-1 \tag{30}$$

The remaining bases is obtained above by finding the Symplectic transform of the standard bases. That is,

$$|\mathcal{B}(\chi_j); v\rangle = \{\mathcal{S}(0, 1|-1, \chi_j)|\mathcal{X}; v\rangle\}, \chi_j = 0, ..., d-1, \in \mathcal{Z}_d \tag{31}$$

$$|\langle\mathcal{B}(i); v|\mathcal{B}(j); m\rangle| = d^{-\frac{1}{2}} \tag{32}$$

The number of mutually unbiased bases in any dimension d is at most $d+1$.

8. Duality between lines in $\mathcal{G}_d$ and MUB in $\mathcal{H}_d$

The duality between lines in phase space and mutually unbiased bases in Hilbert is expressed as follows: Each lines in $\mathcal{G}_d$ consists of set of d points. Likewise each of the mutually unbiased bases has d orthogonal vectors.
The number of lines in $\mathcal{G}_d$ is $\psi(d)$ the same way we have $\psi(d)$ mutually unbiased bases in $\mathcal{H}_d$. Hence established our claim.

The table below shows in brief the relation between the results of our work.

Finite geometry, $\mathcal{G}_d$	Finite Hilbert space, $\mathcal{H}_d$
$\psi(d)$ lines	$\psi(d)(MUB)$
d points in each lines	d orthogonal vectors in each (MUB)

9. Conclusion

Finite quantum systems with variables in $\mathcal{Z}_d$ is our focus in this work. Due to its wide area of use, many works are done in this area. In this work, we found out that there exists a duality between MUB in $\mathcal{H}_d$ and near-linear finite geometry $\mathcal{G}_d$ and the duality is demostrated with examples as we did for the non-near-linear geometry and weak mutually unbiased bases in finite quantum systems in our previous work in [23].

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