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Gronwall-Bellman-Bihari Type Integral Inequalities of a Certain Class of Nonlinear Second Order Differential Equations

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Abstract

In this paper, we obtain nonlinear integral inequalities similar to Gronwall-Bellman-Bihari type. These inequalities are used to establish the boundedness and asymptotic behaviour of solutions of certain class of nonlinear second order differential equations. An example where f is a singular function is presented to show the application of our main results.

1. Introduction

The attaractive Gronwall-Bellman inequality [9] plays a vital role in studying stability, boundedness and asymptotic behaviuor of solutions of differential equations [3, 1]. Many linear and linear generalisations have appeared in the literature [5, 14]. Bihari's inequality [14] is the most important generalisation of the Gronwall-Bellman-Bihari's inequality.
The main role of this method is its effective application in invesgation boundedness and asymptotic behaviour of solutions for nonlinear differential, integro-differential and functional differential equations. Investigation of boundedness and asymptotic behaviour of solution was studied by the following authors such as: Tunc[18], Rogovchenko and Rogovchenko [16, 17], Rogovchenko [15], Huassain [10] Constantine [7],Tian and Fan [19] and Danna [8] and host of others.
In this paper, we used Gronwall-Bellman-Bihari type inequalities which is an extension of Bihari's inequality that was commonly used by some of the aforementioned authors, to consider the boundedness and the asymptotic behaviour of nonlinear second order differential equations.

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2. Preliminary

The development of integral inequalities is based on fundamental results by Gronwall [8], Bellman [3], Bihari [4] and host of others. The aim of this article is to establish some useful integral inequalities which claim their origin to the following integral inequalities proved in [9] and [3].
Before we establish our main results, the following results are needed.

Lemma 2.1. [4], Let $u(t), f(t)$ be positive continuous functions defined on $a \leq t \leq b(b \leq \infty)$ and $K > 0$, $M \geq 0$, further, let $\omega(u)$ be a non negative, nondecreasing continuous function for $u > 0$, then the inequality

$$u(t) \leq K + M \int_a^t f(s)\omega(u(s))ds, \quad a \leq t \leq b \tag{2.1}$$

implies that

$$nu(t) \leq \Omega^{-1}\left(\Omega(K) + M\int_a^t f(s)ds\right) \quad a \leq t \leq b' \leq b \tag{2.2}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{ds}{\omega(s)}, \quad 0 \leq u_0 < u \tag{2.3}$$

In this case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω for t in the subinterval $[a, b']$ of $[a, b]$ such that

$$\Omega(K) + M\int_a^t f(s)ds \in Dom(\Omega^{-1})$$

Theorem 2.2. [11] Let $u(t)$ and $f(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that $K(t, s)$ and its partial derivative $K_t(t, s)$ exist and are continuous for every $t, s \in I$. Suppose $K(t, s) \geq 0$, $K_t(t, 0) \leq 0$ and that the inequality

$$u(t) \leq C + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s K(s,\tau)u(\tau)d\tau\right)ds \; t \in \mathbf{I} \tag{2.4}$$

holds, where C is a nonnegative constant. Then

$$\leq C\left(1 + \int_a^t f(s)\exp\left(f(\tau) + K(\tau,\tau))d\tau\right)ds\right) \quad t \in \mathbf{I} \tag{2.5}$$

3. Development of Nonlinear Integral Inequality

Theorem 3.1. Suppose that $u(t)$ and $f(t)$ are nonnegative and continuous functions on $\mathbf{I} = [a, \infty)$. Suppose that $K(t, s)$ and its partial derivative $K_t(t, s)$ exist and are continuous for every $t, s \in \mathbf{I}$ and $K_t(t, s) \leq 0$. Further, let $\omega \in C(\mathbf{I}, \mathbf{I})$ be nonnegative nondecreasing for $\omega(u) > 0$ on $\mathbf{I}$ for which the inequality

$$u(t) \leq C + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s K(s,\tau)\omega(u(\tau))d\tau\right)ds \; t \in \mathbf{I} \tag{3.1}$$

holds, where C is a nonnegative constant then,

$$\begin{aligned} u(t) \leq \Omega^{-1}\Bigg[C\Bigg(\Omega(C) + \int_a^t f(s)K(s,s)\Bigg(\exp\int_a^s f(\tau)d\tau \\ + \int_a^s \exp\left(\int_\tau^\alpha f(\delta)d(\delta)\right)d\alpha\Bigg)ds\Bigg)\Bigg] \quad t \in \mathbf{I} \end{aligned} \tag{3.2}$$

where Ω is defined in equation (2.3) and $t_1 \in [a, \infty)$ is chosen so that

$$C\left[\left(\Omega(C) + \int_a^t f(s)K(s,s)\left(\exp\int_a^s f(\tau)d\tau + \int_a^s \exp\left(\int_\tau^s f(\delta)d\delta\right)d\tau\right)ds\right)\right] \in Dom(\Omega^{-1})$$

for all t lying in the subinterval $[a, t_1]$ of $\mathbf{I}$. Ω^{-1} is the inverse of the function Ω,. Note Ω^{-1} is also a nondecreasing function.

Proof. Define

$$z(t) = C + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s K(s,\tau)\omega(u(\tau))d\tau\right)ds,\ t \in \mathbf{I}. \tag{3.3}$$

Then,

$$u(t) \le z(t),\ z(a) = C. \tag{3.4}$$

Therefore,

$$z'(t) = f(t)\left(u(t) + \int_a^t K(t,\tau)\omega(u(\tau))d\tau\right). \tag{3.5}$$

It is clear that

$$z'(t) \le f(t)M(t), \tag{3.6}$$

where

$$M(t) = z(t) + \int_a^t K(t,\tau)\omega(z(\tau))d\tau. \tag{3.7}$$

Therefore,

$$M'(t) = z'(t) + K(t,t)\omega(z(t)) + \int_a^t K_t(t,\tau)\omega(z(\tau))d\tau. \tag{3.8}$$

Since $K(t,s)$ and its partial derivative $K_t(t,s)$ exist and are continuous with $K_t(t,s) \le 0$, thus,

$$M'(t) \le z'(t) + P(t), \tag{3.9}$$

where

$$P(t) = K(t,t)\omega(z(t)),\ t \in [a, \infty). \tag{3.10}$$

Let $T \in [a, \infty)$ be any arbitrary number such that

$$M'(t) \le f(t)M(t) + P(T) \text{ for all } a \le t \le T \tag{3.11}$$

Solving equation (3.11) and using the result in equation (3.6), then,

$$z'(t) \le f(t)\left[C\exp\int_a^t f(s)ds + P(T)\int_a^t \exp\left[\int_s^t f(\tau)d\tau\right]ds\right]. \tag{3.12}$$

Let $P(T) > 1$, then

$$z'(t) \le P(T)f(t)\left[C\exp\int_a^t f(s)ds + \int_a^t \exp\left[\int_s^t f(\tau)d\tau\right]ds\right].$$

Defining

$$\Omega(z(t)) = \int_{z_0}^{z(t)} \frac{ds}{\omega(s)}, \quad z \ge z_0 \ge a.$$

and taking $t = T$, therefore,

$$\frac{d\Omega(z(t))}{dt} \le f(t)K(t,t)\left[c\exp\int_a^t f(s)ds + \int_a^t \exp\left(\int_s^t f(\tau)d\tau\right)ds\right]. \tag{3.13}$$

Integrating both sides inequality (3.13) from a to t, we obtain

$$z(t) \le \Omega^{-1}\left[C\left[\Omega(C) + \int_a^t f(s)K(s,s)\left[\exp\int_a^s f(\tau)d\tau + \int_a^s \exp\left(\int_\tau^\alpha f(\sigma)d\sigma\right)d\alpha\right]ds\right]\right].$$

Using the fact that $u(t) \le z(t)$, we arrive at the result (3.2) □

we put $K(t,s) = h(t)g(s)$, to obtain the following result.

Corollary 3.2. Let all conditions in Theorem 3.1 hold with $K(t,s) = h(t)g(s)$ for $h'(t) \le 0$. Then, the inequality (3.1) given as

$$u(t) \le C + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)\left(\int_a^s g(\tau)\omega(u(\tau))d\tau\right)ds \quad t \in \mathrm{I} \tag{3.14}$$

implies

$$u(t) \le \Omega^{-1}\left[C\left[\Omega(C) + \int_a^t f(s)h(s)g(s)\left[\exp\int_a^s f(\tau)d\tau + \int_a^s \exp\left(\int_\tau^\alpha f(\sigma)d\sigma\right)d\alpha\right]ds\right]\right] \quad t \in \mathbf{I} \tag{3.15}$$

where $t_1 \in [a,\infty)$ is chosen such that

$$\left[C\left(\Omega(C) + \int_a^t f(s)h(s)g(s)\left[\exp\int_a^s f(\tau)d\tau + \int_a^s \exp\left(\int_\tau^\alpha f(\sigma)d\sigma\right)d\alpha\right]ds\right]\right] \in Dom(\Omega^{-1})$$

for all t lying in the interval $[a, t_1]$. Also, Ω^{-1} is the inverse of Ω.

Proof. This follows the same argument as in the proof of Theorem 3.1 . □

Remark. If $K(t,s) = 0$ inequality (3.1) reduces to inequality in [2, 3]. If $\omega(u(s)) = u(s)$ inequality (3.14) reduces to inequality in [11]. If $h(t) = 1$ and $\omega(u) = u$, the inequality (3.14) reduces to inequality in [12, 13]. If $K(t,s) = h(t)g(s)$ and $\omega(u) = u$, the inequality (3.14) reduces to inequality in [11].

Theorem 3.3. Suppose that all the conditions of Theorem 3.1 remain valid expect that $K_t(t,s) > 0$, then,

$$u(t) \le \Omega^{-1}\left[C\left(\Omega(C) + \int_a^t f(s)K(s,s)\left(\exp\int_a^s f(\tau)d\tau + r(s)\int_a^s \exp\left(\int_\tau^\alpha f(\delta)d\delta\right)d\alpha\right)ds\right)\right] \quad t \in \mathbf{I} \tag{3.16}$$

where $1 + q(t) \le r(t),\ q(t) = \int_a^t K_t(t,\tau)\omega(z(\tau))d\tau$ and $t_1 \in [a,\infty)$ is chosen so that

$$C\left(\Omega(C) + \int_a^t f(s)K(s,s)\left(\exp\int_a^s f(\tau)d\tau + r(s)\int_a^s \exp\left(\int_\tau^s f(\delta)d\delta\right)\right)ds\right) \in \mathrm{Dom}(\Omega^{-1})$$

for all t lying in the subinterval $[a, t_1]$ of $\mathbf{I}$. Ω^{-1} is the inverse of the function Ω and Ω^{-1} is also a nondecreasing function.

Proof. Since $K_t(t,s) > 0$, using equation (3.8), we set
$q(t) = \int_a^t K_t(t,\tau)\omega(z(\tau))d\tau,\ P(t) = K(t,t)\omega(z(t))$. We write equation (3.11) as

$$M'(t) \le f(t)M(t) + P(T) + q(T) \quad \text{for all} \quad a \le t \le T, \tag{3.17}$$

for any arbitrary $T \in [a,\infty)$.
Solving the inequality using (3.17), we obtain

$$z'(t) \le P(t)f(t)\left(C\exp\int_a^t f(s)ds + (1+q(t))\left(\int_a^t \exp\left(\int_s^t f(\tau)d\tau\right)ds\right)\right)$$

If we let t=T, substituting for P we get

$$\frac{z'(t)}{\omega(z(t))} \le K(t,t)f(t)\left(C\exp\int_a^t f(s)ds + (1+q(t))\left(\int_a^t \exp\left(\int_s^t f(\tau)d\tau\right)ds\right)\right). \tag{3.18}$$

Now with the aid of equation (2.3), we have

$$\frac{d\Omega(z(t))}{dt} \le K(t,t)f(t)\left(C\exp\int_a^t f(s)ds + (1+q(t))\left(\int_a^t \exp\left(\int_s^t f(\tau)d\tau\right)ds\right)\right) \tag{3.19}$$

Integrating and simplifying further yields

$$\begin{aligned} z(t) \le \Omega^{-1}\Bigg[C\Bigg(\Omega(C) + \int_a^t f(s)K(s,s)\Bigg(c\exp\int_a^s f(\tau)d\tau \\ +(1+q(s))\int_a^s \exp\left(\int_\tau^\alpha f(\delta)d\delta\right)d\alpha\Bigg)ds\Bigg)\Bigg]. \end{aligned} \tag{3.20}$$

Since $u(t) \le z(t)$, the result follows. ☐

4. Boundedness and Asymptotic Behaviour of Solution

In this section, our major concern is to discuss the boundedness and asymptotic behaviour of solution of the equation

$$u'' + f(t,u,u') = 0 \tag{4.1}$$

using inequalities established in section 2.

Theorem 4.1. Assume the following hypotheses:

- (C_i) The function $f(t,u,u')$ is continuous on $D = \{(t,u,u'); t \ge 1, u, u' \in \mathbf{R}_+\}$
- (C_{ii}) $|f(t,u,u')| \le h_1(t)\frac{|u|}{t} + h_1(t)h_2(t)\alpha(|u'|)$ for $(t,u,u') \in D$ where $h_1, h_2(t), \alpha$ are nonnegative and continuous functions on $\mathbf{R}_+$.

(C_{iii}) $\alpha(|u'|) \leq \int_1^t h_3(s)\omega\left(\frac{|u|}{s}\right) ds$, where h_3 is a nonnegative and continuous function for every $t \in \mathbf{R}_+$, $\omega(u)$ being a nonnegative, monotonic, nondecreasing, continuous function on $\mathbf{R}_+$

(C_{iv}) $K_1 \exp(K_2) \int_1^t h_1(s)h_2(s)h_3(s)ds = K_3 < \infty$ as $t \to \infty$ for $K_2 = \int_1^t h_1(s)ds$ as $t \to \infty$

Then for every continuous solution $u(t)$ of equation (4.1) is asymptotic to $at+b$ as $t \to \infty$, where a, b are positive constants.

Proof. Suppose equation (4.1) has solution $u(t)$ corresponding to the initial data $u(1) = c_1$ and $u'(1) = c_2$, where c_1 and c_2 are arbitrary numbers. Integrating equation (4.1) twice, over the interval $[1, t]$, we get the following.

$$u'(t) = c_2 - \int_1^t f(s, u, u')ds \tag{4.2}$$

$$u(t) = c_1 + c_2(t-1) - \int_1^t (t-s)f(s, u, u')ds, \tag{4.3}$$

$$|u'(t)| \leq |c_2| + \int_1^t |f(s, u, u')|ds, \tag{4.4}$$

$$|u(t)| \leq (|c_1| + |c_2|)\, t + t\int_1^t |f(s, u, u')|ds, \ t \geq 1. \tag{4.5}$$

Then,

$$\frac{|u|}{t} \leq |c_1| + |c_2| + \int_1^t |f(s, u, u')|ds, \tag{4.6}$$

by condition C_{ii}, it is clear that

$$\frac{|u|}{t} \leq |c_1| + |c_2| + \int_1^t h_1(s)\frac{|u|}{s}ds + \int_1^t h_1(s)h_2(s)\alpha(|u'|)ds, \tag{4.7}$$

for $t \geq s \geq 1$.
Using condition C_{iii} we obtain

$$\frac{|u|}{t} \leq |c_1| + |c_2| + \int_1^t h_1(s)\frac{|u|}{s}ds + \int_1^t h_1(s)h_2(s)\left(\int_1^s h_3(\tau)\omega\left(\frac{|u|}{\tau}\right)d\tau\right)ds. \tag{4.8}$$

Let the right hand side of inequality (4.8) be $z(t)$, then, $\frac{|u(t)|}{t} \leq z(t)$ and since ω is nondecreasing, therefore,

$$\frac{\omega(|u(t)|)}{t} \leq \omega(z(t)),$$

and

$$z(t) = |c_1| + |c_2| + \int_1^t h_1(s)\frac{|u|}{s}ds + \int_1^t h_1(s)h_2(s)\left(\int_1^s h_3(\tau)\omega\left(\frac{|u|}{\tau}\right)d\tau\right)ds. \tag{4.9}$$

It follows that

$$\begin{aligned} z(t) &\leq 1 + |c_1| + |c_2| \\ &+ \int_1^t h_1(s)z(s)ds + \int_1^t h_1(s)h_2(s)\left(\int_1^s h_3(\tau)\omega(z(s))d\tau\right)ds. \end{aligned} \tag{4.10}$$

It is clear that

$$z(t) \le K_1 + \int_1^t h_1(s)z(s)ds + \int_1^t h_1(s)h_2(s)\left(\int_1^s h_3(\tau)\omega(z(\tau))d\tau\right)ds, \tag{4.11}$$

where

$$1 + |c_1| + |c_2| = K_1. \tag{4.12}$$

By applying Corollary 3.2 on inequality (4.12) we get

$$\begin{aligned} z(t) \le \Omega^{-1}\Bigg[K_1\Bigg[\Omega(K_1) + \int_1^t h_1(s)h_2(s)h_3(s)\left(\exp\left(\int_1^s h_1(\tau)d\tau\right)\right. \\ \left. + \int_1^s \exp\left(\int_1^s h_1(\sigma)d\sigma\right)d\tau\right)ds\Bigg]\Bigg], \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} K_1\Bigg[\Omega(K_1) + \int_1^t h_1(s)h_2(s)h_3(s)\left(\exp\left(\int_1^s h_1(\tau)d\tau\right)\right. \\ \left. + \int_1^s \exp\left(\int_1^s h_1(\sigma)d\sigma\right)d\tau\right)ds\Bigg] \in Dom(\Omega^{-1}) \end{aligned} \tag{4.14}$$

By condition C_{iv}, we obtain

$$z(t) \le K_1 \exp(K_2)\int_1^t [h_1(s)h_2(s)h_3(s)]\,ds = K_3 \ \text{ as } \ t \to \infty,$$

where

$$z(t) \le \Omega^{-1}(K_3) = K_4 < \infty.$$

Therefore, the solutions of the nonlinear differential equation (4.1) is bounded. Hence, $\dfrac{|u(t)|}{t} \le z(t) \le K_4$.

To complete the proof, we show that $\int_1^t |f(s,u,u')|ds$ is bounded.

It is clear that

$$\int_1^t |f(s,u,u')|ds \le |c_2| + \int_1^t |f(s,u,u')|ds \le |c_1| + |c_2| + \int_1^t |f(s,u,u')|ds.$$

By equation (4.9), we have

$$\int_1^t |f(s,u,u')|ds \le z(t) \tag{4.15}$$

Thus,

$$\begin{aligned} \int_1^t |f(s,u,u')|ds \le z(t) \le K_1 + \int_1^t h_1(s)z(s)ds \\ + \int_1^t h_1(s)h_2(s)\left(\int_1^s h_3(\tau)\omega(z(\tau))d\tau\right)ds. \end{aligned} \tag{4.16}$$

Application of Corollary 3.2 gives

$$\begin{aligned} z(t) \le \Omega^{-1}\Bigg[K_1\Bigg[\Omega(K_1) + \int_1^t h_1(s)h_2(s)h_3(s)\left(\exp\left(\int_1^s h_1(\tau)d\tau\right)\right. \\ \left. + \int_1^s \exp\left(\int_1^\alpha h_1(\sigma)d\sigma\right)d\alpha\right)ds\Bigg]\Bigg]. \end{aligned} \tag{4.17}$$

Therefore,

$$\int_1^t |f(s,u,u')|ds \le \Omega^{-1}(K_3) = K_4 \quad \text{as } t \to \infty. \tag{4.18}$$

where K_4 is a positive constant. Therefore, the integral $\int_1^\infty |f(s,u,u')|ds$ is absolutely convergent, hence, the $\lim_{t\to\infty} u'(t)$ exits. It is now left for us to show that $\lim_{t\to\infty} u'(t)$ is different from zero by using the method in [6]. Since $\int_1^t |f(s,u,u')|ds \le K_4$ for $t \ge 1$ large enough so that $\int_1^\infty |f(s,u,u')|ds \le \dfrac{1}{2}$, where $K_4 = \frac{1}{2}$. Considering the solution of equation (4.1) with $c_2 = 1$,

$$|u'(t)| \le |1 - \int_1^t f(s,u,u')ds|,$$

$$|u'(t)| \ge 1 - \int_1^t |f(s,u,u')|ds,$$

$$|u'(t)| \ge 1 - \frac{1}{2},$$

$$|u'(t)| > \frac{1}{2} > 0.$$

Hence

$$\lim_{t\to\infty} u'(t) \ne 0.$$

Let

$$\lim_{t\to\infty} u'(t) = a \ne 0,$$

then as $t \to \infty$. $u'(t) \to a$ and so $u(t) \to u(1) + at$ with $u(1) = b$, we have that $u(t) \to at + b, \quad b \ne 0$ as $t \to \infty$.
This concludes the proof. □

Example 4.1. We consider the nonlinear differential equation

$$u'' + \frac{2}{(1+3t)^4 t^2} u' \ln(u') + t^2 u^2 \ln(u^2) = 0 \ \text{ for } \ t \ge 1$$

Setting,
$\omega(u) = u^2, \quad h_1(t) = \frac{2}{(4+3t)^4}, \quad h_2(t) = \frac{1}{t^3} \ \ h_3(t) = t^2$
The conditions (i), (ii), (iii) and (iv) are satisfied and the differential equation has a solution $u(t)$ which is bounded and asymptotic to $at + b$ as $t \to \infty$.

Conclusions: Gronwall-Bellman-Bihari type inequalities is used in this article to consider the boundedness and the asymptotic behaviour of nonlinear second order differential equations.

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