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A Zero-Stable Numerical Model as First, Second, and Third-Order Initial Value Problems Solver

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Abstract

This paper addresses solutions of multi-order ordinary differential equations (ODEs) using a single formulated scheme. The use of a block method to solve initial value problem of a specific order has attracted a lot of attention in the literature. The discrete method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as a numerical integrator for first, second and third-order ODEs. The convergence properties of the method are discussed via zero-stability and consistency, and on investigation, the proposed method is found to be convergent. Solutions of the test problems considered are presented, as well as comparisons to existing approaches in the literature.

1. Introduction

This paper focuses on the development of

$$\sum_{i=0}^{k}\alpha_i(t)y_{n+i}=h\left(\sum_{i=0}^{k}\beta_i(t)f_{n+i}+\beta_j(t)f_{n+j}\right)+h^2\left(\sum_{i=0}^{k}\lambda_i(t)g_{n+i}+\lambda_j(t)g_{n+j}\right)+h^3\left(\sum_{i=0}^{k}\delta_i(t)w_{n+i}+\delta_j(t)w_{n+j}\right). \tag{1.1}$$

to directly integrate ordinary differential equations (ODEs)

$$\begin{cases} y^{'}(x)=f(x,y(x)),y(x_0)=y_0\\ y^{''}(x)=f(x,y(x),y^{'}(x)),y(x_0)=y_0,y^{'}(x_0)=y_0^{'}\\ y^{'''}(x)=f(x,y(x),y^{'}(x),y^{''}(x)),y(x_0)=y_0,y^{'}(x_0)=y_0^{'},y^{''}(x_0)=y_0^{''}\end{cases} \tag{1.2}$$

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where in equation (1.1), either of $\alpha_0(t)$ and $\beta_0(t)$ do not varnish, $\alpha_k(t) = 1$, $\beta_k(t) \neq 0$ and $k = 1$. The traditional method of reducing higher-order ODEs to a set of first-order equations increases the problem's dimension and, as a result, requires more processing time. The use of block methods to solve equation (1.2) has been widely reported, and the resulting solutions have been thoroughly discussed by a number of scholars, including [5]. This block method approach, which generates approximations at different grid points within the integration interval without overlapping sub-intervals, has been reported to avoid the setback commonly experienced in reducing higher-order ODEs to a system of first-order equations and the predictor-corrector approach, see ([11], [6], [15], [2], [17] & [18]).
These efficient approaches have been directly applied as numerical integrators of ODEs focussing at solving IVPs of the same order, but using the same method to integrate IVPs of multiple order has not been widely reported. The goal of this study is to provide a self-starting approach for solving first, second, and third-order ODEs numerically.
In what immediately follows, the derivation of the proposed approach for direct integration of multi-order ODEs is discussed. The method's analysis and implementation are addressed in Section 3, and the results are discussed in Section 4. Finally, in Section 5, the paper's conclusion is presented.

2. Material and Method

In this section, the formulation of the proposed method is considered. We adopt the interpolation and collocation techniques employed in [4] and develop a novel numerical method for solving first, second, and third-order IVPs of ODEs. Using Chebyshev polynomials as the basis function, we develop a novel one-step three-hybrid approach for solving first, second, and third-order ODEs.
Thus,

$$y(x) = \sum_{j=0}^{k+8} a_j T_j \tag{2.1}$$

is introduced as an approximate solution to first, second and third-order ODEs of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{2.2}$$

$$y''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, y'(x_0) = y'_0 \tag{2.3}$$

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(x_0) = y_0, y'(x_0) = y'_0, y^{''}(x_0) = y_0^{''} \tag{2.4}$$

Equation (2.1) is interpolated at$x = x_n$. Collocating the first and second derivatives of equation (2.1) at$x = x_{n+m}, m = 0, \frac{4}{5}, 1, \frac{6}{5}$and the third derivative at$x = x_{n+w}, w = 2$, we have

$$\begin{array}{l} \sum_{j=0}^{k+8} a_j T_j = y_n \\ \sum_{j=1}^{k+8} a_j \, T_j^{'} = f_{n+v} \\ \sum_{j=2}^{k+8} a_j \, T_j^{''} = g_{n+v} \\ \sum_{j=3}^{k+8} a_j \, T_j^{'''} = X_{n+w} \end{array} \tag{2.5}$$

where; $T_j(x)$is the parameters of Chebyshev polynomials, g_{n+v} is the the first derivative of (2.1), $g_{n+v}$is its second derivative, $X_{n+w}$is its third derivative and k is the step number ($k = 2$).

Equation (2.5) is solved using the Gaussian elimination method to obtain the unknown variables $a's$ that are then substituted into equation (2.1). This results in an implicit continuous scheme of the form:

$$\begin{aligned} \alpha_0(t)y_n &= h\left(\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta_{\frac{6}{5}}(t)f_{n+\frac{6}{5}}\right) \\ &+h^2\left(\sum_{j=0}^{1}\lambda_j(t)g_{n+j}+\lambda_{\frac{4}{5}}(t)g_{n+\frac{4}{5}}+\lambda_{\frac{6}{5}}(t)g_{n+\frac{6}{5}}\right)+h^3\left(\delta_2(t)X_{n+2}\right) \end{aligned} \tag{2.6}$$

where $t=\frac{2x-2x_n-h}{h}$, $\alpha_0(t)=1$. Then,

$$A = BC$$

where

$$A=\begin{pmatrix} \beta_0 \\ \beta_{\frac{4}{5}} \\ \beta_1 \\ \beta_{\frac{6}{5}} \\ \lambda_0 \\ \lambda_{\frac{4}{5}} \\ \lambda_1 \\ \lambda_{\frac{6}{5}} \\ \delta_2 \end{pmatrix};$$

$$B=\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -8 & 18 & -32 & 50 & -72 & 98 & -128 & 162 \\ 0 & 2 & \frac{24}{5} & \frac{66}{25} & \frac{-672}{125} & \frac{-1558}{125} & \frac{-30888}{3125} & \frac{56434}{15625} & \frac{1416576}{78125} & \frac{7747938}{390625} \\ 0 & 2 & 8 & 18 & 32 & 50 & 72 & 98 & 128 & 162 \\ 0 & 2 & \frac{56}{5} & \frac{1026}{25} & \frac{16352}{125} & \frac{48682}{125} & \frac{3476088}{3125} & \frac{48256754}{15625} & \frac{656238464}{78125} & \frac{8784645858}{390625} \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 & 5376 & -8640 \\ 0 & 0 & 16 & \frac{288}{5} & \frac{1856}{25} & \frac{-288}{25} & \frac{-23472}{125} & \frac{-915264}{3125} & \frac{2106112}{15625} & \frac{21577536}{78125} \\ 0 & 0 & 16 & 96 & 320 & 800 & 1680 & 3136 & 5376 & 8640 \\ 0 & 0 & 16 & \frac{672}{5} & \frac{17216}{25} & \frac{71008}{25} & \frac{1297488}{125} & \frac{109756864}{3125} & \frac{1760464128}{15625} & \frac{27152483904}{78125} \\ 0 & 0 & 0 & 192 & 4608 & 68160 & 801792 & 8227968 & 77064192 & 675946368 \end{pmatrix};$$

and

$$C=\begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \end{pmatrix}$$

Equation (2.6) is evaluated at $x = x_{n+1}(t = 1)$,$x = x_{n+\frac{4}{5}}(t = \frac{3}{5})$,$x = x_{n+\frac{6}{5}}(t = \frac{7}{5})$ and $x = x_{n+2}(t = 3)$. This produces the following schemes

(2.7)

$$\begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+1} \\ y_{n+\frac{6}{5}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{4}{5}} \\ f_{n+1} \\ f_{n+\frac{6}{5}} \\ f_{n+2} \end{pmatrix} + h^2 E \begin{pmatrix} g_n \\ g_{n+\frac{4}{5}} \\ g_{n+1} \\ g_{n+\frac{6}{5}} \\ g_{n+2} \end{pmatrix} + h^3 F \begin{pmatrix} X_n \\ X_{n+\frac{4}{5}} \\ X_{n+1} \\ X_{n+\frac{6}{5}} \\ X_{n+2} \end{pmatrix}$$

where

$$D = \begin{pmatrix} \frac{3741217}{20314800} & \frac{239909}{1269675} & \frac{16227}{83600} & \frac{640198}{1269675} \\ \frac{4450547}{21067200} & \frac{147286}{329175} & \frac{1151739}{2340800} & \frac{807392}{329175} \\ \frac{427723}{5266800} & \frac{108446}{329175} & \frac{337851}{585200} & \frac{175312}{329175} \\ \frac{13294271}{568814400} & \frac{303598}{8887725} & \frac{553701}{2340800} & \frac{-13238944}{8887725} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} \frac{61118924}{4041140625} & \frac{312941}{20690640} & \frac{3018459}{199562500} & \frac{345583}{5172660} \\ \frac{-11054516}{5028975} & \frac{-113072125}{51496704} & \frac{-4905927}{2235100} & \frac{23973625}{804636} \\ \frac{-2787991552}{628621875} & \frac{-7146695}{1609272} & \frac{-309668832}{69846875} & \frac{22788256}{201159} \\ \frac{-31234432}{45260775} & \frac{-160001125}{231735168} & \frac{-77388}{111755} & \frac{56805500}{1810431} \\ 0 & 0 & 0 & 0 \end{pmatrix} F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{62656}{3143109375} & \frac{2567}{128741760} & \frac{6966}{349234375} & \frac{2798}{1005795} \end{pmatrix}$$

3. Analysis of the Method

The basic features of this approach, such as order, error constant, zero stability, and consistency, are investigated in this section.

3.1. **Order.** Equation (2.7) derived is a discrete scheme belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} \right) + h^2 \left(\sum_{j=0}^{k} \lambda_j(t) g_{n+j} \right) + h^3 \left(\sum_{j=0}^{k} \delta_j(t) X_{n+j} \right) \tag{3.1}$$

Following [8] Fatunla (1988), we define the local truncation error associated with equation (3.2) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh) - h^3 \gamma_j g(x_n + jh) \right] \tag{3.2}$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].
Expanding (3.2) using Taylor series about the point$x$, the expression below is obtained

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{p+2}(x)$$

where the C_0, C_1,$C_2 \ldots C_p \ldots C_{p+1}$ are obtained as $C_0 = \sum_{r=0}^{s} \alpha_r$,$C_1 = \sum_{r=1}^{s} r\alpha_r$,$C_2 = \frac{1}{2!} \sum_{r=1}^{s} r^2 \alpha_r$,$C_q = \frac{1}{q!} \left[\sum_{r=1}^{s} r^q \alpha_r - q(q-1) \sum_{r=1}^{s} \beta_r r^{q-2} - q(q-1)(q-2) \sum_{r=1}^{s} \gamma_r r^{q-3} \right]$.
In the spirit of [13]), equation (3.2) is of order p if $C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$and $C_{p+r} \neq 0$. The $C_{p+r} \neq 0$ is called the error constant and$C_{p+r} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.
Thus, the block (2.7) is of order $p = 8$ and error constant

$$C_{p+2} = \left[\frac{509556736}{27845984619140625}, \frac{33399881}{1824914448000000}, \frac{2797001}{152790039062500}, \frac{3963577}{7128572062500} \right]^T.$$

3.2. Zero Stability of the Method. The linear multistep method (3.1) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation. To analyze the zero-stability of the method, we present (3.2) in vector notation form of column vectors $e = (e_1 \dots e_r)^T$, $d = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$, $F(y_m) = (f_{n+1} \dots f_{n+r})^T$,$G(y_m) = (g_{n+1} \dots g_{n+r})^T$,
$W(y_m) = (T_{n+1} \dots T_{n+r})^T$and matrices $A = (a_{ij})$, $B = (b_{ij})$.
Thus, equation (3.2) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hbf_n + h^2 DG(y_m) + h^2 dg_n + h^3 VW(y_m) + h^3 uX_n \tag{3.3}$$

where h is a fixed mesh size within a block.
In line with (3.3),

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; b = \begin{pmatrix} \frac{2798861662}{12123421875} \\ \frac{-23699458}{1005795} \\ \frac{38973022208}{3143109375} \\ \frac{318614176}{27156465} \\ 0 \end{pmatrix}; d = \begin{pmatrix} \frac{61118924}{4041140625} \\ \frac{-11054516}{5028975} \\ \frac{-2787991552}{628621875} \\ \frac{-31234432}{45260775} \\ 0 \end{pmatrix};$$

$$v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{62656}{3143109375} \end{pmatrix}; B = \begin{pmatrix} \frac{917148937}{3972602880} & \frac{92147253}{399125000} & \frac{18645763}{31035960} \\ \frac{-4835550625}{205986816} & \frac{-20976327}{894040} & \frac{690888125}{1609272} \\ \frac{12577412}{1005795} & \frac{4403973888}{349234375} & \frac{12446464}{1005795} \\ \frac{8161180625}{27156465} & \frac{1321074}{111755} & \frac{-2391358750}{8887725} \\ 0 & 0 & 0 \end{pmatrix};$$

$$D = \begin{pmatrix} \frac{312941}{20690640} & \frac{3018459}{199562500} & \frac{345583}{5172660} \\ \frac{-113072125}{51496704} & \frac{-4905927}{2235100} & \frac{23973625}{804636} \\ \frac{-7146695}{1609272} & \frac{-309668832}{69846875} & \frac{22788256}{201159} \\ \frac{-160001125}{231735168} & \frac{-77388}{111755} & \frac{56805500}{1810431} \\ 0 & 0 & 0 \end{pmatrix}; \text{and } V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2567}{128741760} & \frac{6966}{349234375} & \frac{2798}{1005795} \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{3.4}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{and } A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

substituting A^0 and A^1 in equation (3.4) and solving for R, the values of R are obtained as $0, 0, 0$ and 1.
According to [13], the block method equation (2.7) are zero-stable, since from (3.4), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

3.3. Consistency and convergence of the method. The linear multistep method in equation (3.1) is said to be consistent if it has order $p \geq 1$. Equation (2.7) is of order 8. According to the theorem [7], the necessary and sufficient condition for a linear multistep method to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

3.4. **Numerical Experiments.** Here in this section, different types of differential problems which include first, second and third-order ordinary differential equations are examined to test the effectiveness of the new scheme.

Problem 1: Consider the stiff problem,

$$y_1' = -y_1; \quad y_1(0) = 1 \quad h = 0.1, \quad 0 \leq x \leq 1$$
$$y_2' = -2000y_2; \quad y_2(0) = 1$$
$$y_1(x) = e^{-x}, \quad y_2(x) = e^{-2000x}$$

Table 1: Comparing the error of the new block method with existing methods for solving Problem 1

x- values	Error in new method	Error in new method	Error in [19]	Error in [19]
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	-1.55E-018	-7.60E-004	6.00E-010	9.52E-001
0.2	-2.81E-018	-5.77E-007	9.00E-010	9.10E-001
0.3	-3.82E-018	-7.84E-013	1.10E-009	8.62E-001
0.4	-4.61E-018	-5.95E-009	1.40E-009	8.20E-001
0.5	-5.21E-018	-8.08E-006	1.50E-009	7.80E-001
0.6	-5.66E-018	-6.14E-009	1.70E-009	7.42E-001
0.7	-5.97E-018	-4.67E-012	1.80E-009	7.10E-001
0.8	-6.18E-018	-4.13E-015	1.80E-009	6.72E-001
0.9	-6.29E-018	-3.14E-018	1.80E-009	6.39E-001
1.0	-6.32E-021	-2.38E-021	1.80E-009	6.08E-001

Problem 2: Consider the stiff problem,

$$y''' = 3\sin x; \ y(0) = 1 \ y'(0) = 0 \ y''(0) = -2 \ h = 0.1$$
$$Exact\ Solution : y(x) = 3\cos x + \frac{x^2}{2} - 2$$

Table 2: the error of the new block method with existing methods for solving Problem 2

x- values	Error in the new method	Error in [4]	Error in [12]
0.1	5.4694748E-018	2.000000E-010	6.370460E-13
0.2	1.0864706E-017	4.000000E-010	4.052980E-12
0.3	1.6131788E-017	2.000000E-010	1.00932 6E-11
0.4	2.1218091E-017	2.000000E-010	1.890366E-11
0.5	2.6072798E-017	9.000000E-010	3.033807E-11
0.6	3.0647399E-017	1.100000E-009	4.455258E-11
0.7	3.4896188E-017	1.500000E-009	5.987466E-11
0.8	3.8776710E-017	1.300000E-009	7.711903E-11
0.9	4.2250197E-017	1.500000E-009	9.618412E-11
1.0	4.5281938E-017	2.000000E-009	1.171654E-10

Problem 3:

$$y^{''} = y^{'}; \ y(0) = 0 \ y^{'}(0) = -1 \ h = 0.1$$
$$Exact \ Solution : y(x) = 1 - e^x$$

Table 3: Comparing the error of the new block method with existing methods for solving Problem 3

x- values	Error in the new method	Error in [3]	Error in [12]
0.1	4.600000E-025	2.095826E-010	2.508826E-13
0.2	8.000000E-026	2.092718E-009	6.493175E-11
0.3	5.000000E-026	7.842546E-009	1.683146E-09
0.4	2.500000E-025	2.009500E-008	1.700635E-08
0.5	3.700000E-025	4.199771E-008	1.025454E-07
0.6	1.300000E-025	7.728842E-008	2.558711E-06
0.7	6.000000E-026	1.303844E-007	5.273300E-06
0.8	2.500000E-025	2.064839E-007	8.275935E-06
0.9	3.800000E-025	3.116817E-007	1.161667E-05
1.0	2.000000E-025	4.531001E-007	1.542187E-05

Problem 4:

$$y^{'''} = e^x; \ y(0) = 3; \ y'(0) = 1; \ y''(0) = 5; \ h = 0.1$$
$$Exact \ Solution : y(x) = 2 + 2x^2 + e^x$$

Table 4: Comparing the error of the new block method with existing methods for solving Problem 4

x- values	Error in the new method	Error in [3]	Error in [12]
0.1	1.994384E-018	8.881784E-015	3.369305E-12
0.2	4.198520E-018	3.552714E-014	2.160050E-11
0.3	6.634467E-018	8.304468E-014	5.333245E-11
0.4	9.326604E-018	1.527667E-013	9.988632E-11
0.5	1.230188E-017	2.460254E-013	1.598988E-10
0.6	1.559006E-017	3.668177E-013	2.511404E-10
0.7	1.922407E-017	5.178080E-013	3.961489E-10
0.8	2.324026E-017	7.025491E-013	5.926823E-10
0.9	2.767885E-017	9.254819E-013	8.429168E-10
1.0	3.258424E-017	1.187495E-012	1.144603E-09

Problem 5:

$$y''' = y'(2xy'' + y');\ y(0) = 1;\ y'(0) = \frac{1}{2};\ y''(0) = 0;\ h = 0.01$$
$$Exact\ Solution : y(x) = 1 + \frac{1}{2}\ln\frac{2+x}{2-x}$$

Table 5: Comparing the error of the new block method with existing methods for solving Problem 5

x- values	Error in the new method,	Error in [9]	Error in [16]
0.1	0	1.194048000E-013	2.508826E-13
0.2	0	4.086842000E-013	6.493175E-11
0.3	1.0E-024	1.016689500E-012	1.683146E-09
0.4	0	2.139483600E-012	1.700635E-08
0.5	0	4.083580200E-012	1.025454E-07
0.6	1.0E-024	7.350069300E-012	2.558711E-06
0.7	0	1.279204250E-012	5.273300E-06

Discussion of Results. Tables I-V above show the tabular display of the numerical solutions on the implementation of the newly developed method. It is evident that the block method is more efficient in terms of accuracy when compared with existing methods.

Conclusion: The derivation of a new block approach for directly solving first, second, and third-order ordinary differential equations is investigated in this work. The method's ability to solve three distinct orders of ODEs successfully is one of its most appealing characteristics. The novel approach is applied to several differential equations of order one, two, and three to demonstrate its efficiency. The solutions of test problems solved outperform the existing methods in terms of accuracy, as shown in Tables I-V.

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