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Result on Partially Ordered Compact Spaces

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Abstract

Partially ordered compact spaces were considered in the paper. E denote a compact Harsdorff space, $C(E)$ denote the set of all continuous real valued functions on the compact Harsdorff space $E, C^+(E)$ is the set $\{f \in C(E) : f(t) \geq 0,\ \forall t \in E\}$. $\leq$ denotes a relation of partial order given in E. That is, a binary relation that is reflexive $(x \leq x \forall x \in E)$ antisymmetric $x \leq y$ and $y \leq x implies x = y,\ \forall x, y\ \in E)$ and transitive $(x \leq y$ and $y \leq z implies x \leq z, \forall x, y, z \in E)$ and the class of all monotonic increasing functions belonging to $C^+(E)$, is denoted by $F(E, \leq)$. This expository paper established that $\leq$ is the relation of partial order determined by $F(E, \leq)$ if and only if E is monotonically separated relatives to $\leq$. Also it is shown that there is a one-to-one correspondence $F \to \leq_F$ between the uniformly closed semi-algebras with identity and type 1 in $C(E)$ and the relations of partial order in E relative to which E is monotonically separated.

1. Introduction

This paper studied partially ordered compact spaces. Asiru [1] studied locally compact semi-algebras, Asiru [2] examined semi-simple locally compact semi-algebras, Asiru and Bassey [3] worked on closed type 2 semi-algebras, Bonsall and Tomiuk [4] worked on Semi-algebras generated by Linear operator. Bonsall [5] studied compact linear operator from an algebraic stand point. Bonsall [6] worked on linear operators in a complete positive cones. Bonsall [7] studied locally compact semi-algebras. Bonsall [8] examined positive operators compact in an auxiliary topology. Also, Bonsall [9], [10] studied semi-algebras of continuous functions. Bonsall [11] studied Type 2 semi-algebras of continuous functions

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and Tomiko [12] studied type 1 continuous functions. Also, Nandakumar and Cornelis [13] examined locally compact semi-algebras generated by a commuting operator family.
It was established in [1] that $H = F(E, \leq_H)$ if and only if H is a uniformly closed semi-algebra with identity and of type 1. It is natural to ask under what conditions can a relation $\leq$ of partial order be true that $\leq$ is identical with the relation of partial order determined by $F(E, \leq)$. It is shownin this paper that a necessary and sufficient condition for this is that E be monotonically separated. This establish the fact that there is natural one-to-one correspondence between uniformly closed semi-algebras of type 1 with identity and relations of partial order in E with respect to which E is monotonically separated.

1.1. **Notation and Definitions.**

1.1.1. *Notation.* Let E be a compact Harsdorff space and let $\leq$ denote a relation of partial order given in E(i.e a binary relation that is reflexive $(x \leq x, \forall x \in E)$, antisymmetric $(x \leq y \text{ and } y \leq x$ implies $x=$ y $\forall x, y \in E)$ and transitive $(x \leq z \text{ whenever } x \leq y \text{ and } y \leq z$, $\forall x, y, z \in E)$.

1.1.2. *Definition.*

Definition 1.1. A non-empty subset F of $C(E)$is called a cone if$f + g \in F \text{ and } \alpha f \in F, (\alpha \geq 0)$whenever$f, g \in F$. A non-empty subset $F \text{ of } C(E)$ is called a semi-algebra if $F$is a cone and $fg \in F$ and is said to be closed if it is uniformly closed. It is called a semi-algebra with identity if it containing the unit function 1(x).

Definition 1.2. Given a non-negative integer $n, (n \in \mathbb{Z}^+)$,a semi-algebra F is said to be of type n if $(f^n/(1 + f)) \in F$ whenever $f \in F$. A semi-algebra with identity is called semi-algebra of type 0.

Definition 1.3. Given a subset F of $C(E)$, we define by F a partial order $\leq_F$in E by taking $x \leq_F y$ if and only if $f(x) \leq f(y) \forall f \in F$. Given a partial order $\leq$in E, a real valued function defined on E is said to be monotonically increasing if $f(x) \leq f(y)$ *whenever* $x \leq y$. The class of all monotonically increasing functions belonging to $C^+(E)$ is denoted by $F(E, \leq)$.

Remark. Denote by 1 the unit function: $1(x) = 1$. It has been shown by Bonsall [10] that a closed semi-algebra F with the unit is the class of all $f \in C^+(E)$ monotonically increasing with respect to $\leq_F$. A closed semi-algebra of type n generated by $U_1, U_2, ...U_n \in C^+(E)$is the smallest closed semi-algebra of type n containing $U_1, U_2, ..., U_{n.}$
It has also been shown by Bonsall [11] that the closed semi-algebra of type 2 in $C([0, 1])$ generated by $X(x) = x$ is the class of all $f \in C^+([0, 1])$ non-decreasing, convex, and satisfying $f(0) = 0$.

Definition 1.4. A subset A of E is a lower set if $x \in A$whenever $x \leq y$ for some $y \in A$ and it is an upper set if $y \in A$whenever $x \leq y$ for some $x \in A$. The space E is said to be monotonically separated if, given a, b, $\in E$ with $b \nleq a$, then $\exists$ an open lower set A and an upper set $B : a \in A,\ b \in B \text{ and } A \cap B = \phi$.

Definition 1.5. Given a set H of real valued functions defined on E, the relation of partial order determined by H is the relation $\leq_H$ obtained by taking $x \leq_H y$ if and only if $f(x) \leq f(y), \forall f \in H$.

1.1.3. *Lemmas.*

Lemma 1.6. *Let H be a subset of $C(E)$, and let $\leq_H$ be the relation of partial order determined by H. Then E is monotonically separated relative to $\leq_{H.}$*

Proof. Suppose $b \nleq_H a$. By definition of $\leq_H \exists\, f \in H$ with $f(a) < f(b)$. Choose λ such that $f(a) < \lambda < f(b)$ and take $A = \{t : f(t) < \lambda\}$ *and* $B = \{t : f(t) < \lambda\}$. Clearly, A is an open lower set containing a, B is an open upper set containing b, *and* $A \cap B = \phi$. Thus E is monotonically separated relativeto$\leq_H$. □

Lemma 1.7. *Let E be monotonically separated relative to $\leq$. Given a closed lower set A and a closed upper set B with $A \cap B = \phi, \exists$ an open lower set G and an open set H such that $A \subset G,\ B \subset H,\ G \cap H = \phi$.*

Proof. Given $a \in A$ and $b \in B$, we have $h \nless a$, and so $\exists$ an open lower set $G(a,b)$ and an open upper set

$$H(a,b) : a \in G(a,b),\ b \in H(a,b),\ G(a,b) \cap H(a,b) = \phi \tag{1.1}$$

With b fixed, the compact set A has a finite covering $G(a_i, b)\,(say)$, $(i = 1,\ 2,\ \ldots,\ n)$
Let

$$G(b) = \bigcup_{i=1}^{n} G(a_i, b) \cdots \tag{1.2}$$

and

$$H(b) = \bigcap_{i=1}^{n} H(a_i, b) \cdots \tag{1.3}$$

Then $G(b)$ is an open lower set containing A, $H(b)$ is an open upper set containing b, and $G(b) \cap H(b) = \phi$. Suppose $H(b_1),\ \ldots,\ H(b_p)$ is a covering of B *by* such sets $H(b)$, and take

$$G = \bigcap_{i=1}^{p} G(b_i) \cdots \tag{1.4}$$

$$H = \bigcup_{i=1}^{p} H(b_i), \cdots \tag{1.5}$$

Clearly these sets G *and* H have the required properties. □

Lemma 1.8. *(Urysohn's lemma) A space X is normal if and only if given any two disjoint subsets A and B of $X, \exists$ a continuous function $f : X \longrightarrow [a,b] : f(A) = a$ and $f(B) = b\ \forall\ a,b \in R$.*

Lemma 1.9. *Suppose E is monotonically separated relative to $\leq$. Let A be a closed lower set and B a closed upper set with $A \cap B = \phi$. Then $\exists$ a continuous monotonic increasing function f such that*

$$f(t) = 0,\ (t \in A),\ f(t) = 1,\ (t \in B),\ 0 \leq f(t) \leq 1,\ \forall\ t \in E.$$

Proof. Using the method of Urysohn's lemma we proceed as follows:
Take $A_0^* = A,\ A_1, = E/B$. By Lemma 3.2 $\exists$ an open lower set $A_{1/2}$ and a closed lower set $A_{1/2}^*$ such that

$$A_0^* \subset A_{\frac{1}{2}} \subset A_{\frac{1}{2}}^* \subset A_1 \cdots \tag{1.6}$$

Applying Lemma 3.2 again, we obtain open lower sets $A_{\frac{1}{4}}$. $A_{\frac{3}{4}}$ and close lower sets $A_{\frac{1}{4}}^*$. $A_{\frac{3}{4}}^*$ with

$$A_0^* \subset A_{\frac{1}{4}} \subset A_{\frac{1}{4}}^* \subset A_{\frac{1}{2}}^* \subset A_{\frac{3}{4}} \subset A_{\frac{3}{4}}^* \subset A_1 \cdots \textit{ and so on.} \tag{1.7}$$

Hence by induction we may construct an open lower set A_r and closed lower set A_r^* corresponding to each rational number λ of the form:

$$\frac{m}{2^n},\ (m = 0,\ 1,\ \ldots, 2^n,\ n = 1\ ,\ 2,\ \ldots)$$

$$A_r \subset A_r^*, (0 < r < 1), A_r^* \subset A_{r'}, (0 \leq r' \leq 1) \cdots \tag{1.8}$$

Now let f be the function defined on E by taking $f(t) = 1$, if $t \in B$ and $f(t) = inf\{r : t \in B\}$ if$t \notin B$. Just as in the proof of Urysohn's Lemma except that A_r^* is used in place of $\overline{A_r}$, we have

$$f \in C(E),\ 0 \leq f(t) \leq 1, (\forall\ t \in E),\ f(t) = 0,\ (t \in A), f(t) = 1,\ (t \in B) \cdots \tag{1.9}$$

Next is to show that f is monotonic increasing. If $s \leq s'$ and $s' \in A_r$, then also $s \in A_r$ and so $f(s) \leq f(s')$. If $s \leq s'$ and $s' \in B$, then $f(s') = 1$ and so $f(s) \leq f(s')$ in this case also. Hence f is monotonic increasing and hence the proof. □

2. Main Result

The main result is considered in this section.

Theorem 2.1. *Let $\leq$ be a relation of partial order in a compact Hausdorff space E, and let $F(E, \leq)$ donote the class of all non-negative monotonic increasing continuous function on E.Then $\leq$ is the relation of partial order determined by $F(E, \leq)$ if and only if E is monotonically separated relative to $\leq$.*

Proof. We need to show that $F(E, \leq)$ is a uniformly closed semi-algebra with identity and of type 1. In [1] it is asserts that $H = F(E, \leq)$ if and only if H is a uniformly closed semi-algebra with identity and of type 1. Hence, the result follows from the above assertion.

Suppose E be monotonically separated relative to $\leq$, and that $a,\ b \in E$then there exit a closed lower set X and a closed upper set Y such that $aEX,\ bEY\ and\ X \cap Y = \emptyset$with

$$f(b) \leq f(a),\quad f \in F(E, \leq) \cdots \tag{2.1}$$

We show that $b \leq a$. Suppose on the contrary that

$$b \nleq a \cdots \tag{2.2}$$

and let

$$L(a) = \{s : s \leq a\},\ U(b) = \{s : b \leq s\} \cdots \tag{2.3}$$

Then $L(a)$ is a closed lower set for if $t \notin L(a)$ then $t \not\leq a$, and so there exists an open upper set $G$with $t \in G$. It follows that $L(a) \cap G = \emptyset$, so$L(a)$ is closed, similarly $U(b)$ is a closed upper set.
Applying Lemma 3.2, we can see that $\exists\ f_0 \in F(E, \leq)$ such that$f_0(t) = 0,\ (t \in L(a))$and $f_0(t) = 1,\ (t \in U(b))$. In particular, $f_0(a) < f_0(b)$ which contradicts our hypothesis, hence $b \leq a$.This completes the proof that $\leq$ is the relation determined by $F(E, \leq)$, and the converse was in Lemma 3.2. □

Corollary 2.2. *There is a natural one-to-one correspondence $F \rightarrow\ \leq_F$ between the uniformly closed semi-algebras with identity and type 1 in $C(E)$ and the relation of partial order$\leq$ in E if E is monotonically separated rative to $\leq$.*

Conclusions: This paper studied partially ordered compact spaces and we have established the necessary and sufficient condition for which the compact space Ebe monotonically separated. We also established the fact that there is a natural one-to-one corresponding $F \rightarrow\ \leq_F$ between uniformly closed semi-algebras of type 1 with identity and relations of partial order $(\leq)$ in E with respect to which E is monotonically separated.

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