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Diagonally Implicit 2-Point Block Backward Differentiation Formula With Two Off-Step Points For Solving Stiff Initial Value Problems

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Abstract

A diagonally implicit form of the 2-point block backward differentiation formula (BDF) with two off-step points is developed for solving stiff initial value problems. The method is derived by introducing a lower triangular matrix in the coefficient matrix of the existing 2-point block BDF method with off-step points for solving stiff ordinary differential equations. The method approximates two solutions values with two off-step points simultaneously at each iteration step and its order is 5. The stability analysis of the method indicates that the method is zero-stable. The absolute stability region of the method is plotted and it indicates that the method is A-stable. Some stiff initial value problems are solved and the results obtained show that the new method outperformed an existing method in terms of accuracy. For execution time, the new method outperformed 2BBDF and has no advantage over 2OBBDF.

1. Introduction

Most real life physical and mathematically modelled problems are modelled as differential equations. A differential equation can be ordinary or partial differential equation. Most differential equations cannot be solved analytically. Rather, an alternative approach has to be used to determine an approximate numerical solution to the problems. In this paper, we shall be concerned with the development of a method for approximate numerical solution of stiff initial value problems in ordinary differential equations of the form:

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$$y^{'} = f(x,y), \qquad y(x_0) = y_0, \qquad a \leq x \leq b \tag{1.1}$$

where the function $f(x,y)$ is continuous and differentiable on the interval $a \leq x \leq b$ and assumed to satisfy the liptschiptz conditions for the existence and uniqueness of the solution of ordinary differential equation (1.1). The system of the differential equation (1.1) is usually found in the study of reaction kinetics, nuclear radioactive decay, control systems, vibration of the springs, whether prediction and so on.
Several attempts have been made to define the concept of stiff system of ordinary differential equations. However, [3] defined stiff problems as problems where certain implicit numerical schemes, in particular backward differentiation formulas (BDF) perform better than explicit numerical methods. For simplicity, [4] describes stiff systems as equations that contain a very fast components as well as very slow components. Considerable efforts have been made by researchers to develop the block implicit methods with A-stability property suitable for solving stiff initial value problems such as [1], [11], [6], [8], [12], [9], [10], [2], [7] among others to construct implicit numerical schemes for solving stiff ODEs. The motivation of this research is to construct a diagonally implicit form of the method developed by [1], by introducing a lower triangular matrix in the formula so as to improve its accuracy items of maximum error and computation time.

2. Derivation of the Method

In this section, we shall be concerned with the derivation of the proposed method we shall call Diagonally Implicit 2-point Block Backward Differentiation Formula with two off-step points.

Definition 2.1. The diagonally implicit 2-point block backward differentiation formula with two off-step points (DI2OBBDF) method is defined as:

$$\sum_{j=o}^{1} \alpha_{j,i} y_{n+j-1} + \sum_{j=\frac{3}{2}}^{1+k} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i} f_{n+k}, \qquad k = i = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{2.1}$$

with j increment of 1 in the first sum and increment of $\frac{1}{2}$ in the second sum. k and i are always assumed to have the same value. $k = i = \frac{1}{2}$ represents the first off-step point, $k = i = 1$ represents the first point, $k = i = \frac{3}{2}$ represents the second off-step point and $k = i = 2$ represents the seccond point. The off-step points here are when $k = \frac{1}{2}$ and $k = \frac{3}{2}$ respectively.

The formula (2.1) is derived using Taylor's series expansion about x_n as follows

First Off-Step Point: $k = i = \frac{1}{2}$**.** To derive the first off-step point $y_{n+\frac{1}{2}}$. Define the linear operator as

$$L_{\frac{1}{2}}\left[y\left(x_n\right), h\right]: \ \alpha_{0,\frac{1}{2}} y_{n-1} + \alpha_{1,\frac{1}{2}} y_n + \alpha_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{1}{2}} - h\beta_{\frac{1}{2},\frac{1}{2}} f_{n+\frac{1}{2}} = 0. \tag{2.2}$$

The associated approximate relationship for (2.2) can be written as

$$\alpha_{0,\frac{1}{2}} y(x_n - h) + \alpha_{1,\frac{1}{2}} y(x_n) + \alpha_{\frac{3}{2},\frac{1}{2}} y(x_n + \frac{1}{2}h) - h\beta_{\frac{1}{2},\frac{1}{2}} f\left(x_n + \frac{1}{2}h\right) = 0. \tag{2.3}$$

Expanding (2.3) as a Taylor's series about x_n and collect the like terms gives

$$C_{0,\frac{1}{2}} y(x_n) + C_{1,\frac{1}{2}} h y^{'}(x_n) + C_{\frac{3}{2},\frac{1}{2}} h^2 y^{''}(x_n) + \cdots = 0. \tag{2.4}$$

where,

$$\left.\begin{aligned} C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}} + \alpha_{1,\frac{1}{2}} + \alpha_{\frac{3}{2},\frac{1}{2}} = 0 \\ C_{1,\frac{1}{2}} &= -\alpha_{0,\frac{1}{2}} + \tfrac{1}{2}\alpha_{\frac{3}{2},\frac{1}{2}} - \beta_{\frac{1}{2},\frac{1}{2}} = 0 \\ C_{\frac{3}{2},\frac{1}{2}} &= \tfrac{1}{2}\alpha_{0,\frac{1}{2}} + \tfrac{1}{8}\alpha_{\frac{3}{2},\frac{1}{2}} - \tfrac{1}{2}\beta_{\frac{1}{2},\frac{1}{2}} = 0 \end{aligned}\right\}. \tag{2.5}$$

In deriving the first off-step point $y_{n+\frac{1}{2}}$ the coefficient $\alpha_{\frac{3}{2},\frac{1}{2}}$ is normalized to 1. Solving the simultaneous equation (2.5) for the values of $\alpha_{j,i's}$ and $\beta_{j,i's}$ and Substituting the values in (2.3) yields:

$$y_{n+\frac{1}{2}} = -\frac{1}{8}y_{n-1} + \frac{9}{8}y_n + \frac{3}{8}hf_{n+\frac{1}{2}}. \tag{2.6}$$

Applying the same procedure as in the derivation of first off-step point, the remaining points formulae are obtained. Therefore the diagonally implicit 2-point block backward differentiation formula with two off-step points is obtained as follows:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\tfrac{1}{8}y_{n-1} + \tfrac{9}{8}y_n + \tfrac{3}{8}hf_{n+\frac{1}{2}} \\ y_{n+1} &= \tfrac{1}{21}y_{n-1} - \tfrac{4}{7}y_n + \tfrac{32}{21}y_{n+\frac{1}{2}} + \tfrac{2}{7}hf_{n+1} \\ y_{n+\frac{3}{2}} &= -\tfrac{3}{122}y_{n-1} + \tfrac{25}{61}y_n - \tfrac{75}{61}y_{n+\frac{1}{2}} + \tfrac{225}{122}y_{n+1} + \tfrac{15}{61}hf_{n+\frac{3}{2}} \\ y_{n+2} &= \tfrac{2}{135}y_{n-1} - \tfrac{1}{3}y_n + \tfrac{32}{27}y_{n+\frac{1}{2}} - 2y_{n+1} + \tfrac{32}{15}y_{n+\frac{3}{2}} + \tfrac{2}{9}hf_{n+2} \end{aligned}\right\} \tag{2.7}$$

The method (2.7) is of order 5, with error constant given by:

$$E_6 = \begin{bmatrix} -\frac{37}{20} \\ -\frac{80}{63} \\ -\frac{125}{244} \\ -\frac{4}{45} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3. Stability Analysis of the Method

The stability properties of the method (2.7) based on zero-stability and A-stability are discussed in this section. The region of absolute stability of the method is determined by applying the scalar test differential equation of the form $y' = \lambda y$ where λ is complex constant with negative real part. Substituting the scalar test differential equation $y' = \lambda y$ in the formula (2.7) gives:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\tfrac{1}{8}y_{n-1} + \tfrac{9}{8}y_n + \tfrac{3}{8}\lambda h y_{n+\frac{1}{2}} \\ y_{n+1} &= \tfrac{1}{21}y_{n-1} - \tfrac{4}{7}y_n + \tfrac{32}{21}y_{n+\frac{1}{2}} + \tfrac{2}{7}\lambda h y_{n+1} \\ y_{n+\frac{3}{2}} &= -\tfrac{3}{122}y_{n-1} + \tfrac{25}{61}y_n - \tfrac{75}{61}y_{n+\frac{1}{2}} + \tfrac{225}{122}y_{n+1} + \tfrac{15}{61}\lambda h y_{n+\frac{3}{2}} \\ y_{n+2} &= \tfrac{2}{135}y_{n-1} - \tfrac{1}{3}y_n + \tfrac{32}{27}y_{n+\frac{1}{2}} - 2y_{n+1} + \tfrac{32}{15}y_{n+\frac{3}{2}} + \tfrac{2}{9}\lambda h y_{n+2} \end{aligned}\right\}, \tag{3.1}$$

By rearranging and collecting the like terms we have

$$\left.\begin{aligned} y_{n+\frac{1}{2}} - \tfrac{3}{8}\lambda h y_{n+\frac{1}{2}} &= -\tfrac{1}{8}y_{n-1} + \tfrac{9}{8}y_n \\ -\tfrac{32}{21}y_{n+\frac{1}{2}} + y_{n+1} - \tfrac{2}{7}\lambda h y_{n+1} &= \tfrac{1}{21}y_{n-1} - \tfrac{4}{7}y_n \\ \tfrac{75}{61}y_{n+\frac{1}{2}} - \tfrac{225}{122}y_{n+1} + y_{n+\frac{3}{2}} - \tfrac{15}{61}\lambda h y_{n+\frac{3}{2}} &= -\tfrac{3}{122}y_{n-1} + \tfrac{25}{61}y_n \\ -\tfrac{32}{27}y_{n+\frac{1}{2}} + 2y_{n+1} - \tfrac{32}{15}y_{n+\frac{3}{2}} + y_{n+2} - \tfrac{2}{9}\lambda h y_{n+2} &= \tfrac{2}{135}y_{n-1} - \tfrac{1}{3}y_n \end{aligned}\right\}, \tag{3.2}$$

These equations can be written in matrix form as

(3.3)
$$\begin{bmatrix} (1-\frac{3}{8}\lambda h) & 0 & 0 & 0 \\ -\frac{32}{21} & (1-\frac{2}{7}\lambda h) & 0 & 0 \\ \frac{75}{61} & -\frac{225}{122} & (1-\frac{15}{61}\lambda h) & 0 \\ -\frac{32}{27} & 2 & -\frac{32}{15} & (1-\frac{2}{9}\lambda h) \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{8} & 0 & \frac{9}{8} \\ 0 & \frac{1}{21} & 0 & -\frac{4}{7} \\ 0 & -\frac{3}{122} & 0 & \frac{25}{61} \\ 0 & \frac{2}{135} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

Let $\overline{h} = \lambda h$ in equation (3.3). This leads to

(3.4)
$$\begin{bmatrix} (1-\frac{3}{8}\overline{h}) & 0 & 0 & 0 \\ -\frac{32}{21} & (1-\frac{2}{7}\overline{h}) & 0 & 0 \\ \frac{75}{61} & -\frac{225}{122} & (1-\frac{15}{61}\overline{h}) & 0 \\ -\frac{32}{27} & 2 & -\frac{32}{15} & (1-\frac{2}{9}\overline{h}) \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{8} & 0 & \frac{9}{8} \\ 0 & \frac{1}{21} & 0 & -\frac{4}{7} \\ 0 & -\frac{3}{122} & 0 & \frac{25}{61} \\ 0 & \frac{2}{135} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

Definition 3.1. If m is the number of block and r is the number of points in the block, then $n = mr$.

Here $r = 2$ and $n = 2m$. By [5], we let

$$Y_m = \begin{bmatrix} y_{2m+\frac{1}{2}} \\ y_{2m+1} \\ y_{2m+\frac{3}{2}} \\ y_{2m+2} \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{2(m-1)+\frac{1}{2}} \\ y_{2(m-1)+1} \\ y_{2(m-1)+\frac{3}{2}} \\ y_{2(m-1)+2} \end{bmatrix} = \begin{bmatrix} y_{2m+\frac{3}{2}} \\ y_{2m-1} \\ y_{2m-\frac{1}{2}} \\ y_{2m} \end{bmatrix} = \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

Equation (4.1) can be written in the following form

$$AY_m = BY_{m-1} \tag{3.5}$$

where,

$$A = \begin{bmatrix} (1-\frac{3}{8}\overline{h}) & 0 & 0 & 0 \\ -\frac{32}{21} & (1-\frac{2}{7}\overline{h}) & 0 & 0 \\ \frac{75}{61} & -\frac{225}{122} & (1-\frac{15}{61}\overline{h}) & 0 \\ -\frac{32}{27} & 2 & -\frac{32}{15} & (1-\frac{2}{9}\overline{h}) \end{bmatrix}, B = \begin{bmatrix} 0 & -\frac{1}{8} & 0 & \frac{9}{8} \\ 0 & \frac{1}{21} & 0 & -\frac{4}{7} \\ 0 & -\frac{3}{122} & 0 & \frac{25}{61} \\ 0 & \frac{2}{135} & 0 & -\frac{1}{3} \end{bmatrix}$$

The stability polynomial of the method $R\left(t, \overline{h}\right)$ is obtained by evaluating the determinant of$(At - B)$ to obtain

$$R\left(t,\overline{h}\right) = \det\left(At - B\right) = \left\| \begin{bmatrix} (1-\frac{3}{8}\overline{h}) & 0 & 0 & 0 \\ -\frac{32}{21} & (1-\frac{2}{7}\overline{h}) & 0 & 0 \\ \frac{75}{61} & -\frac{225}{122} & (1-\frac{15}{61}\overline{h}) & 0 \\ -\frac{32}{27} & 2 & -\frac{32}{15} & (1-\frac{2}{9}\overline{h}) \end{bmatrix} t - \begin{bmatrix} 0 & -\frac{1}{8} & 0 & \frac{9}{8} \\ 0 & \frac{1}{21} & 0 & -\frac{4}{7} \\ 0 & -\frac{3}{122} & 0 & \frac{25}{61} \\ 0 & \frac{2}{135} & 0 & -\frac{1}{3} \end{bmatrix} \right\|$$

$$= -\frac{11}{1281}t^2 - \frac{121}{30744}t^2\overline{h} - \frac{1}{1464}t^2\overline{h}^2 - \frac{13595}{15372}t^3\overline{h} - \frac{3127}{15372}t^3\overline{h}^2$$

$$-\frac{10}{1281}t^3\overline{h}^3 - \frac{34705}{30744}t^4\overline{h} + \frac{2069}{4393}t^4\overline{h}^2 - \frac{221}{2562}t^4\overline{h}^3 + \frac{5}{854}t^4\overline{h}^4 + t^4 - \frac{1270}{1281}t^3 \tag{3.6}$$

The absolute stability region of the method (2.7) is determined by plotting the graph of

$$R\left(t,\overline{h}\right) = 0 \tag{3.7}$$

or

$$R\left(t,\overline{h}\right) = -\frac{11}{1281}t^2 - \frac{121}{30744}t^2\overline{h} - \frac{1}{1464}t^2\overline{h}^2 - \frac{13595}{15372}t^3\overline{h} - \frac{3127}{15372}t^3\overline{h}^2$$

$$-\frac{10}{1281}t^3\overline{h}^3 - \frac{34705}{30744}t^4\overline{h} + \frac{2069}{4393}t^4\overline{h}^2 - \frac{221}{2562}t^4\overline{h}^3 + \frac{5}{854}t^4\overline{h}^4 + t^4 - \frac{1270}{1281}t^3$$

$$= 0 \tag{3.8}$$

To show that the method (2.7) is zero stable, we substitute $\overline{h} = 0$ in equation (3.8) to obtain the first characteristics polynomial as:

$$-\frac{11}{1281}t^2 - \frac{1270}{1281}t^3 + t^4 = 0. \tag{3.9}$$

Solving equation (3.9) for t, we obtain the following roots as:

$$t = 0,\ t = 0,\ t = 1,\ t = -\frac{11}{1281}$$

Definition 3.2. A method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

Therefore the values of t above indicate that the method is zero-stable since no magnitude of the root is greater than one and the root $t = 1$ is simple.

Definition 3.3. A method is said to be A-stable if the absolute stability region covers the whole left half plane.

The boundary of the stability region of the method (2.7) is determined by substituting $t = e^{i\theta}$ $0 \leq \theta \leq 2\pi$ into equation (3.8). The graph of the stability region for the method (2.7) using Maple18 software is given below.

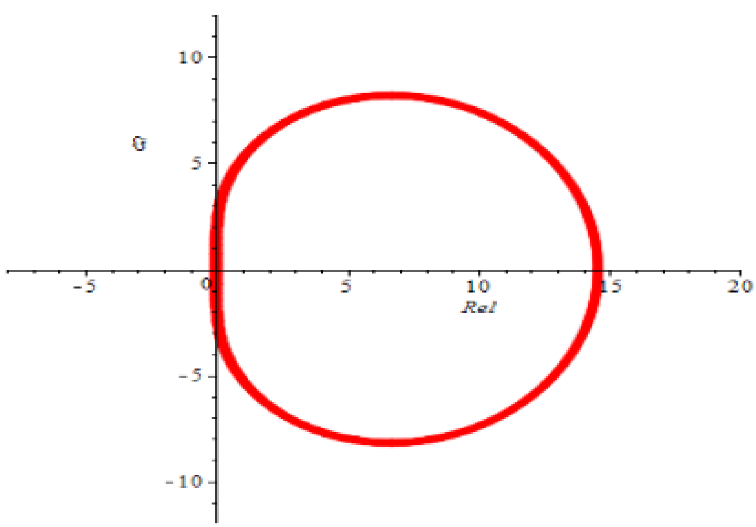

Figure 1: Stability region of the diagonally implicit 2-point BBDF with two off-step points

Thus, the region of stability is the region outside the circular shape. It indicates that the method is A-stable since the region covers the entire negative half plane.

4. Implementation Of the Method

Newton's iteration is applied for the implementation of the method. The description of the iteration is given below we first start by defining the error.

Definition 4.1. Let y_i and $y(x_i)$ be the approximate and exact solutions of (1.1) respectively. Then the absolute error is given by

$$(error_i)_t = \left|(y_i)_t - (y(x_i))_t\right| \tag{4.1}$$

The maximum error is defined by:

$$MAXE = \underbrace{max}_{1\le i\le T}(\underbrace{max(error_i)_t}_{1\le i\le N}), \tag{4.2}$$

where T is the total number of steps and N is the number of equations.
Define

$$\left.\begin{aligned} F_{\frac{1}{2}} &= y_{n+\frac{1}{2}} - \tfrac{3}{8}hf_{n+\frac{1}{2}} - \varepsilon_{\frac{1}{2}} \\ F_1 &= y_{n+1} - \tfrac{32}{21}y_{n+\frac{1}{2}} - \tfrac{2}{7}hf_{n+1} - \varepsilon_1 \\ F_{\frac{3}{2}} &= y_{n+\frac{3}{2}} + \tfrac{75}{61}y_{n+\frac{1}{2}} - \tfrac{225}{122}y_{n+1} - \tfrac{15}{61}hf_{n+\frac{3}{2}} - \varepsilon_{\frac{3}{2}} \\ F_2 &= y_{n+2} - \tfrac{32}{27}y_{n+\frac{1}{2}} + 2y_{n+1} - \tfrac{32}{15}y_{n+\frac{3}{2}} - \tfrac{2}{9}hf_{n+2} - \varepsilon_2 \end{aligned}\right\}, \tag{4.3}$$

where the back values are defined as:

$$\left.\begin{aligned} \varepsilon_{\frac{1}{2}} &= -\tfrac{1}{8}y_{n-1} + \tfrac{9}{8}y_n \\ \varepsilon_1 &= \tfrac{1}{21}y_{n-1} - \tfrac{4}{7}y_n \\ \varepsilon_{\frac{3}{2}} &= -\tfrac{3}{122}y_{n-1} + \tfrac{25}{61}y_n \\ \varepsilon_2 &= \tfrac{2}{135}y_{n-1} - \tfrac{1}{3}y_n \end{aligned}\right\}, \tag{4.4}$$

Let $y_{n+j}^{(i+1)}, \quad j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.5}$$

Newton's iteration for the diagonally implicit 2-point of block BDF with two off-step points takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.6}$$

$$\Rightarrow$$

$$e_{n+j}^{(i+1)} = -\left(F'_j\left(y_{n+j}^{(i)}\right)\right)^{-1}\left(F_j\left(y_{n+j}^{(i)}\right)\right), \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.7}$$

This can be written as

$$\left(F'_j\left(y_{n+j}^{(i)}\right)\right)e_{n+j}^{(i+1)} = -\left(F_j\left(y_{n+j}^{(i)}\right)\right), \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.8}$$

Equation (4.8) is equivalent to the following matrix form:

$$\underbrace{\begin{bmatrix} 1-\frac{3}{8}\frac{\partial F^{(i)}_{n+\frac{1}{2}}}{\partial y^{(i)}_{n+\frac{1}{2}}} & 0 & 0 & 0 \\ -\frac{32}{21} & 1-\frac{2}{7}\frac{\partial F^{(i)}_{n+1}}{\partial y^{(i)}_{n+1}} & 0 & 0 \\ \frac{75}{21} & -\frac{225}{122} & 1-\frac{15}{61}\frac{\partial F^{(i)}_{n+\frac{3}{2}}}{\partial y^{(i)}_{n+\frac{3}{2}}} & 0 \\ -\frac{32}{27} & 2 & -\frac{32}{15} & 1-\frac{2}{9}h\frac{\partial F^{(i)}_{n+2}}{\partial y^{(i)}_{n+2}} \end{bmatrix}}_{Jacobian Matrix} \begin{bmatrix} e^{(i+1)}_{n+\frac{1}{2}} \\ e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+\frac{3}{2}} \\ e^{(i+1)}_{n+2} \end{bmatrix} \tag{4.9}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ \frac{32}{21} & -1 & 0 & 0 \\ -\frac{75}{61} & \frac{225}{122} & -1 & 0 \\ \frac{32}{27} & -2 & \frac{32}{15} & -1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} + h \begin{bmatrix} \frac{3}{8} & 0 & 0 & 0 \\ 0 & \frac{2}{7} & 0 & 0 \\ 0 & 0 & \frac{15}{61} & 0 \\ 0 & 0 & 0 & \frac{2}{9} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{bmatrix}$$

A computer code using C programming language will be written to implement equation (4.9).

5. Problems Tested

The following stiff initial value problems were tested using the method.

Problem 1

$$y' = -20y + 20\sin x + \cos x, y(0) = 1, 0 \le x \le 2$$

Exact solution:

$$y(x) = \sin x + e^{-20x}$$

Source: [1].

Problem 2

$$y_1' = -20y_1 - 19y_2, y_1(0) = 2, 0 \le x \le 20$$
$$y_2' = -19y_1 - 20y_2, y_2(0) = 0,$$

Exact solution:

$$y_1 = e^{-39x} + e^{-x}$$
$$y_2 = e^{-39x} - e^{-x}$$

Eigen values: $\lambda = -1$ and $\lambda = -39$

Source: [1].

Problem 3

$$y_1' = 198y_1 + 199y_2, y_1(0) = 1, 0 \le x \le 10$$
$$y_2' = -398y_1 - 399y_2, y_2(0) = -1,$$

Exact solution:

$$y_1(x) = e^{-x},$$
$$y_2(x) = -e^{-x}$$

Eigen values: $\lambda = -1$ and $\lambda = -200$

Source: [6].

Numerical Results

To determine the efficiency and reliability of the developed method, some stiff problems are solved and the numerical results presented in table 1-3 are compared with those of 2-point block backward differentiation formula and the 2-point block backward differentiation formula with off-step points. The following notations are used in tables.

$2BBDF$: 2-point block backward differentiation formula.

$2OBBDF$: 2-point block backward differentiation formula with off-step points.

$DI2OBBDF$: Diagonally implicit 2-point block backward differentiation formula with off-step two points.

h: Step size.

NS: Total number of steps.

$MAXE$: Maximum error.

$Time$: Computation time in seconds.

Table 1: Numerical results for Problem 1

h	$Method$	NS	$MAXE$	$Time$
10^{-2}	$2BBDF$	100	$7.82684e-002$	$1.97300e+000$
	$2OBBDF$	100	$8.05923e-002$	$6.33001e-004$
	$DI2OBBDF$	100	$1.67159e-002$	$2.30100e-002$
10^{-3}	$2BBDF$	1000	$1.40171e-002$	$1.97400e+000$
	$2OBBDF$	1000	$1.30480e-002$	$1.56188e-003$
	$DI2OBBDF$	1000	$2.93901e-004$	$2.46300e-002$
10^{-4}	$2BBDF$	10000	$1.46435e-003$	$1.99300e+000$
	$2OBBDF$	10000	$1.46355e-003$	$1.90020e-002$
	$DI2OBBDF$	10000	$3.12080e-006$	$6.34600e-002$
10^{-5}	$2BBDF$	100000	$1.47063e-004$	$2.12500e+000$
	$2OBBDF$	100000	$1.47055e-004$	$1.50976e-001$
	$DI2OBBDF$	100000	$3.14064e-008$	$4.54200e-001$
10^{-6}	$2BBDF$	1000000	$1.47126e-005$	$3.31900e+000$
	$2OBBDF$	1000000	$1.47126e-005$	$3.43859e-000$
	$DI2OBBDF$	1000000	$3.14264e-010$	$4.19400e+000$

Table 2: Numerical results for Problem 2

h	$Method$	NS	$MAXE$	$Time$
10^{-2}	$2BBDF$	1000	$6.29433e-002$	$1.87000e+000$
	$2OBBDF$	1000	$7.00088e-002$	$3.07111e-003$
	$DI2OBBDF$	1000	$3.41667e-002$	$2.48700e-002$
10^{-3}	$2BBDF$	10000	$2.61104e-002$	$1.84100e+000$
	$2OBBDF$	10000	$2.58857e-002$	$3.00881e-002$
	$DI2OBBDF$	10000	$1.05482e-003$	$1.23100e-001$
10^{-4}	$2BBDF$	100000	$2.84789e-003$	$2.05000e+000$
	$2OBBDF$	100000	$2.84492e-003$	$2.92691e-001$
	$DI2OBBDF$	100000	$1.17955e-005$	$1.12600e+000$
10^{-5}	$2BBDF$	1000000	$2.87180e-004$	$3.70500e+000$
	$2OBBDF$	1000000	$2.87150e-004$	$4.18554e+000$
	$DI2OBBDF$	1000000	$1.19422e-007$	$1.09400e+001$
10^{-6}	$2BBDF$	10000000	$2.87420e-005$	$2.05300e+001$
	$2OBBDF$	10000000	$2.87417e-005$	$5.30707e-001$
	$DI2OBBDF$	10000000	$1.19569e-009$	$1.10100e+002$

Table 3: Numerical results for Problem 3

h	$Method$	NS	$MAXE$	$Time$
10^{-2}	$2BBDF$	500	$7.18323e-003$	$1.83900e+000$
	$2OBBDF$	500	$7.17251e-003$	$1.22776e-002$
	$DI2OBBDF$	500	$7.58511e-005$	$5.47400e-002$
10^{-3}	$2BBDF$	5000	$7.34012e-004$	$1.87700e+000$
	$2OBBDF$	5000	$7.33813e-004$	$2.66598e-002$
	$DI2OBBDF$	5000	$7.82953e-007$	$6.96400e-002$
10^{-4}	$2BBDF$	50000	$7.35584e-005$	$1.93200e+000$
	$2OBBDF$	50000	$7.35564e-005$	$1.38798e-001$
	$DI2OBBDF$	50000	$7.85438e-009$	$5.64900e-001$
10^{-5}	$2BBDF$	500000	$7.35741e-006$	$2.55700e+000$
	$2OBBDF$	500000	$7.35740e-006$	$2.46586e-000$
	$DI2OBBDF$	500000	$7.85689e-011$	$5.34500e+000$
10^{-6}	$2BBDF$	5000000	$7.35742e-007$	$9.19300e+000$
	$2OBBDF$	5000000	$7.35775e-007$	$2.63500e-001$
	$DI2OBBDF$	5000000	$7.90261e-011$	$5.31700e+001$

In order to give the visual impact on the performance of the methods, the graphs of $Log_{10}(MAXE)$ against h for the problems tested are plotted. Given are the graphs of the scaled maximum error for different problems tested.

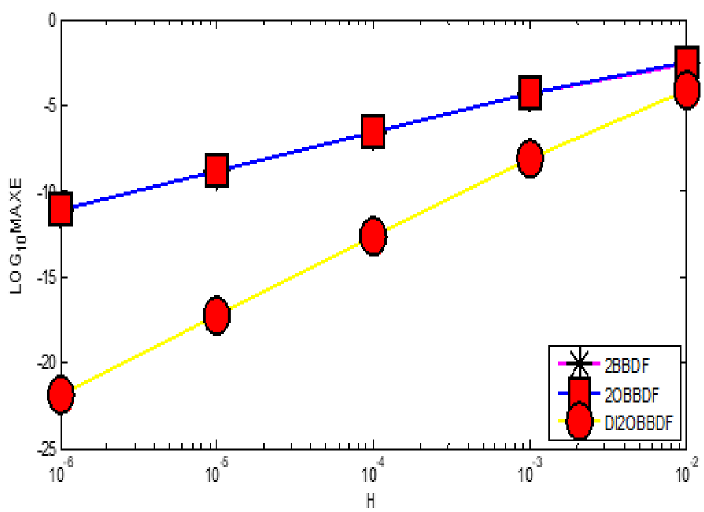

Figure 2: Efficiency Curves for Problem 1

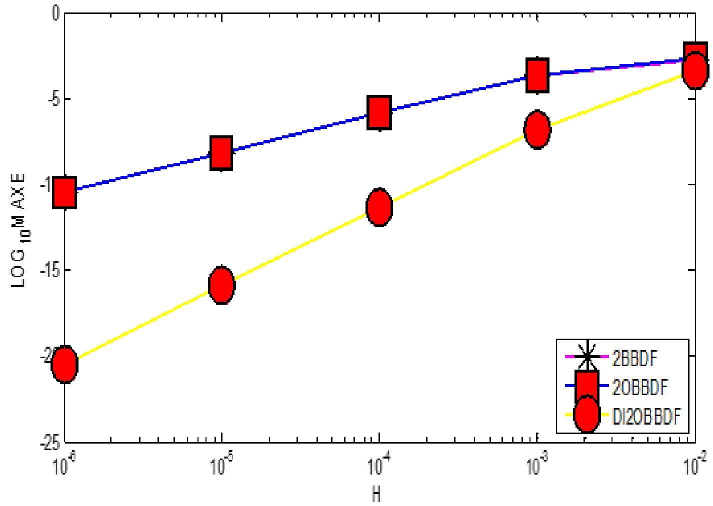

Figure 3: Efficiency Curves for Problem 2

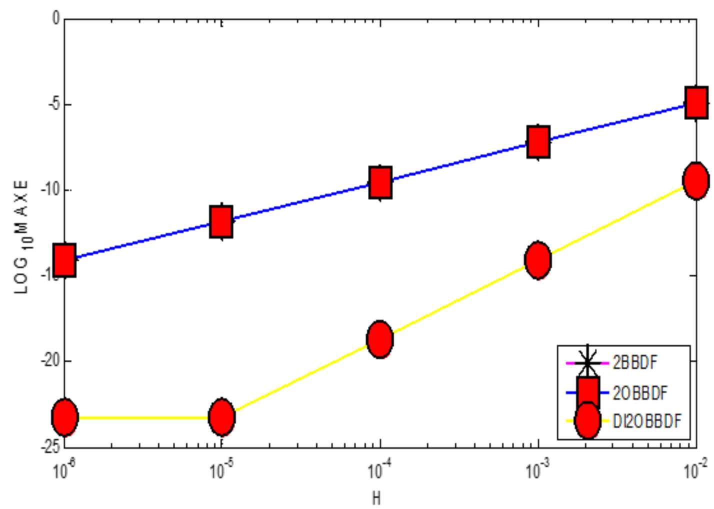

Figure 4: Efficiency Curves for Problem 3

Discussion of Result: From the tables above, the effectiveness and efficiency of the new method developed can be clearly seen from the maximum error (MAXE) and computation time (Time). For all the problems tested, the method is seen to have outperformed the existing 2BBDF and the 2OBBDF methods in terms of Maximum error. However, the computation time of our method is better than that in the 2BBDF method and competes with the one in the 2OBBDF method. Thus, our new method serves as an alternative solver for the numerical integration of first order stiff initial value problems.

Conclusions: A new diagonally implicit block numerical scheme for the integration of stiff systems called diagonally implicit 2-point block backward differentiation formula with two off-step points is derived. The method developed approximates two solution values with two off-step points and it is of order 5. The stability analysis of the method indicates that the method is both zero and A-stable. To demonstrate the effectiveness and efficiency of the method, some stiff initial value problems are solved and compared with some existing algorithms in terms of accuracy and execution time. The numerical results obtained indicate that the new method outperformed the other existing methods in terms of accuracy. For execution time, the new method outperformed 2BBDF and has no advantage over 2OBBDF.

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