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Block Hybrid Method for Solution of Fourth Order Ordinary Differential Equations

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Abstract

In this paper a block hybrid method of order seven for the direct solution of fourth order ordinary differential equations is presented. The derivation of the method uses interpolation and collocation procedures to obtain the main method and additional methods from the same continuous scheme. The continuous representation of the integrators permit us to differentiate and evaluate at both grid and off-grid points. The stability properties of the methods are discussed by expressing them as a one-step method in higher dimension. Numerical results obtained using the proposed method reveal that it compares favorably well with existing methods in the literature.

1. Introduction

Consider the fourth order differential equation of the form:

$$y^{iv} = f\left(y, y', \, y'', \, y'''\right) \quad , \quad y\left(t_0\right) = y_0 \, , \quad y'\left(t_0\right) = y'_0 \tag{1}$$

Equation (1) arise in the fields of mathematical, physical and engineering sciences such as mechanical systems without dissipation, control theory and celestial mechanics. In practice, higher order differential equations are solved by first reducing it to a system of first order differential equations and then applying the

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various methods available for solving systems of first order IVPs (see Lambert (1991), Hairer and Wanner (1996), Adesanya et.al (2008), Awoyemi (2001), Bun and Varsolyer (1992)). Despite the success of this technique, it often involves the use of complicated subroutine which are incorporated to supply the starting values required for the method. Furthermore, this approach does not utilize additional information associated with specific ordinary differential equation, such as the oscillatory nature of the solution (see Vigor-Aguiar and Ramos (2006)). In the literature, several authors have proposed various methods for the direct solution of higher order differential equations which include linear multistep methods (LMMs) (Lambert and Watson (1976), Henrici (1962), Stiefel and Bettis (1969)), multistep collocation methods (see Carpentieri and Paternoster(2005) and D'Ambrosio et.al. (2009)), multiderivative methods (Twizell and Khaliq (1984), Awoyemi (2003) to mention but a few. However, most of these methods are implemented in the predictor corrector mode. According to Conte and de Boor (1981), computer programs associated with predictor-corrector methods are often complicated especially when incorporating subroutines to supply starting values for the methods, thus, resulting in longer computer time and more computational work. Hybrid methods have also been used to solve higher order differential equations, for instance, see Awoyemi and Idowu (2005), Cash (1984), Xiang and Thomas (2002). Gupta (1978), noted that the design of algorithms for hybrid methods is more tedious due to the occurrence of an o_-step function in the methods which increases the number of predictors needed to implement the methods. It was noted in J. Donelson III and E. Hansen(1971) that stability was achieved with higher order methods without additional function evaluations, however, an extra amount of programming was required. The hybrid method proposed in this paper is self-starting and implemented without the use of predictors. We derive a continuous two step hybrid block method with four off-step points through interpolation and collocation, see Lie and Norsett (1989), and Gladwell and Sayers (1976). In recent time, authors have adopted the block method for solving higher order ordinary differential equations.(see Jator and Li (2009), Olabode (2009), Akinfenwa et.al (2011), Jator et.al (2013). We note that block method is capable of giving evaluations at different grids points without overlapping as in the predictor-corrector method, hence does not require the development of separate predictors, or starting values. It is not necessary to make a function evaluation at the start of the new block. Thus, at all blocks except the first, whose function evaluation is already available from the previous block. Hence, as we proceed, we have six function evaluations per block. The paper is presented as follows: In section 2, we discuss the derivation of the method and obtain a continuous representation Y (t) for the exact solution y(t) which is used to generate members of the block method for solving (1). In section 3, the analysis of the new block method is presented . In section 4, we show the accuracy of the new method. Finally, in section 5 we present some concluding remarks.

2. Derivation of the Method

In this section we derive the main Hybrid method of the form:

$$y_{n+k} = \sum_{i=0}^{v} \alpha_j y_{n+\frac{i}{v}} + h^4 \sum_{j=0}^{k} \beta_j f_{n+j} + h^4 \sum_{r=1}^{v} \beta_{\mu_r} f_{n+\mu_r}\ , \qquad i \neq 2 \tag{2}$$

α_j,β_j,β_{μ_r},are coefficients and $\mu_r,\ r = 1, 2, .., v, \quad v = 4$, is $\frac{2r-1}{v}$. In order to obtain equation (2), we assume an approximation of the exact solution y(x) by assuming a continuous solution Y(x) of the form:

$Y(x) = \sum_{j=0}^{p+s-1} b_j \phi_j(x)$ such that x is in the interval $[x_n,\ x_{n+k}]$, b_j are unknown coefficients and $\phi_j(x)$ are polynomial basis function of degree $p+s-1$. The number of interpolation points s and the number of distinct collocation points are respectively chosen to satisfy $1 \leq s < k$ and $r > 0$ where the integer $k = 2$ denotes the step number of the method. We thus construct a k-step continuous multistep method with $\phi_j(x) = x^j$, $p = 7$ and $s = 4$ by imposing the following conditions we have:

$$\sum_{j=0}^{10} b_j x_{n+s}^j = y_{n+s}, s = 0, \frac{1}{4}, \frac{3}{4}, 1. \tag{3}$$

$$\sum_{j=0}^{10} j\,(j-1)\,(j-2)\,(j-3)\, b_j x_{n+p}^{j-4} = f_{n+p}, p = 0, \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{5}{4}, \tfrac{7}{4}, 2. \tag{4}$$

Equation (3) and (4) lead to a system of $r + s$ equations which must be solved to obtain the undetermined coefficient b_j, which are then substituted into $\sum_{j=0}^{p+s-1} b_j \phi_j(x)$. After some algebraic computation the continuous representation of the hybrid method is obtained and given in the form

$$Y(x) = \sum_{i=0}^{4} \alpha_{\frac{i}{4}}(x) y_{n+\frac{i}{4}} + h^4 \sum_{j=0}^{2} \beta_j(x) f_{n+j} + h^4 \sum_{j=1}^{4} \beta_{\mu_j}(x) f_{n+\mu_j} \quad , \quad i \neq 2 \tag{5}$$

where $\alpha_{\frac{i}{4}}(x)$, $i = 0, 1,\ 3\ 4$, $\beta_j(x)$, $j = 0, 1, 2$ and $\beta_{\mu_j}(x) = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$ are continuous coefficients, h is the chosen step-length. We assume that $y_{n+j} = Y\,(x_n + jh)$ is the numerical approximation to the analytical solution $y\,(x_{n+j})$,

$y'_{n+j} = Y'(x_n + jh)$is an approximation to $y'(x_{n+j})$,$y'_{n+j} = Y'(x_n + jh)$ is an approximation to $y'(x_{n+j})$,$y'''_{n+j} = Y'''(x_n + jh)$ is an approximation to $y'''(x_{n+j})$, $f_{n+\frac{i}{4}} = Y^{(iv)}_{\frac{i}{4}}(x_n + jh)$ is an approximation to $y^{(iv)}_{\frac{i}{4}}(x_{n+j})$, where i $= 0, 1,\ 3,\ 5,\ 7,\ 8$.

The main methods are obtained by evaluating (5) at x = $x_{n+\frac{i}{4}}$, i = 5,7 and 8 to obtain (6).

$$(6)\quad \begin{cases} y_{n+\frac{5}{4}} = \frac{2}{3}y_n - \frac{10}{3}y_{n+1} - \frac{5}{3}y_{n+\frac{1}{4}} + \frac{10}{3}y_{n+\frac{3}{4}} + \frac{1}{325140480}h^4\begin{pmatrix} 42606f_n + 756140f_{n+1} - 14010f_{n+2} - 420760f_{n+\frac{1}{4}} \\ -2148300f_{n+\frac{3}{4}} - 388416f_{n+\frac{5}{4}} + 55940f_{n+\frac{7}{4}} \end{pmatrix} \\ y_{n+\frac{7}{4}} = 6y_n - 14y_{n+1} - 14y_{n+\frac{1}{4}} + 21y_{n+\frac{3}{4}} + \frac{1}{25804800}h^4\begin{pmatrix} 30990f_n + 282940f_{n+1} - 9450f_{n+2} - 305060f_{n+\frac{1}{4}} \\ -1704360f_{n+\frac{3}{4}} - 447300f_{n+\frac{5}{4}} + 35440f_{n+\frac{7}{4}} \end{pmatrix} \\ y_{n+2} = \frac{25}{3}y_n - \frac{70}{3}y_{n+1} - \frac{80}{3}y_{n+\frac{1}{4}} + \frac{112}{3}y_{n+\frac{3}{4}} + \frac{1}{11612160}h^4\begin{pmatrix} 26505f_n + 223790f_{n+1} - 5847f_{n+2} - 265060f_{n+\frac{1}{4}} \\ -1554588f_{n+\frac{3}{4}} - 548940f_{n+\frac{5}{4}} + 7340f_{n+\frac{7}{4}} \end{pmatrix} \end{cases}$$

Equation (6) with three equations and six unknown lead to indeterminate hence the need for additional methods which are obtained from the first, second and third derivative of (5) the give the forms

(6)

$$Y'(x) = \frac{d}{dx}\left(\sum_{i=0}^{4}\alpha_{\frac{i}{4}}(x)y_{n+\frac{i}{4}} + h^4\sum_{j=0}^{2}\beta_j(x)f_{n+j} + h^4\sum_{j=1}^{4}\beta_{\mu_j}(x)f_{n+\mu_j} \quad , \quad i \neq 2\right)$$

(7)

$$Y''(x) = \frac{d^2}{dx^2}\left(\sum_{i=0}^{4}\alpha_{\frac{i}{4}}(x)y_{n+\frac{i}{4}} + h^4\sum_{j=0}^{2}\beta_j(x)f_{n+j} + h^4\sum_{j=1}^{4}\beta_{\mu_j}(x)f_{n+\mu_j} \quad , \quad i \neq 2\right)$$

(8)

$$Y'''(x) = \frac{d^3}{dx^3}\left(\sum_{i=0}^{4}\alpha_{\frac{i}{4}}(x)y_{n+\frac{i}{4}} + h^4\sum_{j=0}^{2}\beta_j(x)f_{n+j} + h^4\sum_{j=1}^{4}\beta_{\mu_j}(x)f_{n+\mu_j} \quad , \quad i \neq 2\right)$$

Evaluating (7, 8 and 9) at x = $x_{n+\frac{j}{4}}$, j=0, we obtain equations 10, 11, and 12 respectively

$$hy'_n = 19y_n - 3y_{n+1} - 24y_{n+\frac{1}{4}} + 8y_{n+\frac{3}{4}} - \frac{1}{90316800}h^3\begin{pmatrix} 13161f_n + 1186990f_{n+1} - 16695f_{n+2} - 1287860f_{n+\frac{1}{4}} \\ -1607340f_{n+\frac{3}{4}} - 471996f_{n+\frac{5}{4}} + 66940f_{n+\frac{7}{4}} \end{pmatrix}$$

(9)

$$h^2y''_n = -64y_n + 32y_{n+1} + 112y_{n+\frac{1}{4}} - 80y_{n+\frac{3}{4}} - \frac{1}{270950400}h^2\begin{pmatrix} 39483f_n - 342461003f_{n+1} + 449418f_{n+2} + 51957160f_{n+\frac{1}{4}} \\ +48823992f_{n+\frac{3}{4}} + 13248312f_{n+\frac{5}{4}} - 1820120f_{n+\frac{7}{4}} \end{pmatrix}$$

(10)

$$h^3y'''_n = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \frac{h}{3386880}\begin{pmatrix} 259960f_n - 157164f_{n+1} - 268f_{n+2} + 1151915f_{n+\frac{1}{4}} \\ +405167f_{n+\frac{3}{4}} + 34153f_{n+\frac{5}{4}} - 323f_{n+\frac{7}{4}} \end{pmatrix}$$

(11)

Furthermore, evaluating (7, 8 and 9) at x = $x_{n+\frac{j}{4}}$, j=1…8, j $\neq$2,6. we have

$$(12)\quad\left\{\begin{array}{l}
hy'_{n+\frac{1}{4}} = 6y_n + 2y_{n+1} - 2y_{n+\frac{1}{4}} - 6y_{n+\frac{3}{4}} + \frac{1}{90316800}h^3\begin{pmatrix} 36111f_n + 802550f_{n+1} - 11733f_{n+2} - 11733f_{n+\frac{1}{4}} \\ -1060122f_{n+\frac{3}{4}} - 324156f_{n+\frac{5}{4}} + 46790f_{n+\frac{7}{4}} \end{pmatrix} \\
hy'_{n+\frac{3}{4}} = -2y_n - 6y_{n+1} + 6y_{n+\frac{1}{4}} + 2y_{n+\frac{3}{4}} - \frac{1}{12902400}h^3\begin{pmatrix} 4665f_n + 111410f_{n+1} - 1563f_{n+2} - 47290f_{n+\frac{1}{4}} \\ -180552f_{n+\frac{3}{4}} - 44130f_{n+\frac{5}{4}} + 6260f_{n+\frac{7}{4}} \end{pmatrix} \\
hy'_{n+1} = 3y_n - 19y_{n+1} - 8y_{n+\frac{1}{4}} + 24y_{n+\frac{3}{4}} - \frac{1}{90316800}h^3\begin{pmatrix} 52137f_n + 1170190f_{n+1} - 17367f_{n+2} - 517580f_{n+\frac{1}{4}} \\ -2388708f_{n+\frac{3}{4}} - 484932f_{n+\frac{5}{4}} + 69460f_{n+\frac{7}{4}} \end{pmatrix} \\
hy'_{n+\frac{5}{4}} = 14y_n - 38y_{n+1} - 34y_{n+\frac{1}{4}} + 58y_{n+\frac{3}{4}} + \frac{1}{90316800}h^3\begin{pmatrix} 251219f_n + 3660510f_{n+1} - 82625f_{n+2} - 2473040f_{n+\frac{1}{4}} \\ -13111490f_{n+\frac{3}{4}} - 2333884f_{n+\frac{5}{4}} + 330110f_{n+\frac{7}{4}} \end{pmatrix} \\
hy'_{n+\frac{7}{4}} = 54y_n - 94y_{n+1} - 122y_{n+\frac{1}{4}} + 162y_{n+\frac{3}{4}} + \frac{1}{30105600}h^3\begin{pmatrix} 320231f_n + 2133390f_{n+1} - 73157f_{n+2} - 3193190f_{n+\frac{1}{4}} \\ -18904088f_{n+\frac{3}{4}} - 6926206f_{n+\frac{5}{4}} + 183020f_{n+\frac{7}{4}} \end{pmatrix} \\
hy'_{n+2} = 83y_n - 131y_{n+1} - 184y_{n+\frac{1}{4}} + 232y_{n+\frac{3}{4}} - \frac{1}{90316800}h^3\begin{pmatrix} 1391675f_n + 12171530f_{n+1} - 88229f_{n+2} - 14413020f_{n+\frac{1}{4}} \\ -91813316f_{n+\frac{3}{4}} - 46018420f_{n+\frac{5}{4}} + 3055820f_{n+\frac{7}{4}} \end{pmatrix}
\end{array}\right.$$

$$
(13)\quad \begin{cases}
h^2 y''_{n+\frac{1}{4}} = -40y_n + 80y_{n+1} + 64y_{n+\frac{1}{4}} - 32y_{n+\frac{3}{4}} + \dfrac{1}{270950400}h^2\left(\begin{array}{l} 108333f_n + 6986840f_{n+1} - 101988f_{n+2} + 590840f_{n+\frac{1}{4}} \\ -9732912f_{n+\frac{3}{4}} - 2813832f_{n+\frac{5}{4}} + 406400f_{n+\frac{7}{4}} \end{array}\right) \\
h^2 y''_{n+\frac{3}{4}} = 8y_n - 40y_{n+1} - 32y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} - \dfrac{1}{270950400}h^2\left(\begin{array}{l} 97965f_n - 6943160f_{n+1} + 96708f_{n+2} + 3393520f_{n+\frac{1}{4}} \\ +5661432f_{n+\frac{3}{4}} + 2712192f_{n+\frac{5}{4}} - 386600f_{n+\frac{7}{4}} \end{array}\right) \\
h^2 y''_{n+1} = 32y_n - 64y_{n+1} - 80y_{n+\frac{1}{4}} - 112y_{n+\frac{3}{4}} + \dfrac{1}{270950400}h^2\left(\begin{array}{l} 156411f_n + 34158740f_{n+1} - 559818f_{n+2} - 16833880f_{n+\frac{1}{4}} \\ -85768872f_{n+\frac{3}{4}} - 15373512f_{n+\frac{5}{4}} + 2234120f_{n+\frac{7}{4}} \end{array}\right) \\
h^2 y''_{n+\frac{5}{4}} = 56y_n - 88y_{n+1} - 128y_{n+\frac{1}{4}} + 160y_{n+\frac{3}{4}} + \dfrac{1}{270950400}h^2\left(\begin{array}{l} 753657f_n + 19909400f_{n+1} - 1036068f_{n+2} \\ -30195160f_{n+\frac{1}{4}} - 170611392f_{n+\frac{3}{4}} - 32769912f_{n+\frac{5}{4}} \\ +4160720f_{n+\frac{7}{4}} \end{array}\right) \\
h^2 y''_{n+\frac{7}{4}} = 104y_n - 136y_{n+1} - 224y_{n+\frac{1}{4}} - 256y_{n+\frac{3}{4}} + \dfrac{1}{270950400}h^2\left(\begin{array}{l} 2882079f_n + 45885560f_{n+1} + 950652f_{n+2} \\ -53202880f_{n+\frac{1}{4}} - 365720712f_{n+\frac{3}{4}} \\ -231849072f_{n+\frac{5}{4}} - 14882440f_{n+\frac{7}{4}} \end{array}\right) \\
h^2 y''_{n+2} = 128y_n - 160y_{n+1} - 272y_{n+\frac{1}{4}} + 304y_{n+\frac{3}{4}} + \dfrac{1}{270950400}h^2\left(\begin{array}{l} 4175025f_n + 86111060f_{n+1} - 578058f_{n+2} \\ -63302920f_{n+\frac{1}{4}} - 473659032f_{n+\frac{3}{4}} \\ -368443992f_{n+\frac{5}{4}} - 78944200f_{n+\frac{7}{4}} \end{array}\right)
\end{cases}
$$

$$
(14)\quad \begin{cases}
h^3 y'''_{n+\frac{1}{4}} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} - \dfrac{h}{3386880}\left(\begin{array}{l} 41888f_n + 702380f_{n+1} - 702380f_{n+2} - 471789f_{n+\frac{1}{4}} \\ -861833f_{n+\frac{3}{4}} - 289135f_{n+\frac{5}{4}} + 42469f_{n+\frac{7}{4}} \end{array}\right) \\
h^3 y'''_{n+\frac{3}{4}} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \dfrac{h}{3386880}\left(\begin{array}{l} 24736f_n + 653268f_{n+1} - 8692f_{n+2} - 230323f_{n+\frac{1}{4}} \\ -1070167f_{n+\frac{3}{4}} - 250481f_{n+\frac{5}{4}} + 34939f_{n+\frac{7}{4}} \end{array}\right) \\
h^3 y'''_{n+1} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \dfrac{h}{3386880}\left(\begin{array}{l} 22264f_n + 108052f_{n+1} - 679f_{n+2} - 218709f_{n+\frac{1}{4}} \\ -1454929f_{n+\frac{3}{4}} - 170135f_{n+\frac{5}{4}} + 26813f_{n+\frac{7}{4}} \end{array}\right) \\
h^3 y'''_{n+\frac{5}{4}} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \dfrac{h}{3386880}\left(\begin{array}{l} 24160f_n - 437164f_{n+1} - 9268f_{n+2} - 226835f_{n+\frac{1}{4}} \\ -1374583f_{n+\frac{3}{4}} - 554897f_{n+\frac{5}{4}} + 38427f_{n+\frac{7}{4}} \end{array}\right) \\
h^3 y'''_{n+\frac{7}{4}} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \dfrac{h}{3386880}\left(\begin{array}{l} 4768f_n + 918484f_{n+1} + 5735f_{n+2} \\ -149427f_{n+\frac{1}{4}} - 1914199f_{n+\frac{3}{4}} - 2486897f_{n+\frac{5}{4}} \\ -663685f_{n+\frac{7}{4}} \end{array}\right) \\
h^3 y'''_{n+2} = 32y_n - 32y_{n+1} - 64y_{n+\frac{1}{4}} + 64y_{n+\frac{3}{4}} + \dfrac{h}{3386880}\left(\begin{array}{l} 15736f_n + 373268f_{n+1} + 5735f_{n+2} \\ -191573f_{n+\frac{1}{4}} - 1659217f_{n+\frac{3}{4}} - 2030231f_{n+\frac{5}{4}} \\ -343811f_{n+\frac{7}{4}} \end{array}\right)
\end{cases}
$$

3. Analysis of the Method

In this section, we discuss the zero-stability, local truncation error and order, consistency, and convergence of the method.

3.1. **Local Truncation Error and Order.** Following Fatunla and Lambert we define the local truncation error associated with the method to be the linear difference operator

$$L\left[y(x);h\right]=\sum_{j=0}^{k}\left\{\alpha_j y(x+jh)-h^4\beta_j y^{(iv)}(x+jh)\right\}-h^4\sum_{j=1}^{\nu}\beta_{\mu_j}y^{(iv)}(x+\mu_j h)$$

Assuming that y(x) is sufficiently differentiable, we can expand the terms in () as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right]=C_0 y(x)+C_1 y'(x)+\ldots+C_p h^p y^{(p)}(x)+\ldots,$$

Where the constant coefficients C_p, p=0,1 ... are given as follows:

$$C_0=\sum_{j=0}^{k}\alpha_j$$

$$C_1=\sum_{j=0}^{k} j\alpha_j$$

$$\vdots$$

$$C_p=\frac{1}{p!}\sum_{j=0}^{k} j^p\alpha_j-\frac{1}{(p-4)!}\left(\sum_{j=0}^{k} j^{p-4}\beta_j+\sum_{j=0}^{k}\mu_j^{p-4}\beta_{\mu_j}\right).$$

We say that the method has order p if

$$C_0=C_1=\ldots=C_p=C_{p+1}=C_{p+2}=\ C_{p+3}=0,\ C_{p+4}\neq 0.$$

Therefore, $C_{p+4}$is the error constant and $C_{p+4}h^{p+4}y^{(p+4)}(x_n)$the principal local truncation error at the point x_n.
Thus, we can write the local truncation error (LTE) of the method of order p as

$$LTE=C_{p+4}h^{p+4}y^{(p+4)}(x_n)+O\left(h^{p+5}\right)$$

It is established from our calculation that the hybrid methods given above have order p and error constant C_7 given by the vectors p = $(7,7,7,7,7,7)^T$and

$$C_7=\left(-\frac{563}{38149816320},-\frac{2903311}{1016064000},-\frac{284615}{1826434842624},-\frac{403}{38050725888},-\frac{403}{4227858432},-\frac{403}{2378170368}\right)^T$$

3.1 Zero-Stability
The zero-stability of the method in (**??**) is determined from the first characteristics polynomial as the limit h tends to zero. Thus as h→ 0, the methods (7), (8), (9), (10), (11) and (12) tend to the difference system

$$A^{(1)}Y_{\varpi+1}=A^{(0)}Y_{\varpi}$$

which is normalized to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det\left(R\hat{A}^{(1)} - \hat{A}^{(0)}\right) = R^5\left(R - \frac{1}{2}\right),$$

where $\hat{A}^{(1)}$ is an identity matrix of dimension 6 and $\hat{A}^{(0)}$ is given by

$$\hat{A}^{(0)} = \begin{pmatrix} 0 & 0 & -\frac{1387}{72} & 0 & 0 & \frac{19}{24} \\ 0 & 0 & \frac{292}{15} & 0 & 0 & -\frac{4}{5} \\ 0 & 0 & \frac{73}{6} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{146}{9} & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 146 & 0 & 0 & -6 \\ 0 & 0 & \frac{2555}{9} & 0 & 0 & -\frac{35}{3} \end{pmatrix}$$

Following Fatunla, the block method is zero-stable, since from $\rho(R) = 0$ satisfies $|R_j| = \frac{1}{2}$, j = 1,..6, and for those roots with $|R_j| = \frac{1}{2}$, the multiplicity does not exceed 2. We note that the single members of the block method are not zero-stable, but this property is gained when the methods are combined as numerical integrators in the block form.

Consistency and Convergence

The hybrid method is consistent since each of the method has order p > 1. According to Henrici, convergence = consistency + zero-stability. Hence the method is convergent.

Numerical Examples

In this section we test the new method on some numerical example to show the performance of the new method. All example are solved using our written code in Maple 8.

ExampleI:

$$y^{(iv)} = \left(y'\right)^2 - yy''' - 4x^2 + e^x\left(1 + x^2 - 4x\right), 0 \le x \le 1$$

$$y\left(0\right) = y'\left(0\right) = 1, y''(0) = 3, y'''(0) = 1.$$

Exact: $y\left(x\right) = x^2 + e^x$

This example has also been solved by Adesanya (2012). The new method performed twice as much as that in adesanya (2012) as shown in table 1 below.

Table 1 Comparison of absolute error for example 1

h=0.01				
x	EXACT RESULT	COMPUTED RESULT	ERROR in New method	ERROR IN Adesanya (2012)
0.1	1.115170918	1.115170918	2.18E-27	9.0460(-13)
0.2	1.261402758	1.261402758	1.64E-26	2.1516(-12)
0.3	1.439858808	1.439858808	5.44E-26	3.7549(-12)
0.4	1.651824698	1.651824698	1.29E-25	5.6885(-12)
0.5	1.898721271	1.898721271	2.54E-25	7.8819(-12)
0.6	2.1821188	2.1821188	4.43E-25	1.0212(-11)
0.7	2.503752707	2.503752707	7.06E-25	1.2497(-11)
0.8	2.865540928	2.865540928	1.05E-24	1.4486(-11)
0.9	3.269603111	3.269603111	1.48E-24	1.5849(-11)
1	3.718281828	3.718281828	1.99E-24	1.6159(-11)

Example 2:

$$y^{(iv)} - 4y'' = 0, 0 \leq x \leq 1$$

$$y(0) = 1, y'(0) = 3, y''(0) = 0, y'''(0) = 16.$$

Exact: $y(x) = 1 - x + e^{2x} - e^{-2x}$

Also for this example using the same step size h=1/320 the new method is superior to proposed in Adesanya (2012), as shown in Table 2.

Table 2 : Comparison of absolute error for example 2				
x	EXACT RESULT	COMPUTED RESULT	ERROR In new method	ERROR IN Awoyemi et al. (2005)
0.003125	1.009375081	1.009375081	1.00E-18	0.00E+00
0.00625	1.018750651	1.018750651	2.00E-18	0.00E+00
0.009375	1.028127197	1.028127197	5.20E-17	2.22E-16
0.00125	1.037505208	1.037505208	2.39E-16	2.44E-15
0.015625	1.046885173	1.046885173	5.52E-16	1.15E-14
0.01875	1.056267579	1.056267579	9.57E-16	3.31E-14
0.021875	1.065652916	1.065652916	1.20E-15	7.28E-14
0.025	1.075041672	1.075041672	1.21E-15	1.37E-13
0.028125	1.084434336	1.084434336	6.27E-16	2.31E-13
0.03125	1.093831396	1.093831396	5.54E-16	3.61E-13

Example 3: $y^{(iv)} = y''' + y'' + y' + 2y,\ 0 \leq x \leq 2$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 30.$$

Exact: $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$

Using this example as our final example the new method was compared with methods in Lee et.al. (2015) . again the method is superior to those in Lee et.al. for various values of h.

Table 3 Comparison of absolute error at t = 2 for example 3

Methods				
h	Lee Ken Yap(2015)	Adams	Jator (2008)	New Method
0.1	1.74 (−8)	2.11 (−3)	1.26 (−4)	5.73E-11
0.05	8.45 (−11)	5.37 (−4)	1.91 (−6)	8.45 (−11)
0.025	3.69 (−13)	5.09 (−5)	2.96 (−8)	8.46E-16
0.02	7.11 (−14)	2.25 (−5)	8.65 (−9)	1.42E-16

4. Conclusion

We proposed a two step block hybrid method for direct solution of fourth order differential equations.The methods are implemented without the need for starting values or predictors and hence reducing computational effort and storage usage. We have demonstrated the efficiency of the method on some numerical examples. Details of the numerical results are displayed in Tables 1-3.

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