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Fixed Point Theorems with Rational Expressions over a Closed Subset of Hilbert Spaces

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Abstract

In this paper, we prove the existence and uniqueness of a fixed point for a continuous self- mapping over a closed subset of Hilbert spaces satisfying nonlinear rational contractive-type condition. This result is extended and generalized for some positive integer powers of continuous self-mappings in Hilbert spaces.

1. Introduction

The idea of fixed point theory plays a pivotal role in solving varieties of problems in mathematics and its application. Several works have been done on the applications, generalizations and extensions of the Banach contraction principle in diverse areas.
Several authors, such as [7] investigated the extension of Banach fixed point theorem by removing the completeness of the space with different conditions. [1] considered various contractive conditions for self-mappings in metric space. [2] also investigated the rational type of contractions to obtain a unique fixed point in complete metric space. [4] developed the approach of [7] and proved analogous results involving two mappings on a complete metric space. [3] developed the approach of [6] and proved analogous results involving two mappings on a complete metric space.
In this present work, we establish existence and uniqueness of fixed point for a continuous self- mapping over a closed subset of Banach spaces satisfying nonlinear rational contractive-type condition as an improvement to the works of [9] & [10].

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2. Preliminaries

In this part of the work, we list some useful results that will be useful in the course of our findings.

Theorem 2.1 ([11]). *Let C be non empty closed subset of Hilbert space H. Let$T : C \to C$ be self-mappings satisfying the following condition.*
$\|Tx - Ty\| \leq \alpha \left\{ \frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|} \right\} + \beta \|x-y\|$ *for all* $x, y \in S,\ x \neq y,\ \alpha \in [0, 1/2),\ \ \beta \geq 0$ *and* $2\alpha + \beta < 1$. *Then, T has a common fixed point in C.*

Theorem 2.2 ([8]). *Let C be non empty closed subset of Hilbert space H.Let$T : C \to C$ be self-mappings satisfying the following condition*
$\|Tx - Ty\| \leq \alpha \frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|} + \beta \frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|} + \gamma \|x-y\|$ *for all* $x, y \in C,\ x \neq y,$ $0 \leq \alpha,\ \beta < 1/2,\ 0 \leq \gamma$ *and* $2\alpha + 2\beta + \gamma < 1$. *Then, T has a unique fixed point in C.*

Theorem 2.3 ([5]). *Let S be non empty closed subset of Hilbert space H. Let$T : S \to S$ be self-mappings satisfying the following condition*
$\|Tx - Ty\| \leq a_1 \left\{ \frac{\|x-Tx\|^2+\|y-Ty\|^2+\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Tx\|+\|y-Ty\|+\|x-Ty\|+\|y-Tx\|} \right\} + a_2 \left\{ \frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|} \right\} + a_3 \left\{ \frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|} \right\} + a_4 \|x-y\|$ *for all* $x, y \in S$ *and* $4a_1, a_2, a_3, a_4 \geq 0$ *and* $4a_1 + 2a_2 + 2a_3 + a_4 < 1$. *Then, T has unique fixed point in X.*

Theorem 2.4 ([10]). *Let X be a closed subset of a Hilbert space and T_1, T_2 be two continuous self-mappings on X satisfying contraction condition then, $T_1$and T_2 have unique common fixed point in X:*

$$\|T_1x - T_2y\| \leq a_1 \frac{\|x-T_1x\|\left[1+\|y-T_2y\|\right]}{1+\|x-y\|} + a_2 \frac{\|y-T_2y\|\left[1+\|y-T_1x\|\right]}{1+\|x-y\|} + a_3 \frac{\|x-T_2y\|\left[1+\|y-T_1x\|\right]}{1+\|x-y\|}$$

$$+ a_4 \frac{\|x-y\|\left[1+\|T_1x-T_2y\|\right]}{1+\|x-y\|} + a_5 \frac{\|x-y\|\left[1+\|x-T_1x\|\right]}{1+\|y-T_2y\|} + a_6 \frac{\|x-T_1x\|\left[1+\|x-T_2y\|\right]}{1+\|y-T_2y\|}$$

$$+ a_7 \frac{\|x-T_2y\|\left[1+\|T_1x-T_2y\|\right]}{1+\|x-y\|} + a_8 \frac{\|x-y\|\left[1+\|x-T_2y\|\right]}{1+\|x-y\|} + a_9 \frac{\|x-T_1x\|+\|y-T_2y\|+\|x-y\|}{1+\|x-T_1x\|\,\|x-T_2y\|\,\|y-T_2y\|\,\|x-y\|}$$

$$+ a_{10} \frac{\|x-T_2y\|^2+\|y-T_1x\|^2}{\|x-T_2y\|+\|y-T_1x\|} + a_{11}\left[\|x-T_1x\|+\|y-T_2y\|\right] + a_{12}\left[\|x-T_2y\|+\|y-T_1x\|\right] + a_{13}\|x-y\|.$$

For all $x, y \in X$ and $x \neq y$, where $a_i\,(i = 1, 2, 3, \ldots, 13)$ are non-negative reals with $0 \leq \sum_{i=1}^{8} a_i + 3a_9 + 2\sum_{i=10}^{12} a_i + a_{13} < 1$.

Theorem 2.5 ([9]). *Let T_1 and T_2 be two self-mappings of closed subset X of a Hilbert space satisfying the inequality.*

$$\|T_1x - T_2y\|^2 \leq \alpha \frac{\|x-T_2y\|^2\left[1+\|y-T_1x\|^2\right]}{1+\|x-y\|^2} + \beta \frac{\|x-y\|^2\left[1+\|x-T_2y\|^2\right]}{1+\|x-y\|^2}$$

$$+\gamma\left[\|y-T_1x\|^2 + \|x-T_2y\|^2\right] + \delta\|x-y\|^2,$$

for all $x, y \in X,\ x \neq y$, where $\alpha,\ \beta,\ \gamma, \delta$ are positive real's with $4\alpha + \beta + 4\gamma + \delta < 1$. Then, $T_1$and T_2 have a unique common fixed point.

Theorem 2.6 ([9])**.** *Let X be a closed subset of a Hilbert space and T_1, T_2 be two self-mappings on X satisfying.*

$$\|T_1^p x - T_2^q y\|^2 \leq \alpha \frac{\|x - T_2^q y\|^2 \left[1 + \|y - T_1^p x\|^2\right]}{1 + \|x - y\|^2} + \beta \frac{\|x - y\|^2 \left[1 + \|x - T_2^q y\|^2\right]}{1 + \|x - y\|^2}$$

$$+\gamma \left[\|y - T_1^p x\|^2 + \|x - T_2^q y\|^2\right] + \delta \|x - y\|^2,$$

for all $x, y \in X,\ x \neq y$, where $\alpha,\ \beta,\ \gamma, \delta$ are positive real's with $4\alpha + \beta + 4\gamma + \delta < 1$ and p, q are positive integers. Then, T_1 and T_2 have a unique common fixed point.

3. Main Results

Theorem 3.1. *Let X be a closed subset of a Hilbert space and $T : X \to X$ be a continuous self- mapping satisfying the following inequality.*

$$\|Tx - Ty\| \leq c_1 \frac{\|x - Tx\| \left[1 + \|y - Ty\|\right]}{1 + \|x - y\|} + c_2 \frac{\|y - Ty\| \left[1 + \|y - Tx\|\right]}{1 + \|x - y\|} + c_3 \frac{\|x - Ty\| \left[1 + \|y - Tx\|\right]}{1 + \|x - y\|}$$

$$+c_4 \frac{\|x - y\| \left[1 + \|Tx - Ty\|\right]}{1 + \|x - y\|} + c_5 \frac{\|x - y\| \left[1 + \|x - Tx\|\right]}{1 + \|y - Ty\|} + c_6 \frac{\|x - Tx\| \left[1 + \|x - Ty\|\right]}{1 + \|y - Ty\|}$$

$$+c_7 \frac{\|x - Ty\| \left[1 + \|Tx - Ty\|\right]}{1 + \|x - y\|} + c_8 \frac{\|x - y\| \left[1 + \|x - Ty\|\right]}{1 + \|x - y\|} + c_9 \frac{\|x - Tx\| + \|y - Ty\| + \|x - y\|}{1 + \|x - Tx\| \, \|x - Ty\| \, \|y - Ty\| \, \|x - y\|}$$

$$+c_{10} \frac{\|x - Ty\|^2 + \|y - Tx\|^2}{\|x - Ty\| + \|y - Tx\|} + c_{11} \left[\|x - Tx\| + \|y - Ty\|\right] + c_{12} \left[\|x - Ty\| + \|y - Tx\|\right] + c_{13} \|x - y\|$$

for all $x, y \in X$ and $x \neq y$, where $c_i\ (i = 1, 2, 3, \ldots, 13)$ are non-negative real numbers with $0 \leq \sum_{i=1}^{8} c_i + 3c_9 + 2\sum_{i=10}^{12} c_i + c_{13} < 1$. Then T has a unique fixed point in X.

Proof. We construct a sequence $\{x_n\}$ for an arbitrary point $x_0 \in X$ as follows
$x_{2n+1} = Tx_{2n}, \qquad x_{2n+2} = Tx_{2n+1}$, for $n = 1,\ 2,\ 3,\ \ldots$
We show that the sequence $\{x_n\}$ is a Cauchy sequence in X

$$\|x_{2n+1} - x_{2n}\| = \|Tx_{2n} - Tx_{2n-1}\| \leq c_1 \frac{\|x_{2n} - Tx_{2n}\| \left[1 + \|x_{2n-1} - Tx_{2n-1}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|}$$

$$+c_2 \frac{\|x_{2n-1} - Tx_{2n-1}\| \left[1 + \|x_{2n-1} - Tx_{2n}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|} + c_3 \frac{\|x_{2n} - Tx_{2n-1}\| \left[1 + \|x_{2n-1} - Tx_{2n}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|}$$

$$+c_4 \frac{\|x_{2n} - x_{2n-1}\| \left[1 + \|Tx_{2n} - Tx_{2n-1}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|} + c_5 \frac{\|x_{2n} - x_{2n-1}\| \left[1 + \|x_{2n} - Tx_{2n}\|\right]}{1 + \|x_{2n-1} - Tx_{2n-1}\|}$$

$$+c_6 \frac{\|x_{2n} - Tx_{2n}\| \left[1 + \|x_{2n} - Tx_{2n-1}\|\right]}{1 + \|x_{2n-1} - Tx_{2n-1}\|} + c_7 \frac{\|x_{2n} - Tx_{2n-1}\| \left[1 + \|Tx_{2n} - Tx_{2n-1}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|}$$

$$+c_8 \frac{\|x_{2n} - x_{2n-1}\| \left[1 + \|x_{2n} - Tx_{2n-1}\|\right]}{1 + \|x_{2n} - x_{2n-1}\|}$$

$$+c_9 \frac{\|x_{2n} - Tx_{2n}\| + \|x_{2n-1} - Tx_{2n-1}\| + \|x_{2n} - x_{2n-1}\|}{1 + \|x_{2n} - Tx_{2n}\| \, \|x_{2n} - Tx_{2n-1}\| \, \|x_{2n-1} - Tx_{2n-1}\| \, \|x_{2n} - x_{2n-1}\|}$$

$$+c_{10}\frac{\|x_{2n}-Tx_{2n-1}\|^2+\|x_{2n-1}-Tx_{2n}\|^2}{\|x_{2n}-Tx_{2n-1}\|+\|x_{2n-1}-Tx_{2n}\|}+c_{11}\left[\|x_{2n}-Tx_{2n}\|+\|x_{2n-1}-Tx_{2n-1}\|\right]$$

$$+c_{12}\left[\|x_{2n}-Tx_{2n-1}\|+\|x_{2n-1}-Tx_{2n}\|\right]+c_{13}\|x_{2n}-x_{2n-1}\|$$

This implies

$$\|x_{2n+1}-x_{2n}\|=a(n)\|Tx_{2n}-Tx_{2n-1}\|,$$

where

$$a(n)=\frac{B_1+(c_2+2c_9+c_{10}+c_{11}+c_{12})\|x_{2n}-x_{2n-1}\|}{B_2+(1-c_1-c_2-c_4-c_5-c_9-c_{10}-c_{11}-c_{12})\|x_{2n}-x_{2n-1}\|},$$

$B_1=c_2+c_4+c_5+c_8+2c_9+c_{10}+c_{11}+c_{12}+c_{13}$ and

$$B_2=1-c_1-c_6-c_9-c_{10}-c_{11}-c_{12}$$

Clearly, $\delta=a(n)<1, \forall n=1,\ 2,\ 3,\ ..,$ we get

$$\|x_{n+1}-x_n\|\le\delta\|x_n-x_{n-1}\|,$$

By induction, we have

$$\|x_{n+1}-x_n\|\le\delta^n\|x_1-x_0\|,\quad n\ge1,$$

Taking $n\to\infty$, we have $\|x_{n+1}-x_n\|\to0$.
Hence, $\{x_n\}$ is a Cauchy sequence in X and so it has a limit λ in X. Since the sequences $\{x_{2n+1}\}=\{Tx_{2n}\}$ and $\{x_{2n+2}\}=\{Tx_{2n+1}\}$ are subsequences of $\{x_n\}$, and also these subsequences have the same limit λ in X. We now show that λ is a common fixed point of T. Now consider the following inequality

$$\|\lambda-T\lambda\|=\|\lambda-x_{2n+2}+x_{2n+2}-T\lambda\|=\|(\lambda-x_{2n+2})+(x_{2n+2}-T\lambda)\|\le\|\lambda-x_{2n+2}\|+\|T\lambda-Tx_{2n+1}\|$$

$$\le c_1\frac{\|\lambda-T\lambda\|\left[1+\|x_{2n+1}-Tx_{2n+1}\|\right]}{1+\|\lambda-x_{2n+1}\|}+c_2\frac{\|x_{2n+1}-Tx_{2n+1}\|\left[1+\|x_{2n+1}-T\lambda\|\right]}{1+\|\lambda-x_{2n+1}\|}$$

$$+c_3\frac{\|\lambda-Tx_{2n+1}\|\left[1+\|x_{2n+1}-T\lambda\|\right]}{1+\|\lambda-x_{2n+1}\|}+c_4\frac{\|\lambda-x_{2n+1}\|\left[1+\|T\lambda-Tx_{2n+1}\|\right]}{1+\|\lambda-x_{2n+1}\|}+c_5\frac{\|\lambda-x_{2n+1}\|\left[1+\|\lambda-T\lambda\|\right]}{1+\|x_{2n+1}-Tx_{2n+1}\|}$$

$$+c_6\frac{\|\lambda-T\lambda\|\left[1+\|\lambda-Tx_{2n+1}\|\right]}{1+\|x_{2n+1}-Tx_{2n+1}\|}+c_7\frac{\|\lambda-Tx_{2n+1}\|\left[1+\|T\lambda-Tx_{2n+1}\|\right]}{1+\|\lambda-x_{2n+1}\|}+c_8\frac{\|\lambda-x_{2n+1}\|\left[1+\|\lambda-Tx_{2n+1}\|\right]}{1+\|\lambda-x_{2n+1}\|}$$

$$+c_9\frac{\|\lambda-T\lambda\|+\|x_{2n+1}-Tx_{2n+1}\|+\|\lambda-x_{2n+1}\|}{1+\|\lambda-T\lambda\|\|\lambda-Tx_{2n+1}\|\|x_{2n+1}-Tx_{2n+1}\|\|\lambda-x_{2n+1}\|}+c_{10}\frac{\|\lambda-Tx_{2n+1}\|^2+\|x_{2n+1}-T\lambda\|^2}{\|\lambda-Tx_{2n+1}\|+\|x_{2n+1}-T\lambda\|}$$

$$+c_{11}\left[\|\lambda-T\lambda\|+\|x_{2n+1}-Tx_{2n+1}\|\right]+c_{12}\left[\|\lambda-Tx_{2n+1}\|+\|x_{2n+1}-T\lambda\|\right]+c_{13}\|\lambda-x_{2n+1}\|.$$

Letting $n\to\infty$, we get $\|\lambda-T\lambda\|\le(c_1+c_6+c_9+c_{10}+c_{11}+c_{12})\|\lambda-T\lambda\|$, since $c_1+c_6+c_9+c_{10}+c_{11}+c_{12}<1$, hence $\lambda=T\lambda$.
Similarly, from the hypothesis, we get $\lambda=T\lambda$ by considering the following

$$\|\lambda-T\lambda\|=\|(\lambda-x_{2n+1})+(\ x_{2n+1}-T\lambda)\|.$$

We now show that λ is a unique fixed point of T. Suppose that $t\,(\lambda\neq t)$ is also a common fixed point of T.

Then, by the hypothesis, we get

$$\|\lambda - t\| = \|T\lambda - Tt\| \leq c_1 \frac{\|\lambda - T\lambda\| \left[1 + \|t - Tt\|\right]}{1 + \|\lambda - t\|} + c_2 \frac{\|t - Tt\| \left[1 + \|t - T\lambda\|\right]}{1 + \|\lambda - t\|} + c_3 \frac{\|\lambda - Tt\| \left[1 + \|t - T\lambda\|\right]}{1 + \|\lambda - t\|}$$

$$+ c_4 \frac{\|\lambda - t\| \left[1 + \|T\lambda - Tt\|\right]}{1 + \|\lambda - t\|} + c_5 \frac{\|\lambda - t\| \left[1 + \|\lambda - T\lambda\|\right]}{1 + \|t - Tt\|} + c_6 \frac{\|\lambda - T\lambda\| \left[1 + \|\lambda - Tt\|\right]}{1 + \|t - Tt\|}$$

$$+ c_7 \frac{\|\lambda - Tt\| \left[1 + \|T\lambda - Tt\|\right]}{1 + \|\lambda - t\|} + c_8 \frac{\|\lambda - t\| \left[1 + \|\lambda - Tt\|\right]}{1 + \|\lambda - t\|} + c_9 \frac{\|\ddot{e} - T\lambda\| + \|t - Tt\| + \|\lambda - t\|}{1 + \|\lambda - T\lambda\| \, \|\lambda - Tt\| \, \|t - Tt\| \, \|\lambda - t\|}$$

$$+ c_{10} \frac{\|\lambda - Tt\|^2 + \|t - T\lambda\|^2}{\|\lambda - Tt\| + \|t - T\lambda\|} + c_{11} \left[\|\lambda - T\lambda\| + \|t - Tt\|\right] + c_{12} \left[\|\lambda - Tt\| + \|t - T\lambda\|\right] + c_{13} \|\lambda - t\| .$$

Thus,

$$\|\lambda - t\| \leq (c_3 + c_4 + c_5 + c_7 + c_8 + c_9 + c_{10} + 2c_{12} + c_{13}) \|\lambda - t\| < \|\lambda - t\|$$

a contradiction. Hence $\lambda = t$ (common fixed point λ is unique in X). □

Theorem 3.2. *Let X be a closed subset of a Hilbert space and $T : X \to X$ be a continuous self- mapping satisfying*

$$\|T^r x - T^s y\| \leq c_1 \frac{\|x - T^r x\| \left[1 + \|y - T^s y\|\right]}{1 + \|x - y\|} + c_2 \frac{\|y - T^s y\| \left[1 + \|y - T^r x\|\right]}{1 + \|x - y\|} + c_3 \frac{\|x - T^s y\| \left[1 + \|y - T^r x\|\right]}{1 + \|x - y\|}$$

$$+ c_4 \frac{\|x - y\| \left[1 + \|T^r x - T^s y\|\right]}{1 + \|x - y\|} + c_5 \frac{\|x - y\| \left[1 + \|x - T^r x\|\right]}{1 + \|y - T^s y\|} + c_6 \frac{\|x - T^r x\| \left[1 + \|x - T^s y\|\right]}{1 + \|y - T^s y\|}$$

$$+ c_7 \frac{\|x - T^s y\| \left[1 + \|T^r x - T^s y\|\right]}{1 + \|x - y\|} + c_8 \frac{\|x - y\| \left[1 + \|x - T^s y\|\right]}{1 + \|x - y\|} + c_9 \frac{\|x - T^r x\| + \|y - T^s y\| + \|x - y\|}{1 + \|x - T^r x\| \, \|x - T^s y\| \, \|y - T^s y\| \, \|x - y\|}$$

$$+ c_{10} \frac{\|x - T^s y\|^2 + \|y - T^r x\|^2}{\|x - T^s y\| + \|y - T^r x\|} + c_{11} \left[\|x - T^r x\| + \|y - T^s y\|\right] + c_{12} \left[\|x - T^s y\| + \|y - T^r x\|\right] + c_{13} \|x - y\|$$

for all $x, y \in X$ and $x \neq y$, where c_i $(i = 1, 2, 3, \ldots, 13)$ are non-negative real numbers with $0 \leq \sum_{i=1}^{8} c_i + 3c_9 + 2\sum_{i=10}^{12} c_i + c_{13} < 1$ and r, s are two positive integers.Then T has a unique fixed point in X.

Proof. From Theorem 3.1 T^r and T^s have a unique common fixed point $\lambda \in X$, so that $T^r\lambda = \lambda$ and $T^s\lambda = \lambda$. From $T^r (T\lambda) = T (T^r\lambda) = T\lambda$, it follows that $T\lambda$ is a fixed point of T^r. But λ is a unique fixed point of T^r. Therefore $T\lambda = \lambda$.
Similarly, we can get $T\lambda = \lambda$ from $T^s (T\lambda) = T (T^s\lambda) = T\lambda$. Hence$\lambda$ is a unique fixed point of T.
Now, we show uniqueness. Let t be another fixed point of T, so that $Tt = t$. then from the hypothesis, we have

$$\|\lambda - t\| = \|T^r\lambda - T^s t\| \leq c_1 \frac{\|\lambda - T^r\lambda\| \left[1 + \|t - T^r t\|\right]}{1 + \|\lambda - t\|} + c_2 \frac{\|t - T^s t\| \left[1 + \|t - T^r\lambda\|\right]}{1 + \|\lambda - t\|}$$

$$+ c_3 \frac{\|\lambda - T^s t\| \left[1 + \|t - T^r\lambda\|\right]}{1 + \|\lambda - t\|} + c_4 \frac{\|\lambda - t\| \left[1 + \|T^r\lambda - T^s t\|\right]}{1 + \|\lambda - t\|} + c_5 \frac{\|\lambda - t\| \left[1 + \|\lambda - T^r\lambda\|\right]}{1 + \|t - T^s t\|}$$

$$+ c_6 \frac{\|\lambda - T^r\lambda\| \left[1 + \|\lambda - T^s t\|\right]}{1 + \|t - T^s t\|} + c_7 \frac{\|\lambda - T^s t\| \left[1 + \|T^r\lambda - T^s t\|\right]}{1 + \|\lambda - t\|} + c_8 \frac{\|\lambda - t\| \left[1 + \|\lambda - T^s t\|\right]}{1 + \|\lambda - t\|}$$

$$+c_9\frac{\|\lambda-T^r\lambda\|+\|t-T^st\|+\|\lambda-t\|}{1+\|\lambda-T^r\lambda\|\,\|\lambda-T^st\|\,\|t-T^st\|\,\|\lambda-t\|}+c_{10}\frac{\|\lambda-T^st\|^2+\|t-T^r\lambda\|^2}{\|\lambda-T^st\|+\|t-T^r\lambda\|}+c_{11}\left[\|\lambda-T^r\lambda\|+\|t-T^st\|\right]$$

$$+c_{12}\left[\|\lambda-T^st\|+\|t-T^r\lambda\|\right]+c_{13}\|\lambda-t\|.$$

Therefore, we have

$$\|\lambda-t\|\leq(c_3+c_4+c_5+c_7+c_8+c_9+c_{10}+2c_{12}+c_{13})\|\lambda-t\|<\|\lambda-t\|$$

Implies$\lambda=t$, since $c_3+c_4+c_5+c_7+c_8+c_9+c_{10}+2c_{12}+c_{13}<1$.
Hence, λ is a unique common fixed point of T in X. □

We present the following example to illustrate our main result.

Example 3.3. Let $T:[0,1]\to[0,1]$ be a mapping defined by $Tx=\frac{x^3}{6}$, for all $x\in[0,1]$ with usual norm $\|x-y\|=|x-y|$, for all $x\in[0,1]$.

Proof. From Theorem 3.1, we have

$$\|Tx-Ty\|=\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\leq c_1\frac{\left\|x-\frac{x^3}{6}\right\|\left[1+\left\|y-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}+c_2\frac{\left\|y-\frac{y^3}{6}\right\|\left[1+\left\|y-\frac{x^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_3\frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|y-\frac{x^3}{6}\right\|\right]}{1+\|x-y\|}+c_4\frac{\|x-y\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}+c_5\frac{\|x-y\|\left[1+\left\|x-\frac{x^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}$$

$$+c_6\frac{\left\|x-\frac{x^3}{6}\right\|\left[1+\left\|x-\frac{y^3}{6}\right\|\right]}{1+\left\|y-\frac{y^3}{6}\right\|}+c_7\frac{\left\|x-\frac{y^3}{6}\right\|\left[1+\left\|\frac{x^3}{6}-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}+c_8\frac{\|x-y\|\left[1+\left\|x-\frac{y^3}{6}\right\|\right]}{1+\|x-y\|}$$

$$+c_9\frac{\left\|x-\frac{x^3}{6}\right\|+\left\|y-\frac{y^3}{6}\right\|+\|x-y\|}{1+\left\|x-\frac{x^3}{6}\right\|\left\|x-\frac{y^3}{6}\right\|\left\|y-\frac{y^3}{6}\right\|\|x-y\|}+c_{10}\frac{\left\|x-\frac{y^3}{6}\right\|^2+\left\|y-\frac{x^3}{6}\right\|^2}{\left\|x-\frac{y^3}{6}\right\|+\left\|y-\frac{x^3}{6}\right\|}+c_{11}\left[\left\|x-\frac{x^3}{6}\right\|+\left\|y-\frac{y^3}{6}\right\|\right]$$

$$+c_{12}\left[\left\|x-\frac{y^3}{6}\right\|+\left\|y-\frac{x^3}{6}\right\|\right]+c_{13}\|x-y\|$$

On continuing with the procedure in the proof of Theorem 3.1, we get that 0 is the fixed point of the mapping T. □

Conclusion. In this paper, we prove the existence and uniqueness of a fixed point for a continuous self- mapping T, some positive integers r, s of a pair of continuous self-mappings T^r, T^s of Hilbert spaces. These results generalized and extend the results of some literatures.

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