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An Extended 2-point Super Class of Block Backward Differentiation Formula with off-step Points for Solving Stiff Initial Value Problems

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Abstract

In this paper, we considered an extended 2-point Super class of block backward differentiation formula for solving stiff initial value problems (IVPs) and introduced two off-step points in it. The method produces two solution values together with two off-step points simultaneously at each iteration. The varying free parameter ρ is chosen in the interval $(-1, 1)$ which leads to different forms of the formulae. A careful choice of $\rho = \frac{2}{5}$ is made. Stability analysis shows that the method is A-stable. Numerical results show that the new method performs better than the existing 2BBDF in terms of accuracy and competes with the 2ESBBDF. Thus, suitable for solving stiff initial value problems.

1. Introduction

Most real-world problems common in natural science and engineering are constructed as mathematical models before they are solved. Mathematical models help in translating real-world problems into mathematical formulations which often are in form of differential equations. Differential equations have the remarkable ability of predicting the world around us. They can describe exponential growth and decay, the population growth of species, or the change in investment return over time. In this paper, we are concerned with the numerical solution of IVPs of the form:

$$y' = f(x, y), \qquad y(x_0) = y_0, \qquad a \le x \le b \tag{1.1}$$

Stiff initial value problems are special cases of initial value problem of differential equations. A stiff initial value problem is a differential equation in which some specified numerical methods for solving differential equations are numerically unstable, unless the step-length

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is taken to be very small. Stiff initial value problems have wide variety of applications including the studies of spring and damping systems, chemical kinetics, electrical circuits etc.

The Backward differentiation formula (BDF) is a family of implicit methods for solving stiff initial value problems and is one of the most widely used algorithms for such problems. The method was first introduced by Curtiss and Hirschfelder [4]. The implementation of the BDF was discussed by Gear [5]. The order and accuracy were improved by Cash [3]. Among the earliest research on block backward differentiation formula (BBDF) was the work of Ibrahim *et al.* [6] who came up with r-point BBDF. Suleiman [11] discussed the Super class of BBDF, the method involved a parameter ρ which can be varied over the interval (-1,1) which generates different sets of formulae. The off-step point method for the solution of Stiff IVPs was discussed by Abasi [1] which calculates two solution values together with two off-step points simultaneously at each iteration.

Recent research on block backward differentiation formula (BBDF) include the Fixed coefficient BBDF which was proposed by Ibrahim *et al.* [7]. A two-step hybrid BBDF was discussed by Bakari *et al.* [2] which was derived based on the interpolation and collocation approach and was found to be suitable for solving stiff systems. An extended 2-point Super class of BBDF was proposed by Musa [10]. Zawawi *et. al.* [13] proposed a Variable step BBDF with independent parameter which was derived using the variable step size scheme and was found to outperform existing methods. A 3-step numerical approximant based on block hybrid BDF was developed by Ishaku *et al.* [8] which was obtained through continuous collocation approach with Legendre polynomial as basis function. The method was found to be consistent with order 6.

This paper studied method [10] and introduces two off step points in it to come up with a new numerical method for solving stiff IVPs.

2. Derivation of the Method

The method to be developed computes two solution values y_{n+1} and y_{n+2} together with two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the points where the step size is halved. The formula is computed using two back values y_n and y_{n-1} with step size h.

Definition 2.1. The scheme associated with the extended 2-point Super class of Block backward differentiation formula with off-step points (2ESOBBDF) is defined as:

$$\sum_{\mathrm{j=0}}^{1} \alpha_{\mathrm{j,i}} \mathrm{y}_{\mathrm{n+j-1}} + \sum_{\mathrm{j}=\frac{3}{2}}^{3} \alpha_{\mathrm{j,i}} \mathrm{y}_{\mathrm{n+j-1}} = \mathrm{h}\beta_{\mathrm{k,i}}(\mathrm{f}_{\mathrm{n+k}} - \rho \mathrm{f}_{\mathrm{n+k-\frac{3}{2}}}) \tag{2.1}$$

with j having an increment of 1 in the first sum and $\frac{1}{2}$ in the second sum.
The formula (2.1) is derived from Taylor's series expansion as follows:

Definition 2.2. The Linear operator associated with the extended 2-point Super class of Block backward differentiation formula with off-step points (2ESOBBDF) is defined as:

$$\begin{aligned} L_i[y(x_n), h] &= \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{\frac{3}{2},i} y_{n+\frac{1}{2}} + \alpha_{2,i} y_{n+1} \\ &+ \alpha_{\frac{5}{2},i} y_{n+\frac{3}{2}} + \alpha_{3,i} y_{n+2} - h\beta_{k,i}(f_{n+k} - \rho f_{n+k-\frac{3}{2}}) = 0, \end{aligned} \tag{2.2}$$

where $k = i = \frac{1}{2}, 1, \frac{3}{2}, 2$.

2.1. **First Point.** $k=i=\frac{1}{2}$ To derive the first point $y_{n+\frac{1}{2}}$. We substitute $k=i=\frac{1}{2}$ in (2.2) to obtain

$$\begin{aligned} \alpha_{0,\frac{1}{2}} y_{n-1} + \alpha_{1,\frac{1}{2}} y_n + \alpha_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_{2,\frac{1}{2}} y_{n+1} \\ + \alpha_{\frac{5}{2},\frac{1}{2}} y_{n+\frac{3}{2}} + \alpha_{3,\frac{1}{2}} y_{n+2} - h\beta_{\frac{1}{2},\frac{1}{2}} (f_{n+\frac{1}{2}} - \rho f_{n-1}) = 0, \end{aligned} \tag{2.3}$$

The associated approximate relationship for (2.3) can be written as:

$$\begin{aligned} \alpha_{0,\frac{1}{2}} y(x_n - h) + \alpha_{1\frac{1}{2},} y(x_n) + \alpha_{\frac{3}{2},\frac{1}{2}} y(x_n + \tfrac{1}{2}h) + \alpha_{2,\frac{1}{2}} y(x_n + h) + \\ \alpha_{\frac{5}{2},\frac{1}{2}} y(x_n + \tfrac{3}{2}h) + \alpha_{3,\frac{1}{2}} y(x_n + 2h) - h\beta_{\frac{1}{2},\frac{1}{2}} \left[f(x_n + \tfrac{1}{2}h) - \rho f(x_n - h) \right] = 0, \end{aligned} \tag{2.4}$$

Expanding (2.4) as a Taylor's series about x_n, equating both sides and collecting like terms gives

$$\begin{aligned} C_{0,\frac{1}{2}} y(x_n) + C_{1,\frac{1}{2}} h y'(x_n) + C_{\frac{3}{2},\frac{1}{2}} h^2 y''(x_n) \\ + C_{2,\frac{1}{2}} h^3 y'''(x_n) + C_{\frac{5}{2},\frac{1}{2}} h^4 y'^{v}(x_n) + C_{3,\frac{1}{2}} h^5 y^{v}(x_n) + \cdots = 0, \end{aligned} \tag{2.5}$$

where,

$$\left.\begin{aligned} C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}} + \alpha_{1,\frac{1}{2}} + \alpha_{\frac{3}{2},\frac{1}{2}} + \alpha_{2,\frac{1}{2}} + \alpha_{\frac{5}{2},\frac{1}{2}} + \alpha_{3,\frac{1}{2}} = 0 \\ C_{1,\frac{1}{2}} &= -\alpha_{0,\frac{1}{2}} + \tfrac{1}{2}\alpha_{\frac{3}{2},\frac{1}{2}} + \alpha_{2,\frac{1}{2}} + \tfrac{3}{2}\alpha_{\frac{5}{2},\frac{1}{2}} + 2\alpha_{3,\frac{1}{2}} - \beta_{\frac{1}{2},\frac{1}{2}} + \rho\beta_{\frac{1}{2},\frac{1}{2}} = 0 \\ C_{\frac{3}{2},\frac{1}{2}} &= \tfrac{1}{2}\alpha_{0,\frac{1}{2}} + \tfrac{1}{8}\alpha_{\frac{3}{2},\frac{1}{2}} + \tfrac{1}{2}\alpha_{2,\frac{1}{2}} + \tfrac{9}{8}\alpha_{\frac{5}{2},\frac{1}{2}} + 2\alpha_{3,\frac{1}{2}} - \tfrac{1}{2}\beta_{\frac{1}{2},\frac{1}{2}} - \rho\beta_{\frac{1}{2},\frac{1}{2}} = 0 \\ C_{2,\frac{1}{2}} &= -\tfrac{1}{6}\alpha_{0,\frac{1}{2}} + \tfrac{1}{48}\alpha_{\frac{3}{2},\frac{1}{2}} + \tfrac{1}{6}\alpha_{2,\frac{1}{2}} + \tfrac{9}{16}\alpha_{\frac{5}{2},\frac{1}{2}} + \tfrac{4}{3}\alpha_{3,\frac{1}{2}} - \tfrac{1}{8}\beta_{\frac{1}{2},\frac{1}{2}} + \tfrac{1}{2}\rho\beta_{\frac{1}{2},\frac{1}{2}} = 0 \\ C_{\frac{5}{2},\frac{1}{2}} &= \tfrac{1}{24}\alpha_{0,\frac{1}{2}} + \tfrac{1}{384}\alpha_{\frac{3}{2},\frac{1}{2}} + \tfrac{1}{24}\alpha_{2,\frac{1}{2}} + \tfrac{27}{128}\alpha_{\frac{5}{2},\frac{1}{2}} + \tfrac{2}{3}\alpha_{3,\frac{1}{2}} - \tfrac{1}{48}\beta_{\frac{1}{2},\frac{1}{2}} - \tfrac{1}{6}\rho\beta_{\frac{1}{2},\frac{1}{2}} = 0 \\ C_{3,\frac{1}{2}} &= -\tfrac{1}{120}\alpha_{0,\frac{1}{2}} + \tfrac{1}{3840}\alpha_{\frac{3}{2},\frac{1}{2}} + \tfrac{1}{120}\alpha_{2,\frac{1}{2}+\frac{81}{1280}}\alpha_{\frac{5}{2},\frac{1}{2}} + \tfrac{32}{120}\alpha_{3,\frac{1}{2}} - \tfrac{1}{384}\beta_{\frac{1}{2},\frac{1}{2}} + \tfrac{1}{24}\rho\beta_{\frac{1}{2},\frac{1}{2}} = 0 \end{aligned}\right\} \tag{2.6}$$

The coefficient of the first point $y_{n+\frac{1}{2}}(i.e.\ \alpha_{\frac{3}{2},\frac{1}{2}})$ is normalized to 1. Solving the simultaneous equation (2.6) for the values of $\alpha_{j,i's}$ and $\beta_{j,i's}$ and substituting the values in (2.4) yields:

The first point is therefore obtained as

$$\begin{aligned} y_{n+\frac{1}{2}} = -\tfrac{1}{20}\ \tfrac{174\rho+1}{80\rho-3} y_{n-1} + \tfrac{9}{4}\ \tfrac{20\rho+1}{80\rho-3} y_n + \tfrac{27}{4}\ \tfrac{10\rho-1}{80\rho-3} y_{n+1} \\ -\tfrac{9}{5}\ \tfrac{16\rho-1}{80\rho-3} y_{n+\frac{3}{2}} + \tfrac{1}{4}\ \tfrac{20\rho-1}{80\rho-3} y_{n+2} + \tfrac{3}{80\rho-3} h f_{n+\frac{1}{2}} - \tfrac{3}{80\rho-3} h\rho f_{n-1}. \end{aligned} \tag{2.7}$$

Similar procedure is applied as in the derivation of first point to obtain the remaining points as:

$$\begin{aligned} y_{n+1} = -\tfrac{1}{45}\ \tfrac{134\rho-2}{19\rho+2} y_{n-1} - \tfrac{1}{3}\ \tfrac{17\rho+4}{19\rho+2} y_n + \tfrac{4}{9}\ \tfrac{47\rho+16}{19\rho+2} y_{n+\frac{1}{2}} \\ + \tfrac{4}{15}\ \tfrac{31\rho-16}{19\rho+2} y_{n+\frac{3}{2}} - \tfrac{1}{9}\ \tfrac{13\rho-4}{19\rho+2} y_{n+2} + \tfrac{4}{19\rho+2} h f_{n+1} - \tfrac{4}{19\rho+2} h\rho f_{n-\frac{1}{2}}; \end{aligned} \tag{2.8}$$

$$\begin{aligned} y_{n+\frac{3}{2}} = \tfrac{1}{4}\,\tfrac{4\rho+1}{16\rho-31} \mathrm{y}_{n-1} + \tfrac{5}{4}\ \tfrac{38\rho-5}{16\rho-31} \mathrm{y}_n - \tfrac{5(16\rho-5)}{16\rho-31} \mathrm{y}_{n+\frac{3}{2}} \\ + \tfrac{45}{4}\ \tfrac{4\rho-5}{16\rho-31} \mathrm{y}_{n+1} + \tfrac{5}{4}\ \tfrac{2\rho+5}{16\rho-31} \mathrm{y}_{n+2} - \tfrac{15}{16\rho-31} \mathrm{hf}_{n+\frac{3}{2}} + \tfrac{15}{16\rho-31} \mathrm{h}\rho \mathrm{f}_{n}; \text{ and} \end{aligned} \tag{2.9}$$

$$\begin{aligned} y_{n+2} = -\tfrac{1}{5}\ \tfrac{\rho+4}{\rho-54} y_{n-1} + \tfrac{9(\rho+2)}{\rho-54} y_n + \tfrac{4(3\rho-16)}{\rho-54} y_{n+\frac{1}{2}} \\ -\tfrac{27(\rho-4)}{\rho-54} y_{n+1} + \tfrac{36}{5}\ \tfrac{\rho-16}{\rho-54} y_{n+\frac{3}{2}} - \tfrac{12}{\rho-54} h f_{n+2} + \tfrac{12}{\rho-54} h\rho f_{n+\frac{1}{2}}. \end{aligned} \tag{2.10}$$

Therefore, the extended 2-point Super class of block backward differentiation formula with off-step point is given by:

$$
\text{(2.11)}\quad \left.\begin{aligned}
&y_{n+\frac{1}{2}} = -\frac{1}{20}\,\frac{174\rho+1}{80\rho-3}y_{n-1} + \frac{9}{4}\,\frac{20\rho+1}{80\rho-3}y_n + \frac{27}{4}\,\frac{10\rho-1}{80\rho-3}y_{n+1}\\
&-\frac{9}{5}\,\frac{16\rho-1}{80\rho-3}y_{n+\frac{3}{2}} + \frac{1}{4}\,\frac{20\rho-1}{80\rho-3}y_{n+2} + \frac{3}{80\rho-3}hf_{n+\frac{1}{2}} - \frac{3}{80\rho-3}h\rho f_{n-1}\\[2ex]
&y_{n+1} = -\frac{1}{45}\,\frac{134\rho-2}{19\rho+2}y_{n-1} - \frac{1}{3}\,\frac{17\rho+4}{19\rho+2}y_n + \frac{4}{9}\,\frac{47\rho+16}{19\rho+2}y_{n+\frac{1}{2}}\\
&+\frac{4}{15}\,\frac{31\rho-16}{19\rho+2}y_{n+\frac{3}{2}} - \frac{1}{9}\,\frac{13\rho-4}{19\rho+2}y_{n+2} + \frac{4}{19\tilde{n}+2}hf_{n+1} - \frac{4}{19\rho+2}h\rho f_{n-\frac{1}{2}}\\[2ex]
&y_{n+\frac{3}{2}} = \frac{1}{4}\,\frac{4\rho+1}{16\rho-31}y_{n-1} + \frac{5}{4}\,\frac{38\rho-5}{16\rho-31}y_n - \frac{5(16\rho-5)}{16\rho-31}y_{n+\frac{3}{2}}\\
&+\frac{45}{4}\,\frac{4\rho-5}{16\rho-31}y_{n+1} + \frac{5}{4}\,\frac{2\rho+5}{16\rho-31}y_{n+2} - \frac{15}{16\rho-31}hf_{n+\frac{3}{2}} + \frac{15}{16\rho-31}h\rho f_n\\[2ex]
&y_{n+2} = -\frac{1}{5}\,\frac{\rho+4}{\rho-54}y_{n-1} + \frac{9(\rho+2)}{\rho-54}y_n + \frac{4(3\rho-16)}{\rho-54}y_{n+\frac{1}{2}}\\
&-\frac{27(\rho-4)}{\rho-54}y_{n+1} + \frac{36}{5}\,\frac{\rho-16}{\rho-54}y_{n+\frac{3}{2}} - \frac{12}{\rho-54}hf_{n+2} + \frac{12}{\rho-54}h\rho f_{n+\frac{1}{2}}
\end{aligned}\right\}
$$

For absolute stability of the method, ρ is chosen to be in the interval $(-1, 1)$ as in Suleiman [11]. By choosing and substituting $\rho = \frac{2}{5}$ in (2.11) we obtain the extended 2-point super class of block backward differentiation formula with off-step points method as follows:

$$
\text{(2.12)}\quad \left.\begin{aligned}
y_{n+\frac{1}{2}} &= -\tfrac{353}{2900}y_{n-1} + \tfrac{81}{116}y_n + \tfrac{81}{116}y_{n+1} - \tfrac{243}{725}y_{n+\frac{3}{2}} + \tfrac{7}{116}y_{n+2} + \tfrac{3}{29}hf_{n+\frac{1}{2}} - \tfrac{6}{145}hf_{n-1}\\
y_{n+1} &= -\tfrac{11}{90}y_{n-1} - \tfrac{3}{8}y_n + \tfrac{29}{18}y_{n+\frac{1}{2}} - \tfrac{1}{10}y_{n+\frac{3}{2}} - \tfrac{1}{72}y_{n+2} + \tfrac{5}{12}hf_{n+1} - \tfrac{1}{6}hf_{n-\frac{1}{2}}\\
y_{n+\frac{3}{2}} &= -\tfrac{13}{492}y_{n-1} - \tfrac{85}{164}y_n + \tfrac{35}{123}y_{n+\frac{1}{2}} + \tfrac{255}{164}y_{n+1} - \tfrac{145}{492}y_{n+2} + \tfrac{25}{41}hf_{n+\frac{3}{2}} - \tfrac{10}{41}hf_n\\
y_{n+2} &= \tfrac{11}{670}y_{n-1} - \tfrac{27}{67}y_n + \tfrac{74}{67}y_{n+\frac{1}{2}} - \tfrac{243}{134}y_{n+1} + \tfrac{702}{335}y_{n+\frac{3}{2}} + \tfrac{15}{67}hf_{n+2} - \tfrac{6}{67}hf_{n+\frac{1}{2}}
\end{aligned}\right\}
$$

The method 13 is of order 5.

3. Stability Analysis of the Method

The stability properties the method (2.12) are discussed. We begin with some basic definitions.

Definition 3.1. A Linear multistep method (LMM) is said to be zero stable if no root of the first characteristics polynomial has modulus greater than and any root with modulus 1 is simple.

Definition 3.2. A LMM is said to be A-stable if the stability region covers the entire left half plane.

Rewriting the formula (2.12) in matrix form gives:

$$
\text{(3.1)}\quad \begin{pmatrix} 1 & -\frac{81}{116} & \frac{243}{725} & -\frac{7}{116}\\ -\frac{29}{18} & 1 & \frac{1}{10} & \frac{1}{72}\\ -\frac{35}{123} & -\frac{255}{164} & 1 & \frac{145}{492}\\ -\frac{74}{67} & \frac{243}{134} & -\frac{702}{335} & 1 \end{pmatrix}
\begin{pmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2} \end{pmatrix}
=
\begin{pmatrix} 0 & -\frac{353}{2900} & 0 & \frac{81}{116}\\ 0 & -\frac{11}{90} & 0 & -\frac{3}{8}\\ 0 & -\frac{13}{492} & 0 & -\frac{85}{164}\\ 0 & \frac{11}{670} & 0 & -\frac{27}{67} \end{pmatrix}
\begin{pmatrix} y_{n-\frac{3}{2}}\\ y_{n-1}\\ y_{n-\frac{1}{2}}\\ y_n \end{pmatrix}
$$

$$+h\begin{pmatrix} 0 & -\frac{6}{145} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{10}{41} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} + h\begin{pmatrix} \frac{3}{29} & 0 & 0 & 0 \\ 0 & \frac{5}{12} & 0 & 0 \\ 0 & 0 & \frac{25}{41} & 0 \\ -\frac{6}{67} & 0 & 0 & \frac{15}{67} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}$$

Definition 3.3. Let Y_m and F_m be vectors defined by $Y_m = [y_{n+1}, y_{n+2}, \ldots, y_{n+r}]^T$, $F_m = [f_{n+1}, f_{n+2}, \ldots, f_{n+r}]^T$,$r = 2$ and $n = 2m$.

The equation (2.12) can be represented in the following form:

(3.2) $$A_0 Y_m = A_1 Y_{m-1} + h\left(\mathrm{B_0 F_{m-1} + B_1 F_m}\right)$$

where

$$\mathrm{A}_0 = \begin{pmatrix} 1 & -\frac{81}{116} & \frac{243}{725} & -\frac{7}{116} \\ -\frac{29}{18} & 1 & \frac{1}{10} & \frac{1}{72} \\ -\frac{35}{123} & -\frac{255}{164} & 1 & \frac{145}{492} \\ -\frac{74}{67} & \frac{243}{134} & -\frac{702}{335} & 1 \end{pmatrix}; \qquad \mathrm{A}_1 = \begin{pmatrix} 0 & -\frac{353}{2900} & 0 & \frac{81}{116} \\ 0 & -\frac{11}{90} & 0 & -\frac{3}{8} \\ 0 & -\frac{13}{492} & 0 & -\frac{85}{164} \\ 0 & \frac{11}{670} & 0 & -\frac{27}{67} \end{pmatrix};$$

$$\mathrm{B}_0 = \begin{pmatrix} 0 & -\frac{6}{145} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{10}{41} \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \mathrm{B}_1 = \begin{pmatrix} \frac{3}{29} & 0 & 0 & 0 \\ 0 & \frac{5}{12} & 0 & 0 \\ 0 & 0 & \frac{25}{41} & 0 \\ -\frac{6}{67} & 0 & 0 & \frac{15}{67} \end{pmatrix};$$

$$Y_{m-1} = \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} = \begin{pmatrix} y_{2m-\frac{3}{2}} \\ y_{2m-1} \\ y_{2m-\frac{1}{2}} \\ y_{2m} \end{pmatrix} = \begin{pmatrix} y_{2(m-1)+\frac{1}{2}} \\ y_{2(m-1)+1} \\ y_{2(m-1)+\frac{3}{2}} \\ y_{2(m-1)+2} \end{pmatrix};$$

$$F_{m-1} = \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} = \begin{pmatrix} f_{2m-\frac{3}{2}} \\ f_{2m-1} \\ f_{2m-\frac{1}{2}} \\ f_{2m} \end{pmatrix} = \begin{pmatrix} f_{2(m-1)+\frac{1}{2}} \\ f_{2(m-1)+1} \\ f_{2(m-1)+\frac{3}{2}} \\ f_{2(m-1)+2} \end{pmatrix};$$

$$Y_m = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} y_{2m+\frac{1}{2}} \\ y_{2m+1} \\ y_{2m+\frac{3}{2}} \\ y_{2m+2} \end{pmatrix}; \quad \text{and} \quad F_m = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} = \begin{pmatrix} f_{2m+\frac{1}{2}} \\ f_{2m+1} \\ f_{2m+\frac{3}{2}} \\ f_{2m+2} \end{pmatrix}.$$

Applying the scalar test equation $y' = \lambda y$ into equation (3.2) and letting $\overline{h} = \lambda h$ gives

(3.3) $$A_0 Y_m = A_1 Y_{m-1} + \overline{h}\left(\mathrm{B_0 Y_{m-1} + B_1 Y_m}\right)$$

To find the stability polynomial, the determinant of the following equation is evaluated:

(3.4) $$\left|\left(A_0 - \overline{h}B_1\right)t - \left(A_1 + \overline{h}B_0\right)\right| = 0$$

this implies that

(3.5) $$Det[tA - B] = 0$$

where,

$$A = \left(\begin{pmatrix} 1 & -\frac{81}{116} & \frac{243}{725} & -\frac{7}{116} \\ -\frac{29}{18} & 1 & \frac{1}{10} & \frac{1}{72} \\ -\frac{35}{123} & -\frac{255}{164} & 1 & \frac{145}{492} \\ -\frac{74}{67} & \frac{243}{134} & -\frac{702}{335} & 1 \end{pmatrix} - \begin{pmatrix} \frac{3\overline{h}}{29} & 0 & 0 & 0 \\ 0 & \frac{5\overline{h}}{12} & 0 & 0 \\ 0 & 0 & \frac{25\overline{h}}{41} & 0 \\ -\frac{6\overline{h}}{67} & 0 & 0 & \frac{15\overline{h}}{67} \end{pmatrix} \right)$$

and

$$B = \left(\begin{pmatrix} 0 & -\frac{353}{2900} & 0 & \frac{81}{116} \\ 0 & -\frac{11}{90} & 0 & -\frac{3}{8} \\ 0 & -\frac{13}{492} & 0 & -\frac{85}{164} \\ 0 & \frac{11}{670} & 0 & -\frac{27}{67} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{6\overline{h}}{145} & 0 & 0 \\ 0 & 0 & -\frac{\overline{h}}{6} & 0 \\ 0 & 0 & 0 & -\frac{10\overline{h}}{41} \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

which is equivalent to the stability polynomial:

$$(3.6)\qquad \begin{aligned} R(t,\overline{h}) = {} & \tfrac{104211}{398315}t^2 + \tfrac{1875}{318652}t^4\overline{h}^4 - \tfrac{67275}{637304}t^4\overline{h}^3 + \tfrac{300501}{637304}t^4\overline{h}^2 - \tfrac{1596837}{1593260}t^4\overline{h} \\ & -\tfrac{12}{79663}t\overline{h}^4 - \tfrac{279}{47560}t^2\overline{h}^3 + \tfrac{927}{796630}t\overline{h}^3 + \tfrac{15987}{1593260}t\overline{h} + \tfrac{9009}{1274608}t\overline{h}^2 + \tfrac{24102}{398315}t^2\overline{h}^2 \\ & +\tfrac{293157}{1593260}t^2\overline{h} + \tfrac{2997}{79663}t^3\overline{h}^3 - \tfrac{2712807}{6373040}t^3\overline{h}^2 - \tfrac{708147}{1593260}t^3\overline{h} + \tfrac{353691}{398315}t^4 - \tfrac{457902}{398315}t^3 = 0, \end{aligned}$$

The boundary of the stability region of (2.12) will be determined by substituting $t = e^{-i\theta}$ into (3.6). The graph of the stability region is plotted using Maple and is given in Figure 1.

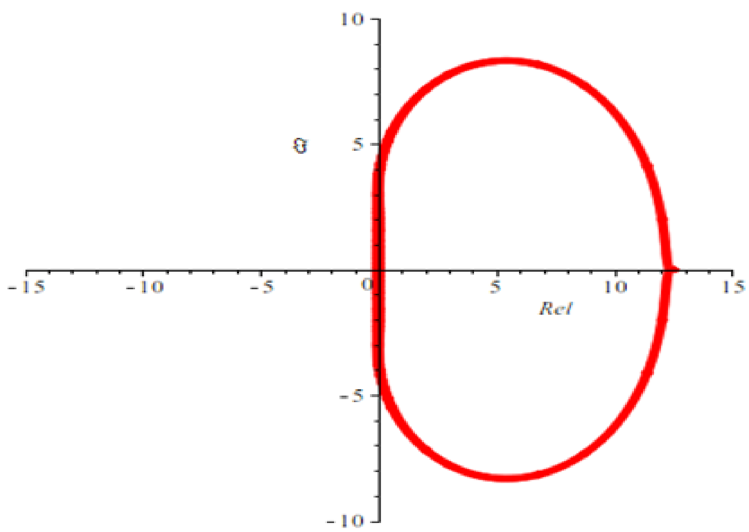

Figure 1: Stability region of 2ESOBBDF

The region of stability is the region outside the circle and it indicates that method (2.12) is A-stable since the stability region covers the entire left half plane.
To show that the method (2.12) is zero stable, we substitute $\overline{h} = 0$ in equation (3.6) to obtain the following first characteristic polynomial as:

$$(3.7)\qquad \frac{104211}{398315}t^2 - \frac{457902}{398315}t^3 + \frac{353691}{398315}t^4 = 0$$

Solving equation (3.7) for t, we obtain the following roots as:

$$t = 0,\ t = 0,\ \ t = 0.2946385404\ ,\ \ t = 1 \tag{3.8}$$

Hence, the values of t indicate that the method (2.12) is zero stable since no root of the first characteristics polynomial has modulus greater than one and the root $t = 1$ is simple.

4. Implementation of the Method

Newton's iteration is employed to implement the method. We consider the implementation when the free parameter $\rho = \frac{2}{5}$ and the same applies for any value of $\rho\ \epsilon\ (-1, 1)$. The iteration is described below.

Definition 4.1. Let y_i and $y(x_i)$ be the approximate and exact solutions of (1.1) respectively. The absolute error is defined by

$$(error_i)_t = \ |(y_i)_t - (y(x_i\ )_t| \tag{4.1}$$

The maximum error is given by

$$MAXE = \underbrace{max}_{1\le i\le T}\,(\underbrace{max(error_i)_t}_{1\le i\le N}) \tag{4.2}$$

where T is the total number of steps and N is the number of equations.
Let $y_{n+j}^{(i+1)}$ denotes the $(i+1)^{th}$ iteration and

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.3}$$

Let

$$\begin{aligned} F_1 &= y_{n+\frac{1}{2}} - \tfrac{81}{116}y_{n+1} + \tfrac{243}{725}y_{n+\frac{3}{2}} - \tfrac{7}{116}y_{n+2} - \tfrac{3}{29}hf_{n+\frac{1}{2}} + \tfrac{6}{145}hf_{n-1} - \varepsilon_1 \\ F_2 &= y_{n+1} - \tfrac{29}{18}y_{n+\frac{1}{2}} + \tfrac{1}{10}y_{n+\frac{3}{2}} + \tfrac{1}{72}y_{n+2} - \tfrac{5}{12}hf_{n+1} + \tfrac{1}{6}hf_{n-\frac{1}{2}} - \varepsilon_2 \\ F_3 &= \ y_{n+\frac{3}{2}} - \tfrac{35}{123}y_{n+\frac{1}{2}} - \tfrac{255}{164}y_{n+1} + \tfrac{145}{492}y_{n+2} - \tfrac{25}{41}hf_{n+\frac{3}{2}} + \tfrac{10}{41}hf_n - \varepsilon_3 \\ F_4 &= y_{n+2} - \tfrac{74}{67}y_{n+\frac{1}{2}} + \tfrac{243}{134}y_{n+1} - \tfrac{702}{335}y_{n+\frac{3}{2}} - \tfrac{15}{67}hf_{n+2} + \tfrac{6}{67}hf_{n+\frac{1}{2}} - \varepsilon_4 \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \varepsilon_1 &= -\frac{353}{2900}y_{n-1} + \frac{81}{116}y_n \\ \varepsilon_2 &= -\frac{11}{90}y_{n-1} - \frac{3}{8}y_n \\ \varepsilon_3 &= -\frac{13}{492}y_{n-1} - \frac{85}{164}y_n \\ \varepsilon_4 &= \frac{11}{670}y_{n-1} - \frac{27}{67}y_n \end{aligned}$$

are the back values.
Then, the Newton's iteration for the 2ESOBBDF takes the form:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left[F_J(y_{n+j}^{(i)})\right]\left[F'_J(y_{n+j}^{(i)})\right]^{-1}, \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.5}$$

which can be written as

$$\left[F'_J(y_{n+j}^{(i)})\right] e_{n+j}^{(i+1)} = -\left[F_J\left(y_{n+j}^{(i)}\right)\right], \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{4.6}$$

where $e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}$
which is equivalent to the following matrix

$$\begin{pmatrix} 1 - \frac{3}{29} h \frac{\partial f_{n+\frac{1}{2}}{}^{(i)}}{\partial y_{n+\frac{1}{2}}{}^{(i)}} & -\frac{81}{116} & \frac{243}{725} & -\frac{7}{116} \\ -\frac{29}{18} & 1 - \frac{5}{12} h \frac{\partial f_{n+1}{}^{(i)}}{\partial y_{n+1}{}^{(i)}} & \frac{1}{10} & \frac{1}{72} \\ -\frac{35}{123} & -\frac{255}{164} & 1 - \frac{25}{41} h \frac{\partial f_{n+\frac{3}{2}}{}^{(i)}}{\partial y_{n+\frac{3}{2}}{}^{(i)}} & \frac{145}{492} \\ -\frac{74}{67} + \frac{6}{67} h \frac{\partial f_{n+2}}{\partial y_{n+2}} & \frac{243}{134} & -\frac{702}{335} & 1 - \frac{15}{67} h \frac{\partial f_{n+2}{}^{(i)}}{\partial y_{n+2}{}^{(i)}} \end{pmatrix} \times \tag{4.7}$$

$$\begin{pmatrix} e_{n+\frac{1}{2}}{}^{(i+1)} \\ e_{n+1}{}^{(i+1)} \\ e_{n+\frac{3}{2}}{}^{(i+1)} \\ e_{n+2}{}^{(i+1)} \end{pmatrix} = \begin{pmatrix} -1 & \frac{81}{116} & -\frac{243}{725} & \frac{7}{116} \\ \frac{29}{18} & -1 & -\frac{1}{10} & -\frac{1}{72} \\ \frac{35}{123} & \frac{255}{164} & -1 & \frac{-145}{492} \\ \frac{74}{67} & -\frac{243}{134} & \frac{702}{335} & -1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}}{}^{(i)} \\ y_{n+1}{}^{(i)} \\ y_{n+\frac{3}{2}}{}^{(i)} \\ y_{n+2}{}^{(i)} \end{pmatrix} + \begin{pmatrix} \frac{3}{29} & 0 & 0 & 0 \\ 0 & \frac{5}{12} & 0 & 0 \\ 0 & 0 & \frac{25}{41} & 0 \\ -\frac{6}{67} & 0 & 0 & \frac{15}{67} \end{pmatrix} \times$$

$$\begin{pmatrix} f_{n+\frac{1}{2}}{}^{(i)} \\ f_{n+1}{}^{(i)} \\ f_{n+\frac{3}{2}}{}^{(i)} \\ f_{n+2}{}^{(i)} \end{pmatrix} + h \begin{pmatrix} 0 & -\frac{6}{145} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{10}{41} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{2}}{}^{(i)} \\ f_{n-1}{}^{(i)} \\ f_{n-\frac{1}{2}}{}^{(i)} \\ f_n{}^{(i)} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}.$$

5. Tested Problems

To validate the efficiency of the method, the following Stiff IVPs are solved:

Problem 1 (Ibrahim *et al.* [6])

$$y' = 100(\sin x - y), y(0) = 0, 0 \ \le x \le 3.$$

Exact Solution:

$$y(x) = \frac{\sin x - 0.01 \cos x + 0.01 e^{-100x}}{1.0001}$$

Problem 2 (Zawawi *et al.* [12])

$$y_1' = 9y_1 + 24y_2 + 5\cos x - \frac{1}{3}\sin x, y_1(0) = \frac{4}{3}, 0 \ \le x \le 10,$$

$$y_2' = -24y_1 - 51y_2 - 9\cos x + \frac{1}{3}\sin x, y_2(0) = \frac{2}{3}$$

Exact Solution:

$$y_1(x) = 2e^{-3x} - e^{-39x} + \frac{1}{3}\cos x,$$

$$y_2(x) = -e^{-3x} + 2e^{-39x} - \frac{1}{3}\cos x.$$

Eigenvalues: -3 and -39

Problem 3 (Musa *et al.* [9])

$$y^{'} = -10y + 10, y(0) = 2, 0 \ \leq x \leq 10.$$

Exact Solution:

$$y\left(x\right) = 1 + e^{-10x}$$

5.1. **Numerical results.** The numerical results for the test problems in section 5 are tabulated below. The problems are solved and compared with two other stiff solvers. The maximum error and the time taken to solve the problems using each of the methods are presented. The graph of $\log_{10}\left(MAXE\right)$ against h for each problem is plotted. The notations used in the tables are listed below:

2ESOBBDF = Extended 2-point Super class of block backward differentiation formula with off-step points;

2ESBBDF = Extended 2-point Super class of block backward differentiation formula;

2BBDF = 2-point block backward differentiation formula;

h = step size;

NS = total number of integration steps;

MAXE = maximum error; and

Time = computation time in seconds.

Table 1: Numerical results for problems 1, 2 and 3

	h	**Method**	**NS**	**MAXE**	**Time**
1.	10^{-2}	2BBDF	150	7.32490e-004	1.8360e-001
		2ESBBDF	150	1.81217e-004	6.3214e-002
		2ESOBBDF	150	2.37665e-004	1.19e-001
	10^{-4}	2BBDF	15000	7.18301e-005	1.13400e-001
		2ESBBDF	15000	1.26692e-006	1.327e+000
		2ESOBBDF	15000	9.61694e-007	5.8e-001
	10^{-6}	2BBDF	1500000	7.35563e-007	3.07600e+001
		2ESBBDF	1500000	1.33363e-010	1.9923e+000
		2ESOBBDF	1500000	1.04513e-010	1.087e+001
2.	10^{-2}	2BBDF	500	1.24297e-001	1.7950e-001
		2ESBBDF	500	7.10662e-002	2.708e-001
		2ESOBBDF	500	7.07357e-002	0.6409e+000
	10^{-4}	2BBDF	5000	5.64465e-003	1.5100e+001
		2ESBBDF	5000	3.96606e-005	1.288e-002
		2ESOBBDF	5000	3.05398e-005	2.761e+000
	10^{-6}	2BBDF	50000	5.69705e-005	8.9500e+001
		2ESBBDF	50000	4.04661e-009	8.426e-002
		2ESOBBDF	50000	3.17310e-009	2.278e+002
3.	10^{-2}	2BBDF	100	7.82684e-002	1.4330e-001
		2ESBBDF	100	1.98228e-003	8.141e+000
		2ESOBBDF	100	1.76065e-002	0.1103e+000
	10^{-4}	2BBDF	10000	1.46435e-003	7.8600e-001
		2ESBBDF	10000	5.28209e-006	4.883e-001
		2ESOBBDF	10000	4.09585e-006	0.6171e+000
	10^{-6}	2BBDF	1000000	1.47126e-005	4.4900e+001
		2ESBBDF	1000000	5.33672e-011	8.166e-001
		2ESOBBDF	1000000	4.18558e-010	7.774e+000

The graph of $\log_{10}(MAXE)$ against h against h is plotted to visualize the performance of the method developed against other methods. Given below are the graphs of the scaled maximum error for the problems tested.

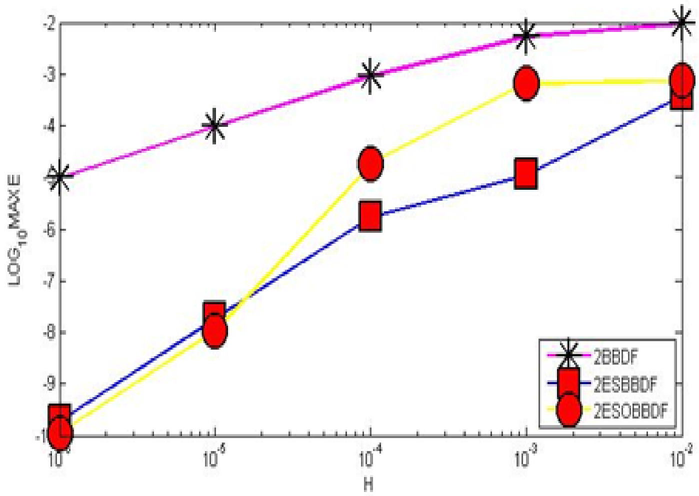

Figure 2: Graph of $\log_{10}(MAXE)$ against h for problem 1

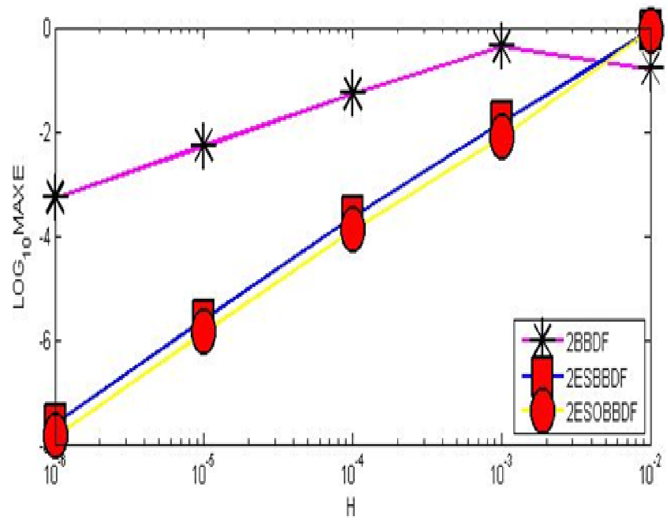

Figure 3: Graph of $\log_{10}(MAXE)$ against h for problem 2

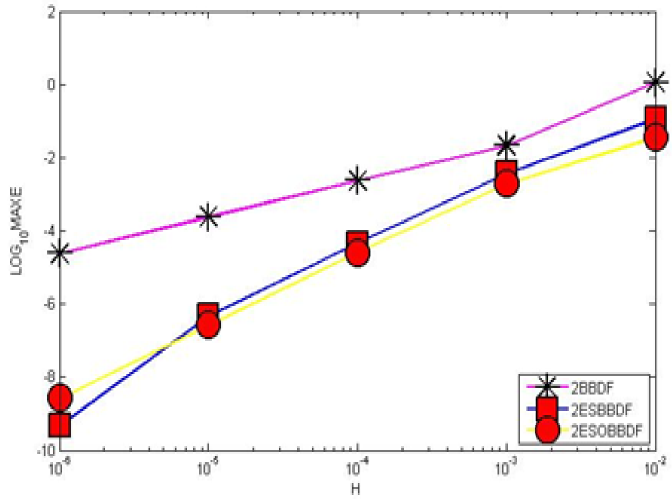

Figure 4: Graph of $\log_{10}(MAXE)$ against h for problem 3

Discussion of Results. It can be seen that the new method (2ESOBBDF) outperformed the 2BBDF in terms of accuracy and competes with the 2ESBBDF. Convergence is evident by the decrease in error as the step size h decreases. This can be seen from the tables when h is reduced (from 0.01, 0.001, 0.0001, 0.00001 and 0.000001). The maximum error decreases when the step size becomes smaller which shows consistency of the method. Thus, the computed solution tends to the exact solution as the step length tends to zero. It is therefore concluded that a new method called an extended 2-point super class of block backward differentiation formula with off-step points for solving stiff initial value problems is developed. It is proven to have outperform the 2BBDF and competes with the 2ESBBDF, which serves as an alternative for solving stiff initial value problems.

Conclusions: An extended 2-point super class of block backward differentiation formula with off-step points (2ESOBBDF) for solving stiff initial value problems is developed. The method computes two solution values with two off-step points at each iteration. A careful choice of $\rho = \frac{2}{5}$ is made and the method is found to be zero stable and A-stable. Numerical results show that the new method is superior to the existing 2BBDF in terms of accuracy and competes with the 2ESBBDF. Thus, suitable for solving stiff initial value problems.

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