Nigerian Journal of Mathematics and Applications
Volume **32** (**II**), (2022), 211 − 240. *Printed by Unilorin press*
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Special Issue in Honor of Prof. J. A. Gbadeyan's Retirement

An Orthogonal-Based Block Hybrid Method for the Solution of Third Order Ordinary Differential Equations

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Abstract

An orthogonal polynomial of the weight function $w(x) = x^2+x+1$ in the interval [-1, 1] was created as the basis function in block methods using the collocation and interpolation methodology. For third-order problems, an efficient class of continuous and discrete numerical integration techniques of implicit hybrid form were devised and effectively implemented. These schemes were used to address three separate problems and their performances evaluated. Using the existing theorems to investigate our new methods, the inquiry proves that the methods are consistent, zero-stable, and are therefore convergent.

1. Introduction

Many issues in science and engineering produce third-order Ordinary Differential Equations (ODEs) of the type - Initial Value Problems (IVPs),

$$y''' = f(x, y, y', y''); y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma, x \in (a, b) \tag{1.1}$$

where f is a continuous function in $[a, b]$. Other numerical approaches for solving equation (1) exist in the literature, such as Euler's method, the Runge-Kutta methods and Linear Multistep Methods (LMMs). Implicit LMMs in the predictor-corrector mode are by nature prone to error propagation. As a result, linear multi-step approaches were used to construct block methods. [18] was the first to suggest Block methods for solving ODEs in 1953. The block technique employed a collocation and interpolation strategy. Many researchers have employed block approaches, including [21], [2], [22], [7] & [20]. Also, [4] utilized Chebyshev as the basis function, while [5] used power series. Varied number of researchers built orthogonal polynomials with various weight functions as the basis function for the generation of numerical schemes. [19] & [1], for example, employed orthogonal

Received: 22/06/2022, Accepted: 15/07/2022, Revised: 02/08/2022. * Corresponding author.
2015 *Mathematics Subject Classification.* 00000 & 00000.
Keywords and phrases. Orthogonal polynomial, Hybrid, Interpolation, Collocation, Block Method
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polynomials with the weight function $w(x) = x^2$ for the interval $[-1, 1]$, whereas [11] used orthogonal polynomials with weight function $w(x) = x$. With regards to the weight function $w(x) = 1 + \frac{x}{2}$, [13] created an orthogonal basis in the interval $[-1, 1]$. Majority of these papers dealt with first-order differential equations. [15] worked on second order differential equations.
In a bid to further investigate versatility of these methods and to create avenue for more efficient approaches, this study uses orthogonal basis function with weight function $w(x) = x^2 + x + 1$ this is situated within the interval $[-1, 1]$. This is deployed to solve third order differential equations.

2. Orthogonal Polynomials

When the inner product of two polynomials vanishes within their defining interval, they are said to be orthogonal. This is illustrated as follows:

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = h_n\delta_{mn} \tag{2.1}$$

for

$$\delta_{mn} = \begin{cases} 0, m \neq n \\ 1, m = 1 \end{cases}$$

where $w(x)$ is a continuous and positive weight function in the range $[a, b]$, and the moments are

$$\mu = \int_a^b w(x)x^n dx, \; n = 0, 1, 2, ... \tag{2.2}$$

exists.
The Integral

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx \tag{2.3}$$

is the inner product of ϕ_m and ϕ_n.
polynomials. When it comes to orthogonality,

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \; m \neq n \tag{2.4}$$

2.1. Construction of Orthogonal Polynomials.

Consider the orthogonal polynomial class $\{\phi n(x)\}$, which is defined as

$$\phi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \tag{2.5}$$

For the creation of orthogonal polynomials, the following requirements must be satisfied.

$$\phi_n(1) = 1 \tag{2.6}$$

$$< \phi_m(x), \phi_n(x) >= 0 \; (m \neq n) \tag{2.7}$$

The procedure for constructing these orthogonal polynomials valid within the interval [-1, 1] is described as follows: let $w(x) = 1 + x + x^2$ and $[a, b] = [-1, 1]$ in (2.5)-(2.7). Then for $n = 0$, we have:

$$\phi_0(x) = C_0^{(0)} \text{ and } \phi_0(1) = 1 = C_0^{(0)}$$

Hence, $\phi_0(x) = 1$
For $n = 1$, we have two equations

$$\phi_1(1) = C_0^{(1)} + C_1^{(1)} = 1 \tag{2.8}$$

$$< \phi_0(x), \phi_1(x) >= \frac{8}{3}C_0^{(1)} + \frac{2}{3}C_1^{(1)} = 0 \tag{2.9}$$

From these two equations

$$\phi_1(x) = -\frac{1}{3} + \frac{4}{3}x \tag{2.10}$$

For $n = 2$, we solve

$$C_0^{(1)} + C_1^{(2)} + C_2^{(2)} = 1 = \phi_2(1) \tag{2.11}$$

$$\frac{8}{3}C_0^{(1)} + \frac{2}{3}C_1^{(2)} + \frac{16}{5}C_2^{(2)} = 0 =< \phi_0(x), \phi_2(x) > \tag{2.12}$$

$$\frac{6}{5}C_1^{(2)} + \frac{8}{45}C_2^{(2)} = 0 =< \phi_1(x), \phi_2(x) > \tag{2.13}$$

Which yield the polynomial

$$\phi_2(x) = -\frac{49}{66} - \frac{10}{33}x + \frac{45}{22}x^2 \tag{2.14}$$

Following this approach, we have the class:

(2.15)

$$\left.\begin{aligned}
&Q_0(x) = 1\\
&Q_1(x) = \frac{4}{3}x - \frac{1}{3}\\
&Q_2(x) = \frac{45}{22}x^2 - \frac{10}{33} - \frac{49}{66}\\
&Q_3(x) = \frac{1484}{419}x^3 - \frac{483}{838}x^2 - \frac{930}{419}x + \frac{213}{838}\\
&Q_4(x) = \frac{49427}{7880}x^4 - \frac{994}{985}x^3 - \frac{4347}{788}x^2 + \frac{2002}{2995}x + \frac{13609}{23640}\\
&Q_5(x) = \frac{169587}{14882}x^5 - \frac{108625}{59528}x^4 - \frac{13735}{1063}x^3 + \frac{50445}{29764}x^2 + \frac{127805}{44646}x - \frac{5285}{25512}\\
&Q_6(x) = \frac{21640593}{1029296}x^6 - \frac{865683}{257324}x^5 - \frac{29995065}{1029296}x^4 + \frac{511695}{128662}x^3 + \frac{112595805}{11322256}x^2\\
&\quad - \frac{2700945}{2830564}x - \frac{5509685}{11322256}\\
&Q_7(x) = \frac{912217735}{23254947}x^7 - \frac{2332825495}{372079152}x^6 - \frac{663966303}{10335532}x^5 + \frac{3345316975}{372079152}x^4\\
&\quad + \frac{1383496345}{46509894}x^3 - \frac{134149785}{41342128}x^2 - \frac{315107135}{93019788}x + \frac{66696665}{372079152}\\
&Q_8(x) = \frac{206238477159}{2794077824}x^8 - \frac{4632559789}{392917194}x^7 - \frac{876515334695}{6286675104}x^6 + \frac{867587721}{43657466}x^5\\
&\quad + \frac{1026917752861}{12573350208}x^4 - \frac{3774934163}{392917194}x^3 - \frac{10578992655}{698519456}x^2 + \frac{462916223}{392917194}x + \frac{10792421983}{25146700416}\\
&Q_9(x) = \frac{7592947736959}{54306345568}x^9 - \frac{4849132400307}{21722582272}x^8 - \frac{4061887568739}{13576586392}x^7\\
&\quad + \frac{2347732919259}{54306345568}x^6 + \frac{5760948177621}{27153172784}x^5 - \frac{2848740151257}{108612691136}x^4 - \frac{750682757643}{13576586392}x^3\\
&\quad + \frac{278681373243}{54306345568}x^2 + \frac{208683420559}{54306345568}x - \frac{34811913843}{217225382272}
\end{aligned}\right\}$$

3. Development of Numerical Block Algorithms for Third-Order Problems

The experimental solution that approximates the analytical solution of (1.1) takes the form:

$$Y(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x) \tag{3.1}$$

where $x \in [a, b]$, r and s are the number of collocation and interpolation points respectively. The function $\phi_j(x)$ is the j^{th} degree orthogonal polynomial valid in the range of integration of $[a, b]$. The third derivative of (3.1) is given by

$$y'''(x) = \sum_{j=0}^{r+s-1} a_j \phi_j'''(x) = f(x, y, y', y'') \tag{3.2}$$

By standard convention, to approximate the solution of (1.1), (3.1) must be interpolated three times, this is done at $xn+s$ points. Equation (3.2) is thereafter collocated at $xn+r$ points yielding a system of equations to be solved using Gaussian elimination method.

4. One-step Method Development using $x_{n+\frac{1}{2}}$ and $x_{n+\frac{2}{3}}$ Off-step Points

We interpolate (3.1) at $x = x_{n+s}$, $s = 0$, $\frac{1}{2}$ and $\frac{2}{3}$; and collocate (3.2) at $x = x_{n+r}$, $r = 0, \frac{1}{2}$, $\frac{2}{3}$ and 1; to get a system of seven equations as follows:

$$\begin{bmatrix} 1 & \frac{-5}{3} & \frac{53}{33} & \frac{-689}{419} & \frac{983}{591} & \frac{-5339}{3189} & \frac{631}{375} \\ 1 & \frac{-1}{3} & \frac{-49}{66} & \frac{213}{838} & \frac{13609}{23640} & \frac{-5285}{25512} & \frac{-382}{785} \\ 1 & \frac{1}{9} & \frac{-61}{99} & \frac{-4735}{11313} & \frac{18241}{79785} & \frac{41431}{86103} & \frac{394}{3833} \\ 0 & 0 & 0 & \frac{71232}{419} & \frac{-246792}{197} & \frac{140399}{27} & \frac{-112087}{7} \\ 0 & 0 & 0 & \frac{71232}{419} & \frac{-47712}{985} & \frac{-659280}{1063} & \frac{11263}{59} \\ 0 & 0 & 0 & \frac{71232}{419} & \frac{347704}{985} & \frac{-2197}{17} & \frac{-40923}{37} \\ 0 & 0 & 0 & \frac{71232}{419} & \frac{9247}{8} & \frac{17997}{4} & \frac{105317}{8} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{2}{3}} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{2}} \\ h^3 f_{n+\frac{2}{3}} \\ h^3 f_{n+1} \end{bmatrix} \tag{4.1}$$

Solving (4.1) we get the values of $a's$ as:

$$\left.\begin{aligned} a_0 =& \frac{19}{80} y_n - \frac{7}{10} y_{n+\frac{1}{2}} + \frac{117}{80} y_{n+\frac{2}{3}} + \frac{45}{121342} f_n h^3 + \frac{105}{17324} f_{n+\frac{1}{2}} h^3 \\ &+ \frac{16}{79891} h^3 f_{n+\frac{2}{3}} h^3 + \frac{17}{54424} f_{n+1} h^3 \\ a_1 =& \frac{-5}{48} y_n - \frac{11}{6} y_{n+\frac{1}{2}} + \frac{31}{16} y_{n+\frac{2}{3}} + \frac{7}{90675} f_n h^3 + \frac{45}{11834} f_{n+\frac{1}{2}} h^3 \\ &+ \frac{36}{14701} f_{n+\frac{2}{3}} h^3 + \frac{7}{40020} f_{n+1} h^3 \\ a_2 =& \frac{11}{30} y_n - \frac{22}{15} y_{n+\frac{1}{2}} + \frac{11}{10} y_{n+\frac{2}{3}} + \frac{13}{30705} f_n h^3 + \frac{43}{6541} f_{n+\frac{1}{2}} h^3 \\ &+ \frac{5}{4753} f_{n+\frac{2}{3}} h^3 + \frac{31}{77820} f_{n+1} h^3 \\ a_3 =& h^3 \left(\frac{31}{107905 f_n} + \frac{289}{61701} f_{n+\frac{1}{2}} + \frac{53}{91267} f_{n+\frac{2}{3}} + \frac{2}{6055} f_{n+1} \right) \\ a_4 =& h^3 \left(\frac{-5}{30282} f_n - \frac{34}{18469} f_{n+\frac{1}{2}} + \frac{121}{60163} f_{n+\frac{2}{3}} - \frac{1}{193560} f_{n+1} \right) \\ a_5 =& h^3 \left(\frac{21}{244397} f_n - \frac{18}{129551} f_{n+\frac{1}{2}} - \frac{6}{121543} f_{n+\frac{2}{3}} + \frac{11}{107442} f_{n+1} \right) \\ a_6 =& h^3 \left(\frac{-9}{484408} f_n + \frac{9}{60551} f_{n+\frac{1}{2}} - \frac{26}{155489} f_{n+\frac{2}{3}} + \frac{9}{242204} f_{n+1} \right) \end{aligned}\right\} \tag{4.2}$$

Putting (4.2) in (3.1) we get the continuous implicit one-step method:

(4.3)

$$\bar{y}(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + \alpha_{\frac{1}{2}}(x) y_{n+\frac{1}{2}} + \alpha_{\frac{2}{3}}(x) y_{n+\frac{2}{3}} + h^3 \left(\sum_{j=0}^{1} \beta_j(x) f_{n+j} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} \right)$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients from (3.1). The parameters $\alpha_j(x)$ and $\beta_j(x)$ are gotten as:

$$\left.\begin{aligned}
\alpha_0(t) &= \left(\frac{3t^2}{4} - \frac{t}{4}\right) \\
\alpha_{\frac{1}{2}} &= \left(1 - 2t - 3t^2)\right) \\
\alpha_{\frac{2}{3}} &= \left(\frac{9t^2}{4} + \frac{9t}{4}\right) \\
\beta_0(t) &= h^3 \left(\frac{-t^6}{2560} + \frac{t^5}{960} - \frac{t^4}{1536} + \frac{t^3}{110119583041086400000} + \frac{27t^2}{17174} \right. \\
&\qquad \left. - \frac{31t}{60643} + \frac{1}{424650447444326970000} \right) \\
\beta_{\frac{1}{2}} &= h^3 \left(\frac{t^6}{320} - \frac{t^5}{480} - \frac{t^4}{64} + \frac{t^3}{48} + \frac{287t^2}{12960} - \frac{59t}{6480} + \frac{1}{12639407946839695000} \right) \\
\beta_{\frac{2}{3}} &= -h^3 \left(\frac{9t^6}{2560} + \frac{t^5}{1112698757569671600} - \frac{9t^4}{512} + \frac{t^3}{6246666900901564400} + \right. \\
&\qquad \left. \frac{127t^2}{11520} - \frac{7t}{2304} - \frac{1}{1436334531570795500 0} \right) \\
\beta_1 &= -h^3 \left(\frac{-t^6}{1280} - \frac{t^5}{960} + \frac{t^4}{768} + \frac{t^3}{117704828271805010000} - \frac{31t^2}{25920} \right. \\
&\qquad \left. + \frac{19t}{51840} + \frac{1}{271988660179635340000} \right)
\end{aligned}\right\} \tag{4.4}$$

where $t = \frac{2x - 2x_n - h}{h}$.

By evaluating (4.3) at x_{n+1}, the main method is gotten as:

$$y_{n+1} = \frac{y_n}{2} - 4y_{n+\frac{1}{2}} + \frac{9y_{n+\frac{2}{3}}}{2} + h^3 \left(\frac{11}{10368} f_n + \frac{25}{1296} f_{n+\frac{1}{2}} + \frac{7}{1152} f_{n+\frac{2}{3}} + \frac{7}{5184} f_{n+1} \right) \tag{4.5}$$

By differentiating (4.3) we get the continuous coefficients:

(4.6)

$$\left.\begin{aligned}
\alpha_0'(t) &= \left(\frac{6t-1}{h}\right)\\
\alpha_{\frac{1}{2}}'(t) &= -\left(\frac{12t+4}{h}\right)\\
\alpha_{\frac{2}{3}}'(t) &= \left(\frac{9t+\frac{9}{2}}{h}\right)\\
\beta_0'(t) &= h^2\left(\frac{-3t^5}{640}-\frac{t^4}{96}-\frac{t^3}{192}-\frac{t^2}{1736844139283195600}+\frac{54t}{8587}-\frac{53}{51840}\right)\\
\beta_{\frac{1}{2}}'(t) &= h^2\left(\frac{3t^5}{80}-\frac{t^4}{48}-\frac{t^3}{8}+\frac{t^2}{8}+\frac{287t}{3240}-\frac{59}{3240}\right)\\
\beta_{\frac{2}{3}}'(t) &= -h^2\left(\frac{27t^5}{640}+\frac{t^4}{105413671262544000}-\frac{9t^3}{64}+\frac{t^2}{1507204900423584000}+\frac{127t}{2880}-\frac{7}{1152}\right)\\
\beta_1'(t) &= h^2\left(\frac{3t^5}{320}+\frac{t^4}{96}-\frac{t^3}{96}-\frac{t^2}{1736844139283195600}+\frac{31t}{6480}-\frac{19}{25920}\right)
\end{aligned}\right\}$$

From the second derivatives of the continuous functions (4.3) we also have:

(4.7)

$$\left.\begin{aligned}
a_0''(t) &= \frac{6}{h^2}\\
a_{\frac{1}{2}}''(t) &= \frac{-24}{h^2}\\
a_{\frac{2}{3}}''(t) &= \frac{18}{h^2}\\
\beta_0''(t) &= h\left(\frac{-3t^4}{8}-\frac{t^3}{12}-\frac{t^2}{32}+\frac{t}{43262638360365280}+\frac{163}{12960}\right)\\
\beta_{\frac{1}{2}}(t) &= h\left(-\frac{3t^4}{8}-\frac{t^3}{6}-\frac{3t^2}{4}+\frac{t}{2}+\frac{287}{1620}\right)\\
\beta_{\frac{2}{3}}''(t) &= h\left(-\frac{27t^4}{64}-\frac{t^3}{13176708907817998}+\frac{27t^2}{32}-\frac{t}{376801225105896000}-\frac{127}{1440}\right)\\
\beta_1''(t) &= h\left(\frac{3t^4}{32}+\frac{t^3}{12}-\frac{t^2}{16}-\frac{t}{434211034820798910}+\frac{31}{3240}\right)
\end{aligned}\right\}$$

The two derivatives of (4.3) are evaluated at xn, $xn+\frac{1}{2}$, $xn+\frac{2}{3}$ and $xn+1$, respectively, to produce the supplementary techniques to be coupled with the primary method (4.5):

(4.8) $$hy_n' + \frac{7}{2}y_n - 8y_{n+\frac{1}{2}} + \frac{9}{2}y_{n+\frac{2}{3}} = h^3\left(\frac{-93}{7153}f_n + \frac{55}{648}f_{n+\frac{1}{2}} - \frac{139}{2880}f_{n+\frac{2}{3}} + \frac{77}{12960}f_{n+1}\right)$$

(4.9)
$$hy'_{n+\frac{1}{2}} + \frac{1}{2}y_n + 4y_{n+\frac{1}{2}} + \frac{9}{2}y_{n+\frac{2}{3}} = h^3\left(-\frac{53}{51840}f_n - \frac{59}{3240}f_{n+\frac{1}{2}} + \frac{7}{1152}f_{n+\frac{2}{3}} - \frac{19}{12960}f_{n+1}\right)$$

(4.10)
$$hy'_{n+\frac{2}{3}} - \frac{1}{2}y_n + 8y_{n+\frac{1}{2}} - \frac{15}{2}y_{n+\frac{2}{3}} = h^3\left(\frac{23}{23227}f_n + \frac{199}{9720}f_{n+\frac{1}{2}} - \frac{31}{8640}f_{n+\frac{2}{3}} + \frac{5}{7778}f_{n+1}\right)$$

(4.11)
$$hy'_{n+1} - \frac{5}{2}y_n + 16y_{n+\frac{1}{2}} - \frac{27}{2}y_{n+\frac{2}{3}} = h^3\left(\frac{5}{864}f_n + \frac{47}{540}f_{n+\frac{1}{2}} + \frac{29}{480}f_{n+\frac{2}{3}} + \frac{29}{2160}f_{n+1}\right)$$

(4.12)
$$h^2y''_n - 6y_n + 24y_{n+\frac{1}{2}} - 18y_{y+\frac{2}{3}} = h^3\left(-\frac{173}{1162}f_n - \frac{1721}{3240}f_{n+\frac{1}{2}} + \frac{961}{2880}f_{n+\frac{2}{3}} - \frac{551}{12960}f_{n+1}\right)$$

(4.13)
$$h^2y''_{n+\frac{1}{2}} - 6y_n + 24y_{n+\frac{1}{2}} - 18y_{y+\frac{2}{3}} = h^3\left(\frac{163}{12960}f_n + \frac{287}{1620}f_{n+\frac{1}{2}} - \frac{127}{1440}f_{n+\frac{2}{3}} + \frac{31}{3240}f_{n+1}\right)$$

(4.14)
$$h^2y''_{n+\frac{2}{3}} - 6y_n + 24y_{n+\frac{1}{2}} - 18y_{y+\frac{2}{3}} = h^3\left(\frac{62}{5339}f_n + \frac{311}{1201}f_{n+\frac{1}{2}} + \frac{1}{2880}f_{n+\frac{2}{3}} + \frac{89}{12960}f_{n+1}\right)$$

(4.15)
$$h^2y''_{n+1} - 6y_n + 24y_{n+\frac{1}{2}} - 18y_{y+\frac{2}{3}} = h^3\left(\frac{164}{9221}f_n + \frac{439}{3240}f_{n+\frac{1}{2}} + \frac{961}{2880}f_{n+\frac{2}{3}} + \frac{128}{1031}f_{n+1}\right)$$

Equations (4.5) and (4.8)-(4.15) are solved by [23] block formula below:

(4.16)
$$Ay_m = hBF(y_m) + Ey_n + hDf_n$$

From equations (4.3) and (4.8)-(4.15); we obtained A, B, D and E, using (4.16), as follows:

$$A = \begin{bmatrix} 4 & \frac{-9}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & \frac{-9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & \frac{-15}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 16 & \frac{-27}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 24 & -18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & -18 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 24 & -18 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 24 & -18 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{25}{1296} & \frac{7}{1152} & \frac{7}{5184} \\ \frac{55}{648} & \frac{-139}{2880} & \frac{77}{12960} \\ \frac{-59}{3240} & \frac{7}{1152} & \frac{-19}{2592} \\ \frac{199}{9720} & \frac{-31}{8640} & \frac{5}{7776} \\ \frac{47}{540} & \frac{29}{480} & \frac{29}{2160} \\ \frac{-1721}{3240} & \frac{961}{2880} & \frac{-551}{12960} \\ \frac{287}{1620} & \frac{-127}{1440} & \frac{31}{3240} \\ \frac{311}{1201} & \frac{1}{2880} & \frac{89}{12960} \\ \frac{439}{3240} & \frac{961}{2880} & \frac{128}{1031} \end{bmatrix}, D = \begin{bmatrix} \frac{11}{10368} \\ \frac{-93}{7153} \\ \frac{-53}{51840} \\ \frac{23}{23227} \\ \frac{5}{864} \\ \frac{-173}{1162} \\ \frac{163}{12960} \\ \frac{62}{5339} \\ \frac{164}{9221} \end{bmatrix},$$

$$E = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{-7}{2} & -1 & 0 \\ \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{5}{2} & 0 & 0 \\ 6 & 0 & -1 \\ 6 & 0 & 0 \\ 6 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix}.$$

Putting A, B, D and E into the equation (4.16) the explicit schemes are gotten as:

(4.17)
$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3}y'_n + \frac{1}{18}y''_n + + \frac{69}{18185}f_n + h^3\left(\frac{35}{71801}f_{n+\frac{1}{3}} - \frac{2}{173719}f_{n+\frac{2}{3}} + \frac{115}{31462}f_{n+1} - \frac{53}{30240}f_{n+2}\right)$$

(4.18)
$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}y'_n + \frac{2}{9}y''_n + \frac{233}{1749}f_n + h^3\left(\frac{16}{5703}f_{n+\frac{1}{3}} - \frac{19}{255150}f_{n+\frac{2}{3}} + \frac{176}{4725}f_{n+1} - \frac{61}{5670}f_{n+2}\right)$$

(4.19)
$$y_{n+1} = y_n + y'_n + \frac{1}{2}y''_n + \frac{27}{560}f_n + h^3\left(\frac{135}{5174}f_{n+\frac{1}{3}} + \frac{113}{6013}f_{n+\frac{2}{3}} + \frac{35}{348}f_{n+1} - \frac{181}{1676}f_{n+2}\right)$$

$$y'_{n+\frac{1}{3}} = y'_{n+\frac{1}{3}} + \frac{187}{6480}f_n + h^2\left(\frac{1}{216}f_{n+\frac{1}{3}} - \frac{7}{64800}f_{n+\frac{2}{3}} + \frac{211}{5400}f_{n+1} - \frac{73}{4320}f_{n+2}\right) \tag{4.20}$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{2}{3}y''_n + \frac{1}{5}f_n + h^2\left(\frac{4}{405}f_{n+\frac{1}{3}} - \frac{1}{4050}f_{n+\frac{2}{3}} + \frac{116}{675}f_{n+1} - \frac{7}{270}f_{n+2}\right) \tag{4.21}$$

$$y'_{n+1} = y'_n + y''_n + \frac{5}{48}f_n + h^2\left(\frac{1}{40}f_{n+\frac{1}{3}} - \frac{1}{2400} + \frac{63}{200}f_{n+1} + \frac{9}{160}f_{n+2}\right) \tag{4.22}$$

$$y'_{n+2} = y'_n + 2y''_n + \frac{1}{15} f_n + h^2 \left(\frac{187}{5039} f_{n+\frac{1}{3}} + \frac{3}{50} + \frac{36}{25} f_{n+1} - \frac{9}{10} f_{n+2} \right) \tag{4.23}$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{143}{1620} f_n + h \left(\frac{83}{3240} f_{n+\frac{1}{3}} - \frac{19}{32400} f_{n+\frac{1}{3}} + \frac{57}{200} f_{n+1} - \frac{23}{240} f_{n+2} \right) \tag{4.24}$$

$$y''_{n+\frac{2}{3}} = y''_n + \frac{44}{405} f_n + h \left(\frac{2}{405} f_{n+\frac{1}{3}} - \frac{1}{4050} f_{n+\frac{2}{3}} + \frac{34}{75} f_{n+1} + \frac{1}{10} f_{n+2} \right) \tag{4.25}$$

$$y''_{n+1} = y''_n + \frac{7}{60} f_n + h \left(\frac{17}{120} f_{n+\frac{1}{3}} - \frac{1}{1200} f_{n+\frac{2}{3}} + \frac{81}{1200} f_{n+1} + \frac{27}{80} f_{n+2} \right) \tag{4.26}$$

$$y''_{n+2} = y''_n + \frac{263}{540} f_n + h \left(\frac{38}{15} f_{n+\frac{1}{3}} + \frac{41}{150} f_{n+\frac{2}{3}} + \frac{54}{25} f_{n+1} - \frac{27}{10} f_{n+2} \right) \tag{4.27}$$

5. Creation of a one-step approach based on $xn + \frac{1}{5}$, $xn + \frac{2}{5}$, $xn + \frac{3}{5}$ and $xn + \frac{4}{5}$ as the off-step Points

Interpolate (3.1) at $x = x_{n+s}$, $s = 0, \frac{1}{5}$ and $\frac{2}{5}$ and collocate (3.2) at all the points $x = x_{n+r}$, $r = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and 1 to get:

$$\begin{bmatrix}
1 & \frac{-5}{3} & \frac{53}{33} & \frac{-689}{419} & \frac{983}{591} & \frac{-5339}{3189} & \frac{631}{375} & \frac{10287}{6091} & \frac{3401}{2008} \\
1 & \frac{-17}{15} & \frac{29}{165} & \frac{2453}{3999} & \frac{-1046}{1331} & \frac{393}{1111} & \frac{2735}{10028} & \frac{-781}{1277} & \frac{157}{349} \\
1 & \frac{-3}{5} & \frac{-3}{5} & \frac{745}{1152} & \frac{773}{3253} & \frac{-799}{1299} & \frac{199}{7659} & \frac{931}{1780} & \frac{-693}{3155} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{-246792}{197} & \frac{140399}{27} & \frac{-112087}{7} & \frac{203929}{5} & \frac{-181945}{2} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{-29299}{38} & \frac{67043}{43} & \frac{-65472}{47} & \frac{-38189}{44} & \frac{58236}{13} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{-26905}{93} & \frac{-22200}{67} & \frac{23845}{22} & \frac{2571}{940} & \frac{-38085}{17} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{17703}{92} & \frac{-16502}{35} & \frac{-105569}{127} & \frac{26265}{44} & \frac{54478}{27} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{31685}{47} & \frac{75155}{66} & \frac{38566}{63} & \frac{53071}{38} & \frac{-84367}{23} \\
0 & 0 & 0 & \frac{35701}{210} & \frac{9247}{8} & \frac{17997}{4} & \frac{105317}{8} & \frac{64401}{2} & \frac{695129}{10}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}
=
\begin{bmatrix} y_n \\ y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ h^3 f_n \\ h^3 f_{n+\frac{1}{5}} \\ h^3 f_{n+\frac{2}{5}} \\ h^3 f_{n+\frac{3}{5}} \\ h^3 f_{n+\frac{4}{5}} \\ h^3 f_{n+1} \end{bmatrix} \tag{5.1}$$

The unknown parameters aj, $j = 0(1)8$ are determined by solving the system of equations (5.1), this yields:

$$
\left.
\begin{aligned}
a_0 =& \frac{9}{4}y_n - \frac{45}{8}y_{n+\frac{1}{5}} + \frac{35}{8}y_{n+\frac{2}{5}} + h^3\left(\frac{f_n}{351306} - \frac{20}{115167}f_{n+1} + -\frac{113}{13039}f_{n+\frac{1}{5}}\right.\\
&\left. + -\frac{143}{11221}f_{n+\frac{2}{5}} + -\frac{67}{22328}f_{n+\frac{3}{5}} + -\frac{67}{56949}f_{n+\frac{4}{5}}\right)\\
a_1 =& \frac{20}{9}y_n - \frac{455}{72}y_{n+\frac{1}{5}} + \frac{295}{72}y_{n+\frac{2}{5}} - h^3\left(\frac{7}{837843}f_n - \frac{17}{92165}f_{n+1} + -\frac{74}{8063}f_{n+\frac{1}{5}}\right.\\
&\left. + -\frac{281}{19748}f_{n+\frac{2}{5}} + -\frac{31}{7675}f_{n+\frac{3}{5}} + -\frac{256}{175701}f_{n+\frac{4}{5}}\right)\\
a_2 =& \frac{55}{36}y_n - \frac{55}{18}y_{n+\frac{1}{5}} + \frac{55}{36}y_{n+\frac{2}{5}} - h^3\left(\frac{5}{354787}f_n + \frac{10}{74773}f_{n+1} + -\frac{33}{5989}f_{n+\frac{1}{5}}\right.\\
&\left. + -\frac{31}{3382}f_{n+\frac{2}{5}} + -\frac{135}{33337}f_{n+\frac{3}{5}} + -\frac{59}{41760}f_{n+\frac{4}{5}}\right)\\
a_3 =& h^3\left(\frac{23}{36083}f_{n+\frac{1}{5}} - \frac{16}{517681}f_{n+1} - \frac{3}{105691}f_n + \frac{151}{74204}f_{n+\frac{2}{5}}\right.\\
&\left. + \frac{37}{15864}f_{n+\frac{3}{5}} + \frac{37}{39496}f_{n+\frac{4}{5}}\right)\\
a_4 =& h^3\left(\frac{51}{114116}f_{n+\frac{3}{5}} - \frac{8}{630049}f_{n+1} - \frac{13}{37013}f_{n+\frac{1}{5}} - \frac{9}{13049}f_{n+\frac{2}{5}}\right.\\
&\left. - \frac{5}{195341}f_n + \frac{45}{71167}f_{n+\frac{4}{5}}\right)\\
a_5 =& h^3\left(\frac{3}{1185568}f_{n+1} - \frac{1}{598972}f_n + \frac{7}{27261}f_{n+\frac{1}{5}} - \frac{14}{73097}f_{n+\frac{2}{5}}\right.\\
&\left. - \frac{58}{153411}f_{n+\frac{3}{5}} + \frac{25}{80139}fn + \frac{4}{5}\right)\\
a_6 =& h^3\left(\frac{8}{640001}f_{n+1} - \frac{4}{403907}f_n - \frac{7}{91015}f_{n+\frac{1}{5}} + \frac{17}{61228}f_{n+\frac{2}{5}} -\right.\\
&\left. \frac{10}{36703}f_{n+\frac{3}{5}} + \frac{13}{188078}f_{n+\frac{4}{5}}\right)\\
a_7 =& h^3\left(\frac{4}{316409}f_n + \frac{1}{64742}f_{n+1} - \frac{9}{256253}f_{n+\frac{1}{5}} + \frac{4}{284343}f_{n+\frac{2}{5}} +\right.\\
&\left. \frac{7}{166239}f_{n+\frac{3}{5}} - \frac{11}{223842}f_{n+\frac{4}{5}}\right)\\
a_8 =& h^3\left(\frac{2}{401087}f_{n+1} - \frac{3}{888238}f_n + \frac{4}{216259}f_{n+\frac{1}{5}} - \frac{23}{571988}f_{n+\frac{2}{5}}\right.\\
&\left. + \frac{13}{299342}f_{n+\frac{3}{5}} - \frac{3}{128627}f_{n+\frac{4}{5}}\right)
\end{aligned}
\right\}
\tag{5.2}
$$

putting (5.2) in (3.1) yields the continuous implicit one-step method

(5.3)

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_{\frac{1}{5}}(x) y_{n+\frac{1}{5}} + \alpha_{\frac{2}{5}}(x) y_{n+\frac{2}{5}}(x) + \alpha_{\frac{3}{5}}(x) y_{n+\frac{3}{5}}(x) + \alpha_{\frac{4}{5}}(x) y_{n+\frac{4}{5}}(x)$$
$$+ h\left(\sum_{j=0}^{1} \beta_j(x) f_{n+j} + \beta_{\frac{1}{5}}(x) f_{n+\frac{1}{5}} + \beta_{\frac{2}{5}}(x) f_{n+\frac{2}{5}} + \beta_{\frac{3}{5}}(x) f_{n+\frac{3}{5}} + \beta_{\frac{4}{5}}(x) f_{n+\frac{4}{5}}\right)$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients from (5.3). We get the parameters $\alpha_j(x)$ and $\beta_j(x)$ as:

$$
\left.\begin{aligned}
\alpha_0(t) &= \frac{25}{8}t^2 \\
\alpha_{\frac{1}{5}}(t) &= -\frac{25}{4}t^2 - \frac{15}{2}t - \frac{5}{4} \\
\alpha_{\frac{2}{5}}(t) &= \frac{25}{8}t^2 + 5t + \frac{15}{8} \\
\beta_0(t) &= h^3\left(-\frac{9}{36101}t^8 + \frac{14}{26133}t^7 + \frac{18}{98131}t^6 - \frac{31}{35840}t^5 - \frac{1}{32153}t^4 + \frac{23}{72789}t^3\right. \\
&\qquad \left. + \frac{9}{239671}t^2 - \frac{1}{299216}t + \frac{1}{7929323}\right) \\
\beta_{\frac{1}{5}}(t) &= h^3\left(\frac{24}{17579}t^8 - \frac{23}{14413}t^7 - \frac{32}{8049}t^6 + \frac{70}{12051}t^5 + \frac{26}{34007}t^4 - \frac{23}{10295}t^3\right. \\
&\qquad \left. + \frac{227}{18388}t^2 + \frac{55}{5576}t + \frac{39}{26677}\right) \\
\beta_{\frac{2}{5}}(t) &= h^3\left(-\frac{38}{12803}t^8 + \frac{27}{26318}t^7 + \frac{97}{8542}t^6 - \frac{97}{20127}t^5 - \frac{131}{8604}t^4 + \frac{354}{28811}t^3\right. \\
&\qquad \left. + \frac{125}{5127}t^2 + \frac{111}{10774}t + \frac{78}{64693}\right) \\
\beta_{\frac{3}{5}}(t) &= h^3\left(\frac{68}{21213}t^8 + \frac{73}{64050}t^7 - \frac{168}{13945}t^6 - \frac{101}{19298}t^5 + \frac{199}{12957}t^4 + \frac{171}{13739}t^3\right. \\
&\qquad \left. + \frac{5}{15203}t^2 - \frac{31}{19025}t - \frac{5}{18872}\right) \\
\beta_{\frac{4}{5}}(t) &= h^3\left(-\frac{31}{18007}t^8 - \frac{27}{16337}t^7 + \frac{139}{27727}t^6 + \frac{131}{21776}t^5 - \frac{26}{26969}t^4 - \frac{41}{17720}t^3\right. \\
&\qquad \left. + \frac{31}{52493}t^2 + \frac{45}{54236}t + \frac{5}{39389}\right) \\
\beta_1(t) &= h^3\left(-\frac{13}{35320}t^8 - \frac{21}{38384}t^7 + \frac{22}{41567}t^6 + \frac{46}{50751}t^5 - \frac{1}{10245}t^4 - \frac{46}{138589}t^3\right. \\
&\qquad \left. + \frac{17}{93029}t^2 + \frac{14}{71231}t + \frac{3}{100840}\right)
\end{aligned}\right\} \tag{5.4}
$$

where $t = \frac{2x-2x_n-h}{h}$.
By evaluating (5.3) at $x_{n+\frac{3}{5}}$, $x_{n+\frac{4}{5}}$ and x_{n+1} the main methods are obtained as:

(5.5)

$$
y_{n+\frac{3}{5}} = y_n - 3y_{n+\frac{1}{5}} + 3y_{n+\frac{2}{5}} + h^3\left(\frac{1}{314562}f_n + \frac{111}{28364}f_{n+\frac{1}{5}} + \frac{141}{32680}f_{n+\frac{2}{5}} - \frac{29}{63603}f_{n+\frac{3}{5}} + \frac{35}{117186}f_{n+\frac{4}{5}} - \frac{13}{175978}f_{n+1}\right)
$$

$$
\begin{aligned}
y_{n+\frac{4}{5}} &= 3y_n - 8y_{n+\frac{1}{5}} + 6y_{n+\frac{2}{5}} + \\
& h^3\left(\frac{2}{71507}f_n + \frac{71}{6076}f_{n+\frac{1}{5}} + \frac{135}{7951}f_{n+\frac{2}{5}} + \frac{38}{14225}f_{n+\frac{3}{5}} + \frac{19}{22642}f_{n+\frac{4}{5}} - \frac{34}{68001}f_{n+1}\right)
\end{aligned} \tag{5.6}
$$

(5.7)

$$
\begin{aligned}
y_{n+1} &= 6y_n - 15y_{n+\frac{1}{5}} + 10y_{n+\frac{2}{5}} \\
&+ h^3\left(-\frac{8}{105415}f_n + \frac{95}{3991}f_{n+\frac{1}{5}} + \frac{326}{8683}f_{n+\frac{2}{5}} + \frac{179}{13455}f_{n+\frac{3}{5}} + \frac{76}{12829}f_{n+\frac{4}{5}} - \frac{34}{68001}f_{n+1}\right)
\end{aligned}
$$

Differentiating (5.3) gives the continuous coefficients:

$$
\left.
\begin{aligned}
\alpha_0'(t) =& \frac{25t+10}{2h}y_n \\
\alpha_{\frac{1}{5}}'(t) =& \frac{-(25t+15)}{h}y_{\frac{1}{5}} \\
\alpha_{\frac{2}{5}}'(t) =& \left(\frac{25t+10}{2h}\right)y_{n+\frac{2}{5}} \\
\beta_0'(t) =& h^2\left(\frac{196}{26133}t^6 - \frac{67}{16797}t^7 + \frac{29}{13175}t^5 - \frac{31}{3584}t^4 - \frac{8}{32153}t^3\right. \\
& \left. + d\frac{46}{24263}t^2 + \frac{19}{126493}t - \frac{1}{149608}\right) \\
\beta_{\frac{1}{5}}'(t) =& h^2\left(\frac{131}{5997}t^7 - \frac{46}{2059}t^6 - \frac{128}{2683}t^5 + \frac{547}{9417}t^4 + \frac{103}{16840}t^3\right. \\
& \left. - \frac{138}{10295}t^2 + \frac{227}{4597}t + \frac{55}{2788}\right) \\
\beta_{\frac{2}{5}}'(t) =& h^2\left(-\frac{139}{2927}t^7 + \frac{181}{12602}t^6 + \frac{582}{4271}t^5 - \frac{487}{10105}t^4 - \frac{262}{2151}t^3\right. \\
& \left. + \frac{1031}{13985}t^2 + \frac{626}{6419}t + \frac{111}{5387}\right) \\
\beta_{\frac{3}{5}}'(t) =& h^2\left(\frac{181}{3529}t^7 + \frac{73}{4575}t^6 - \frac{169}{1169}t^5 - \frac{159}{3038}t^4 + \frac{389}{3166}t^3\right. \\
& \left. + \frac{371}{4968}t^2 + \frac{20}{15203}t - \frac{62}{19025}\right) \\
\beta_{\frac{4}{5}}'(t) =& h^2\left(-\frac{657}{23852}t^7 - \frac{41}{1772}t^6 - \frac{169}{1169}t^5 - \frac{159}{3038}t^4 - \frac{41}{5316}t^3\right. \\
& \left. - \frac{92}{6627}t^2 + \frac{124}{52493}t + \frac{45}{27118}\right) \\
\beta_1'(t) =& -h^2\left(-\frac{26}{4415}t^7 - \frac{52}{6789}t^6 + \frac{51}{8030}t^5 + \frac{67}{7392}t^4 - \frac{8}{10245}t^3\right. \\
& \left. - \frac{97}{48707}t^2 + \frac{27}{36938}t + \frac{28}{71231}\right)
\end{aligned}
\right\} \tag{5.8}
$$

The second derivatives of the continuous functions are given as:

(5.9)
$$\left.\begin{aligned}
\alpha''_0(t) &= \frac{25}{h^2}\\
\alpha''_{\frac{1}{5}}(t) &= -\frac{50}{h^2}\\
\alpha''_{\frac{2}{5}}(t) &= \frac{25}{h^2}\\
\beta''_0(t) &= h\left(-\frac{345}{6178}t^6+\frac{748}{8711}t^5+\frac{58}{2635}t^4-\frac{31}{448}t^3-\frac{41}{27464}t^2+\frac{103}{13582}t+\frac{17}{56589}\right)\\
\beta''_{\frac{1}{5}}(t) &= h\left(\frac{2575}{842}t^6-\frac{941}{351}t^5-\frac{1894}{397}t^4+\frac{1882}{405}t^3+\frac{309}{842}t^2-\frac{1104}{2059}t+\frac{1573}{1613}\right)\\
\beta''_{\frac{2}{5}}(t) &= h\left(-\frac{365}{549}t^6+\frac{197}{1143}t^5+\frac{2198}{1613}t^4-\frac{475}{1232}t^3-\frac{524}{717}t^2+\frac{969}{3286}t+\frac{1189}{6096}\right)\\
\beta''_{\frac{3}{5}}(t) &= h\left(\frac{545}{759}t^6+\frac{292}{1525}t^5-\frac{1171}{810}t^4-\frac{533}{1273}t^3+\frac{1167}{1583}t^2+\frac{371}{1242}t+\frac{40}{15203}\right)\\
\beta''_{\frac{4}{5}}(t) &= -h\left(\frac{703}{1823}t^6+\frac{123}{443}t^5 * t^4-\frac{655}{1361}t^3+\frac{41}{886}t^2+\frac{123}{2215}t-\frac{251}{53128}\right)\\
\beta''_1(t) &= h\left(\frac{364}{4415}t^6+\frac{208}{2263}t^5-\frac{51}{803}t^4-\frac{67}{924}t^3+\frac{16}{3415}t^2+\frac{388}{48707}t-\frac{27}{18469}\right)
\end{aligned}\right\}$$

The extra techniques to be used in conjunction with the main method are produced by evaluating the two derivatives of (5.3) at $x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}, x_{n+\frac{4}{5}}$ and x_{n+1} , respectively to obtain

$$hy'_n+\frac{15}{2}y_n-10y_{n+\frac{1}{5}}+\frac{5}{2}y_{n+\frac{2}{5}} = h^3\left(\frac{46}{17517}f_n+\frac{94}{7559}f_{n+\frac{1}{5}} -\frac{50}{12493}f_{n+\frac{2}{5}}+\frac{111}{26854}f_{n+\frac{3}{5}}-\frac{44}{17845}f_{n+\frac{4}{5}}+\frac{28}{46185}f_{n+1}\right) \tag{5.10}$$

$$hy'_{n+\frac{1}{5}}+\frac{5}{2}y_n-0y_{n+\frac{1}{5}}-\frac{5}{2}y_{n+\frac{2}{5}} = h^3\left(-\frac{6}{31397}f_n-\frac{109}{16864}f_{n+\frac{1}{5}} +\frac{47}{478414}f_{n+\frac{2}{5}}+\frac{3}{47023}f_{n+\frac{3}{5}}-\frac{32}{114455}f_{n+\frac{4}{5}}+\frac{17}{161207}f_{n+1}\right) \tag{5.11}$$

$$hy'_{n+\frac{2}{5}}-\frac{5}{2}y_n+10y_{n+\frac{1}{5}}-\frac{15}{2}y_{n+\frac{2}{5}} = h^3\left(-\frac{7}{258415}f_n+\frac{94}{7559}f_{n+\frac{1}{5}} -\frac{53}{10806}f_{n+\frac{2}{5}}-\frac{41}{26368}f_{n+\frac{3}{5}}+\frac{58}{75399}f_{n+\frac{4}{5}}-\frac{11}{59287}f_{n+1}\right) \tag{5.12}$$

$$hy'_{n+\frac{3}{5}} - \frac{15}{2}y_n + 20y_{n+\frac{1}{5}} - \frac{25}{2}y_{n+\frac{2}{5}} = h^3\left(\frac{7}{82844}f_n + \frac{145}{4967}f_{n+\frac{1}{5}} + \frac{155}{3686}f_{n+\frac{2}{5}} + \frac{43}{50829}f_{n+\frac{3}{5}} + \frac{19}{11667}f_{n+\frac{4}{5}} - \frac{20}{42613}f_{n+1}\right) \tag{5.13}$$

$$hy'_{n+\frac{4}{5}} - \frac{25}{2}y_n + 30y_{n+\frac{1}{5}} - \frac{35}{2}y_{n+\frac{2}{5}} = h^3\left(\frac{1}{1518105}f_n + \frac{181}{3676}f_{n+\frac{1}{5}} + \frac{118}{1421}f_{n+\frac{2}{5}} + \frac{56}{1595}f_{n+\frac{3}{5}} + \frac{46}{6537}f_{n+\frac{4}{5}} - \frac{7}{6408}f_{n+1}\right) \tag{5.14}$$

$$hy'_{n+1} - \frac{35}{2}y_n + 40y_{n+\frac{1}{5}} - \frac{45}{2}y_{n+\frac{2}{5}} = h^3\left(-\frac{5}{4631}f_n + \frac{263}{3668}f_{n+\frac{1}{5}} + \frac{2566}{20529}f_{n+\frac{2}{5}} + \frac{79}{1198}f_{n+\frac{3}{5}} + \frac{393}{7549}f_{n+\frac{4}{5}} - \frac{7}{32071}f_{n+1}\right) \tag{5.15}$$

$$h^2y''_n - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\left(-\frac{152}{2397}f_n - \frac{605}{3384}f_{n+\frac{1}{5}} + \frac{761}{9469}f_{n+\frac{2}{5}} - \frac{221}{3728}f_{n+\frac{3}{5}} + \frac{181}{6877}f_{n+\frac{4}{5}} - \frac{65}{12474}f_{n+1}\right) \tag{5.16}$$

$$h^2y''_{n+\frac{1}{5}} - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\left(\frac{33}{9683}f_n + \frac{52}{3049}f_{n+\frac{1}{5}} - \frac{79}{2676}f_{n+\frac{2}{5}} + \frac{251}{23929}f_{n+\frac{3}{5}} - \frac{19}{18881}f_{n+\frac{4}{5}} - \frac{12}{28289}f_{n+1}\right) \tag{5.17}$$

$$h^2y''_{n+\frac{2}{5}} - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\left(-\frac{59}{81990}f_n + \frac{1350}{12667}f_{n+\frac{1}{5}} + \frac{349}{3116}f_{n+\frac{2}{5}} - \frac{93}{3496}f_{n+\frac{3}{5}} + \frac{124}{11115}f_{n+\frac{4}{5}} - \frac{53}{21961}f_{n+1}\right) \tag{5.18}$$

$$h^2y''_{n+\frac{3}{5}} - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\left(\frac{26}{20565}f_n + \frac{487}{5271}f_{n+\frac{1}{5}} + \frac{727}{3247}f_{n+\frac{2}{5}} + \frac{554}{6419}f_{n+\frac{3}{5}} - \frac{840}{237721}f_{n+\frac{4}{5}} - \frac{17}{51779}f_{n+1}\right) \tag{5.19}$$

$$h^2y''_{n+\frac{4}{5}} - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\left(-\frac{61}{18007}f_n + \frac{493}{4411}f_{n+\frac{1}{5}} + \frac{218}{1181}f_{n+\frac{2}{5}} + \frac{188}{863}f_{n+\frac{3}{5}} + \frac{203}{2091}f_{n+\frac{4}{5}} - \frac{35}{4433}f_{n+1}\right) \tag{5.20}$$

$$
\begin{aligned}
&h^2y''_{n+1} - 25y_n + 50y_{n+\frac{1}{5}} - 25y_{n+\frac{2}{5}} = h^3\Bigg(-\frac{68}{10247}f_n + \frac{297}{2771}f_{n+\frac{1}{5}} \\
&+\frac{380}{1559}f_{n+\frac{2}{5}} + \frac{1899}{22688}f_{n+\frac{3}{5}} + \frac{317}{983}f_{n+\frac{4}{5}} + \frac{209}{4220}f_{n+1}\Bigg)
\end{aligned} \tag{5.21}
$$

(5.5)-(5.7) and (5.10)-(5.21) are solved using equation (4.16). A, B, D and E are obtained from the equations as:

$$
A = \begin{bmatrix}
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & -10 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-5}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & \frac{-15}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & \frac{-25}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & \frac{-35}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & \frac{45}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
\frac{111}{28364} & \frac{141}{32680} & \frac{-29}{63603} & \frac{35}{117186} & \frac{-13}{175978} \\
\frac{71}{6076} & \frac{135}{7951} & \frac{38}{14225} & \frac{19}{22642} & \frac{-11}{54242} \\
\frac{95}{3991} & \frac{326}{8683} & \frac{179}{13455} & \frac{76}{12829} & \frac{-34}{68001} \\
\frac{94}{7559} & \frac{-50}{12493} & \frac{111}{26854} & \frac{-44}{17845} & \frac{28}{46185} \\
\frac{-109}{16864} & \frac{47}{478414} & \frac{3}{47023} & \frac{-32}{114455} & \frac{17}{161207} \\
\frac{237}{25286} & \frac{53}{10806} & \frac{-41}{26368} & \frac{58}{75399} & \frac{-11}{59287} \\
\frac{145}{4967} & \frac{155}{3686} & \frac{43}{50829} & \frac{19}{11667} & \frac{-20}{42613} \\
\frac{181}{3676} & \frac{118}{1421} & \frac{56}{1595} & \frac{46}{6537} & -\frac{7}{6408} \\
\frac{263}{3668} & \frac{2566}{20529} & \frac{79}{1198} & \frac{393}{7549} & \frac{-7}{32071} \\
\frac{-605}{3384} & \frac{761}{9469} & \frac{-221}{3728} & \frac{181}{6877} & \frac{-65}{12474} \\
\frac{52}{3049} & \frac{-79}{2676} & \frac{251}{23929} & \frac{-19}{18881} & \frac{-12}{28289} \\
\frac{1350}{12667} & \frac{349}{3116} & \frac{-93}{3496} & \frac{124}{11115} & \frac{-53}{21961} \\
\frac{487}{5271} & \frac{727}{3247} & \frac{554}{6419} & \frac{-840}{237721} & \frac{-17}{51779} \\
\frac{493}{4411} & \frac{218}{1181} & \frac{188}{863} & \frac{203}{2091} & \frac{-35}{4433} \\
\frac{297}{2771} & \frac{380}{1559} & \frac{1899}{22688} & \frac{317}{983} & \frac{209}{4220}
\end{bmatrix}
D = \begin{bmatrix}
\frac{1}{314562} \\
\frac{2}{71507} \\
\frac{-8}{105415} \\
\frac{46}{17517} \\
\frac{-6}{31397} \\
\frac{7}{258415} \\
\frac{7}{82844} \\
\frac{1}{1518105} \\
\frac{-5}{4361} \\
\frac{-152}{2397} \\
\frac{33}{9683} \\
\frac{-59}{81990} \\
\frac{26}{20565} \\
\frac{-61}{18007} \\
\frac{-68}{10247}
\end{bmatrix}
E = \begin{bmatrix}
1 & 0 & 0 \\
3 & 0 & 0 \\
6 & 0 & 0 \\
\frac{-15}{2} & -1 & 0 \\
\frac{-5}{2} & 0 & 0 \\
\frac{5}{2} & 0 & 0 \\
\frac{15}{2} & 0 & 0 \\
\frac{25}{2} & 0 & 0 \\
\frac{35}{2} & 0 & 0 \\
25 & 0 & -1 \\
25 & 0 & 0 \\
25 & 0 & 0 \\
25 & 0 & 0 \\
25 & 0 & 0 \\
25 & 0 & 0
\end{bmatrix}
$$

Substituting A, B, D and E into (4.16) yields the explicit schemes:

$$\begin{aligned} y_{n+\frac{1}{5}} &= y_n + \frac{1}{5}y_n'' + \frac{1}{50}y_n'' + \frac{31}{41720}f_n \\ &\quad + h^3\left(\frac{29}{26641}f_{n+\frac{1}{5}} - \frac{13}{16111}f_{n+\frac{2}{5}} - \frac{18}{50149}f_{n+\frac{3}{5}} - \frac{7}{210482}f_{n+\frac{4}{5}} - \frac{4}{234815}f_{n+1}\right) \end{aligned} \tag{5.22}$$

$$\begin{aligned} y_{n+\frac{2}{5}} &= y_n + \frac{2}{5}y_n' + \frac{2}{25}y_n'' + \frac{42}{10441}f_n + \frac{451}{48347}f_{n+\frac{1}{5}} \\ &\quad + h^3\left(-\frac{29}{6006}f_{n+\frac{2}{5}} - \frac{39}{12625}f_{n+\frac{3}{5}} - \frac{12}{10721}f_{n+\frac{4}{5}} + \frac{8}{45881}f_{n+1}\right) \end{aligned} \tag{5.23}$$

$$\begin{aligned} y_{n+\frac{3}{5}} =& y_n + \frac{3}{5}y_n' + \frac{9}{50}y_n'' + \frac{257}{26113}f_n \\ &+ h^3\left(\frac{133}{48347}f_{n+\frac{1}{5}} - \frac{71}{9161}f_{n+\frac{2}{5}} + \frac{38}{4913}f_{n+\frac{3}{5}} - \frac{10}{3379}h^3f_{n+\frac{4}{5}} + \frac{17}{33978}f_{n+1}\right) \end{aligned} \tag{5.24}$$

$$\begin{aligned} y_{n+\frac{4}{5}} &= y_n + \frac{4}{5}y_n' + \frac{8}{25}y_n'' + \frac{257}{14106}f_n + \frac{28}{475}h^3f_{n+\frac{1}{5}} - \frac{87}{15713}h^3f_{n+\frac{2}{5}} \\ &\quad + \frac{251}{13690}h^3f_{n+\frac{3}{5}} - \frac{81}{14437}h^3f_{n+\frac{4}{5}} + \frac{65}{66349}f_{n+1} \end{aligned} \tag{5.25}$$

$$\begin{aligned} y_{n+1} &= y_n + y_n' + \frac{}{2}y_n'' + \frac{178}{6137}f_n + \frac{491}{4873}h^3f_{n+\frac{1}{5}} + \frac{34}{24943}f_{n+\frac{2}{5}} + \frac{171}{14406}f_{n+\frac{3}{5}} \\ &\quad - \frac{53}{11111}h^3f_{n+\frac{4}{5}} + \frac{28}{18677}f_{n+1} \end{aligned} \tag{5.26}$$

$$\begin{aligned} y'_{n+\frac{1}{5}} &= y_n' + \frac{1}{5}y_n'' + \frac{140}{14191}f_n \\ &\quad + h^2\left(\frac{53}{3144}f_{n+\frac{1}{5}} - \frac{71}{5930}f_{n+\frac{2}{5}} + \frac{61}{7834}f_{n+\frac{3}{5}} - \frac{73}{23718}f_{n+\frac{4}{5}} + \frac{11}{20319}f_{n+1}\right) \end{aligned} \tag{5.27}$$

$$\begin{aligned} y'_{n+\frac{2}{5}} &= y_n' + \frac{2}{5}y_n'' + \frac{293}{12870}f_n \\ &\quad + h^2\left(\frac{197}{2878}f_{n+\frac{1}{5}} - \frac{103}{4432}f_{n+\frac{2}{5}} + \frac{241}{13371}f_{n+\frac{3}{5}} - \frac{42}{5759}f_{n+\frac{4}{5}} + \frac{97}{75046}f_{n+1}\right) \end{aligned} \tag{5.28}$$

$$\begin{aligned} y'_{n+\frac{3}{5}} &= y_n' + \frac{3}{5}y_n'' + \frac{67}{1887}f_n \\ &\quad + h^2\left(\frac{223}{1798}f_{n+\frac{1}{5}} - \frac{56}{25839}f_{n+\frac{2}{5}} + \frac{45}{1394}f_{n+\frac{3}{5}} - \frac{41}{3505}f_{n+\frac{4}{5}} + \frac{56}{27305}f_{n+1}\right) \end{aligned} \tag{5.29}$$

$$\begin{aligned} y'_{n+\frac{4}{5}} &= y_n' + \frac{4}{5}y_n'' + \frac{349}{7255}f_n \\ &\quad + h^2\left(\frac{189}{1051}f_{n+\frac{1}{5}} + \frac{49}{2154}f_{n+\frac{2}{5}} + \frac{102}{1301}f_{n+\frac{3}{5}} - \frac{124}{310733}f_{n+\frac{4}{5}} + \frac{34}{10765}f_{n+1}\right) \end{aligned} \tag{5.30}$$

$$y'_{n+1} = y'_n + y''_n + \frac{116}{1945}f_n + h^2\left(\frac{722}{3033}f_{n+\frac{1}{5}} + \frac{289}{5943}f_{n+\frac{2}{5}} + \frac{333}{2750}f_{n+\frac{3}{5}} + \frac{269}{9537}f_{n+\frac{4}{5}} + \frac{55}{12539}f_{n+1}\right) \tag{5.31}$$

$$y''_{n+\frac{1}{5}} = y''_n + \frac{376}{5627}f_n + h\left(\frac{207}{1057}f_{n+\frac{1}{5}} - \frac{1319}{12003}f_{n+\frac{2}{5}} + \frac{529}{7582}f_{n+\frac{3}{5}} - \frac{383}{14016}f_{n+\frac{4}{5}} + \frac{117}{24443}f_{n+1}\right) \tag{5.32}$$

$$y''n + \frac{2}{5} = y''_n + \frac{203}{3238}f_n + h\left(\frac{573}{12008}f_{n+\frac{1}{5}} + \frac{95}{3003}f_{n+\frac{2}{5}} + \frac{528}{16157}f_{n+\frac{3}{5}} - \frac{134}{8837}f_{n+\frac{4}{5}} + \frac{73}{26095}f_{n+1}\right) \tag{5.33}$$

$$y''n + \frac{3}{5} = y''_n + \frac{1051}{16250}f_n + h\left(\frac{397}{1464}f_{n+\frac{1}{5}} + \frac{821}{5720}f_{n+\frac{2}{5}} + \frac{259}{1779}f_{n+\frac{3}{5}} - \frac{185}{6197}f_{n+\frac{4}{5}} + \frac{197}{40348}f_{n+1}\right) \tag{5.34}$$

$$y''n + \frac{4}{5} = y''_n + \frac{144}{2399}f_n + h\left(\frac{249}{857}f_{n+\frac{1}{5}} + \frac{79}{758}f_{n+\frac{2}{5}} + \frac{189}{682}f_{n+\frac{3}{5}} + \frac{281}{3971}f_{n+\frac{4}{5}} - \frac{53}{19743}f_{n+1}\right) \tag{5.35}$$

$$y''n + 1 = y''_n + \frac{388}{5539}f_n + h\left(\frac{491}{1717}f_{n+\frac{1}{5}} + \frac{265}{1622}f_{n+\frac{2}{5}} + \frac{1164}{1147}f_{n+\frac{3}{5}} + \frac{409}{1381}f_{n+\frac{4}{5}} + \frac{26}{475}f_{n+1}\right) \tag{5.36}$$

6. Methodology Evaluation

The following are the essential characteristics for numerical integration schemes of ODEs: order, error constant, zero stability, and consistency.
The derived main methods are discrete schemes that fall under the LMMs of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{6.1}$$

We use the difference operator to define the Local Truncation Error(LTE) associated with (6.1) in accordance with the approach of [12] & [16].

$$L[y(x):h] = \sum_{j=0}^{k} [\alpha_j y(x_n + jh) - h^3 \beta_j f(x_n + jh)] \tag{6.2}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$.

Expanding (6.1) in Taylor's Series about the point x, we obtain the expression

$$L[y(x):h] = c_o y(x) + c_1 h y^{'}(x) + ... c_{p+3} h^{p+3} y^{p+3}(x) \tag{6.3}$$

where the $c_o, c_1, c_2 ... c_p ... c_{p+3}$ are obtained

$$c_0 = \sum_{j=0}^{k} \alpha_j \tag{6.4}$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j \tag{6.5}$$

$$c_3 = \frac{1}{3!} \sum_{j=1}^{k} j^3 \alpha_j \tag{6.6}$$

$$c_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)(q-2)(q-3) \sum_{j=1}^{k} \beta_j j^{q-3}] \tag{6.7}$$

If $co = c1 = c2 = c2 = ...cp = cp+1 = cp+2 = 0$ and $cp+3 \neq 0$. equation (6.1) is of order p [16].
The error constant is $cp+3 \neq 0$ and the Principal Local truncation error at the location xn is $cp + 3h^{p+3}y^{p+3}(xn)$. The error constants in equations (4.5) & (5.5)-(5.7) are $C_{p+3} = \frac{-1}{24248957}$, $\frac{-1}{54338274}$, $\frac{-1}{12713718}$ and $\frac{-8}{97197}$, respectively, with order $p = 8$.

6.0.1. *Zero stability.* If no root of the first characteristic polynomial $rho(R)$ has modulus more than one and if and only if every root of modulus one has multiplicity not greater than the order of the differential equation, the LMM (6.1) is said to be zero-stable.

6.0.2. *Consistency.* If the LMM (6.1) has order $p \geq 1$ and the first and second characteristic polynomials are specified respectively as below, then it is said to be consistent.

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{6.8}$$

and

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{6.9}$$

where z is the principal root, satisfy the following conditions:

$$\sum_{j=0}^{k} \alpha_j = 0 \tag{6.10}$$

$$\rho(1) = \rho'(1) = 0 \tag{6.11}$$

and

$$\rho'''(1) = 3!\sigma(1) \tag{6.12}$$

(See, [14]).

6.0.3. *Convergence.* Dahlquist Equivalence Theorem. "A multi-step method is convergent if and only if it is consistent and stable". The necessary and sufficient condition for an LMM to be convergent, according to Dahlguist's theorem, is that it be consistent and zero-stable.The theorem is supported in [9] & [10].

6.1. **Zero Stability.** To examine the zero-stability of the method, we express equations (4.17)-(4.27) and (5.22)-(5.36) respectively in block form as follows:

$$A^0 ym = hBf(ym) + A'ynhDfn$$

where h is a constant mesh size inside a block. In accordance with this, equations (4.17)-(4.27) result in

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{23}{960} & \frac{-9}{512} & \frac{3}{1280} \\ \frac{224}{3645} & \frac{-17}{405} & \frac{4}{729} \\ \frac{1}{5} & \frac{-9}{80} & \frac{1}{60} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & \frac{31}{2560} \\ 0 & 0 & \frac{89}{3645} \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

First characteristic polynomial of the hybrid block method is given below

$$\rho(R) = det(RA^0 - A')$$

$$\rho(R) = det\left(RA^0 - A'\right) \tag{6.13}$$

$$\begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{pmatrix}$$

$$e(R) = R^3 - R^2 = 0 \; R^2(R-1) = 0$$

$$R^2 = 0 \text{ or } R - 1 = 0$$

$$R = 0 \text{ or } R = 1$$

When A^0 and A' are substituted in equation (6.13) and R is solved for, the values of R are 0 and 1. The block method equations (4.17)-(4.27) are zero-stable, since from (6.13), $\rho(R) = 0$, fulfils $|Rj| \leq 1, j = 1$ and the multiplicity does not exceed three for those roots with $|Rj| = 1$.

The zero stability of equations (5.22)-(5.36) gives

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{29}{26641} & \frac{-13}{16111} & \frac{18}{50149} & \frac{-7}{210482} & \frac{-42}{34815} \\ \frac{451}{48347} & \frac{-29}{6006} & \frac{39}{12625} & \frac{-12}{10721} & \frac{8}{45881} \\ \frac{133}{4645} & \frac{-71}{9161} & \frac{38}{4913} & \frac{-10}{3379} & \frac{17}{33978} \\ \frac{28}{475} & \frac{-87}{15713} & \frac{251}{13690} & \frac{-81}{14437} & \frac{65}{66349} \\ \frac{491}{4873} & \frac{34}{24943} & \frac{171}{4406} & \frac{-53}{11111} & \frac{28}{18677} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{31}{41720} \\ 0 & 0 & 0 & 0 & \frac{42}{10441} \\ 0 & 0 & 0 & 0 & \frac{257}{26113} \\ 0 & 0 & 0 & 0 & \frac{257}{14106} \\ 0 & 0 & 0 & 0 & \frac{178}{6137} \end{bmatrix}$$

(6.14)
$$\rho(R) = det\left(RA^0 - A'\right)$$

$$\begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{pmatrix}$$

$$e(R) = R^3 - R^2 = 0 \; R^2(R-1) = 0$$

$$R^2 = 0 \text{ or } R - 1 = 0$$

$$R = 0 \text{ or } R = 1$$

When A^0 and A' are substituted in equation (6.14) and R is solved for, the values of R are 0 and 1. The block method equations (5.22)-(5.36) are zero-stable because, starting with (6.14), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1$, and the multiplicity for those roots with $|R_j| = 1$ does not exceed three.

6.2. **Consistency.** The schemes (4.5) & (5.5)-(5.7) are of order $rho = 8 > 1$ and they have been evaluated to meet the requirements (6.10)-(6.12). As a result, the methods are consistent

6.2.1. *Convergence.* The methodology are convergent because they meet the two requirements for convergence described above.

6.3. **Region of Absolute Stability (RAS).** For the one-step with Off-step Points $\frac{1}{2}$ and $\frac{2}{3}$, we have

$$y_{n+1} + 4y_{n+\frac{1}{2}} - \frac{9}{2}y_{n+\frac{2}{3}} - \frac{1}{2}y_n = \frac{h^3}{10368}\left(11f_n + 200f_{n+\frac{1}{2}} + 63f_{n+\frac{2}{3}} + 14f_{n+2}\right)$$

$$\overline{h}(z) = \frac{10368\left(z + 4z^{\frac{1}{2}} - \frac{9}{2}z^{\frac{2}{3}} - \frac{1}{2}\right)}{14z + 200z^{\frac{1}{2}} + 63z^{\frac{2}{3}} - 11}$$

$$\overline{h}(\theta) = \frac{10368\left(e^{i\theta} + 4e^{i\frac{1}{2}\theta} - \frac{9}{2}e^{i\frac{2}{3}\theta} - \frac{1}{2}\right)}{14e^{i\theta} + 200e^{i\frac{1}{2}\theta} + 63e^{i\frac{2}{3}\theta} + 11}$$

The RAS is shown in the figure below

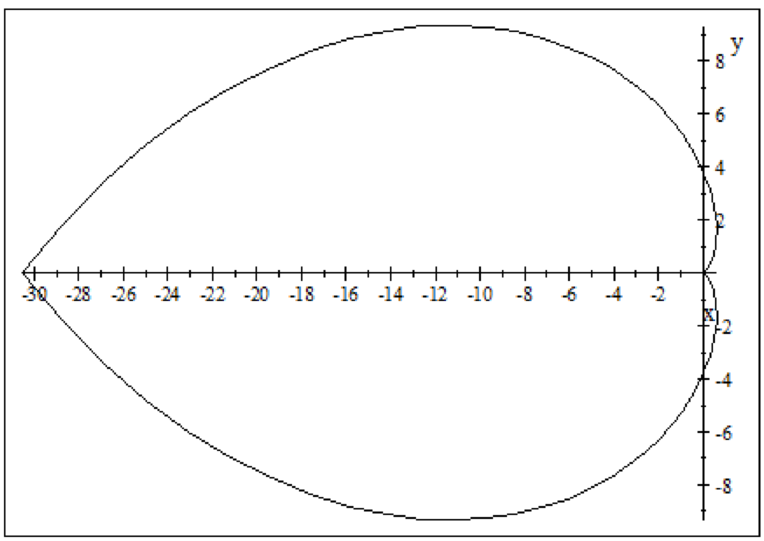

Figure 1: Region of Absolute Stability for One-Step with Off-step Points $\frac{1}{2}$ and $\frac{2}{3}$

For the one-step with Off-step Points $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$, we have

$$y_{n+\frac{3}{5}} - 3y_n + 3y_{n+\frac{1}{5}} - 3y_{n+\frac{2}{5}}$$
$$= h\left(\frac{1}{314562}f_n + \frac{111}{28364}f_{n+\frac{1}{5}} + \frac{141}{32680}f_{n+\frac{2}{5}} + \frac{29}{63603}f_{n+\frac{3}{5}} + \frac{35}{117186}f_{n+\frac{4}{5}} - \frac{13}{175987}f_{n+1}\right)$$

$$\overline{h}(z) = \frac{\left(z^{\frac{3}{5}} + 3z^{\frac{1}{5}} - 3z^{\frac{2}{5}}\right)}{\left(\frac{111}{28364}z^{\frac{1}{5}} + \frac{141}{32680}z^{\frac{2}{5}} - \frac{29}{63603}z^{\frac{3}{5}} + \frac{35}{117186}z^{\frac{4}{5}} - \frac{13}{175987}z + \frac{1}{314562}\right)}$$

$$\overline{h}(\theta) = \frac{\left(e^{i\frac{3}{5}\theta} + 3e^{i\frac{1}{5}\theta} - 3e^{i\frac{2}{5}\theta}\right)}{\left(\frac{111}{28364}e^{i\frac{1}{5}\theta} + \frac{141}{32680}e^{i\frac{2}{5}\theta} - \frac{29}{63603}z^{i\frac{3}{5}\theta} + \frac{35}{117186}e^{i\frac{4}{5}\theta} - \frac{13}{175987}e^{i\theta} + \frac{1}{314562}\right)}$$

The RAS is shown in the figure below

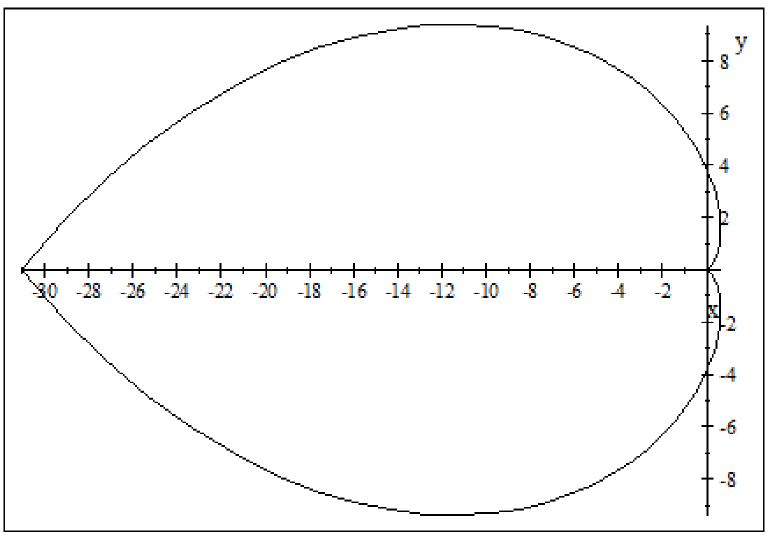

Figure 2: Region of Absolute Stability for One-Step with Off-step Points $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$

$$y_{n+\frac{4}{5}} - 6y_{n+\frac{2}{5}} + 8y_{n+\frac{1}{5}} - 3y_n$$
$$= h^3\left(\frac{2}{71507}f_n + \frac{71}{6076}f_{n+\frac{1}{5}} + \frac{135}{7951}f_{n+\frac{2}{5}} + \frac{38}{14225}f_{n+\frac{3}{5}} + \frac{19}{22642}f_{n+\frac{4}{5}} - \frac{34}{68001}f_{n+1}\right)$$

$$\overline{h}(z) = \frac{\left(z^{\frac{4}{5}} - 6z^{\frac{2}{5}} + 8z^{\frac{1}{5}} - 3\right)}{\left(\frac{2}{71507} + \frac{71}{6076}z^{\frac{1}{5}} + \frac{135}{7951}z^{\frac{2}{5}} + \frac{38}{14225}z^{\frac{3}{5}} + \frac{19}{22642}z^{\frac{4}{5}} - \frac{34}{68001}z\right)}$$

$$\overline{h}(\theta) = \frac{\left(e^{i\frac{4}{5}\theta} - 6e^{i\frac{2}{5}\theta} + 8e^{i\frac{1}{5}\theta} - 3\right)}{\left(\frac{71}{6076}e^{i\frac{1}{5}\theta} + \frac{135}{7951}e^{i\frac{2}{5}\theta} + \frac{38}{14225}z^{i\frac{3}{5}\theta} + \frac{19}{22642}e^{i\frac{4}{5}\theta} - \frac{34}{68001}e^{i\theta} + \frac{2}{71507}\right)}$$

The RAS is shown in the figure below

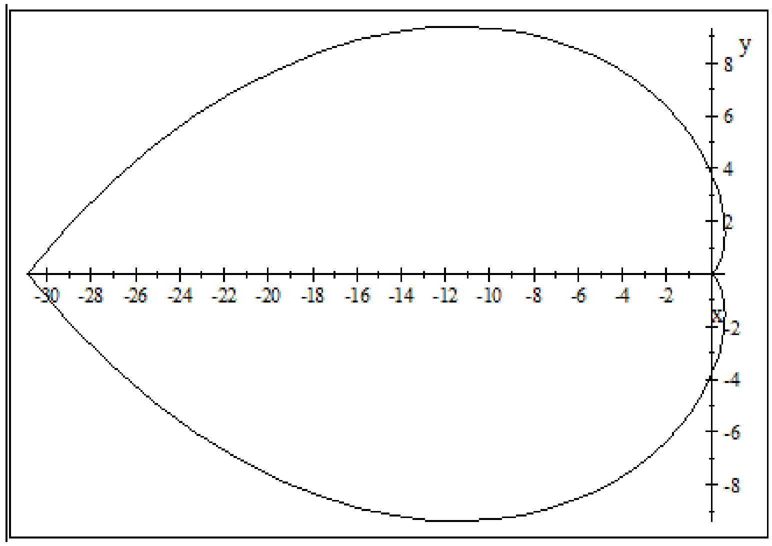

Figure 3: Region of Absolute Stability for One-Step with Off-step Points $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$

$$y_{n+1} + 15y_{n+\frac{1}{5}} - 10y_{n+\frac{2}{5}} - 6y_n$$
$$= h^3\left(-\frac{8}{105415}f_n + \frac{95}{3991}f_{n+\frac{1}{5}} + \frac{326}{8683}f_{n+\frac{2}{5}} + \frac{179}{3455}f_{n+\frac{3}{5}} + \frac{76}{12829}f_{n+\frac{4}{5}} - \frac{34}{68001}f_{n+1}\right)$$

$$\overline{h}(z) = \frac{\left(z - 10z^{\frac{2}{5}} + 15z^{\frac{1}{5}} - 6\right)}{\left(\frac{95}{3991}z^{\frac{1}{5}} + \frac{326}{8683}z^{\frac{2}{5}} + \frac{179}{3455}z^{\frac{3}{5}} + \frac{76}{12829}z^{\frac{4}{5}} - \frac{34}{68001}z - \frac{8}{105415}\right)}$$

$$\overline{h}(\theta) = \frac{\left(e^{i\theta} - 10e^{i\frac{2}{5}\theta} + 15e^{i\frac{1}{5}\theta} - 6\right)}{\left(\frac{95}{3991}e^{i\frac{1}{5}\theta} + \frac{326}{8683}e^{i\frac{2}{5}\theta} + \frac{179}{3455}z^{i\frac{3}{5}\theta} + \frac{76}{12829}e^{i\frac{4}{5}\theta} - \frac{34}{68001}e^{i\theta} - \frac{8}{105415}\right)}$$

The RAS is shown in the figure below:

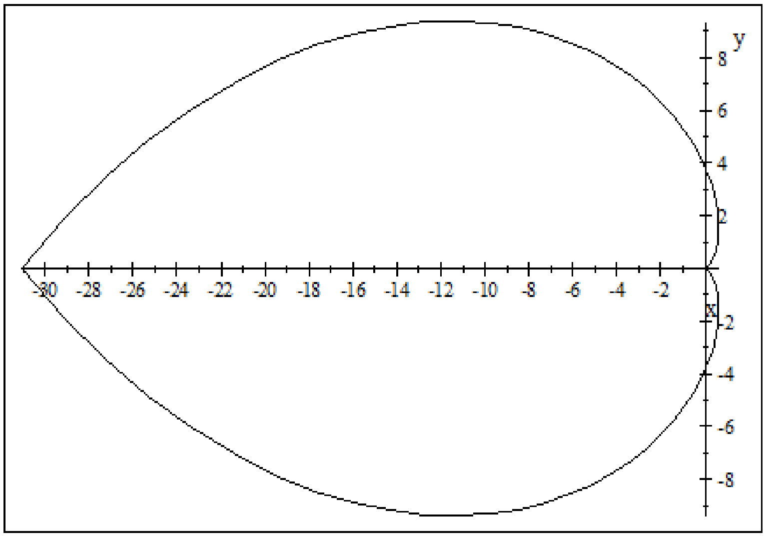

Figure 4: Region of Absolute Stability for One-Step with Off-step Points $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$

7. Application of the Methods

Three problems characterized by different features will be considered.

Problem 1

Constant coefficient homogeneous problem sourced from [17]:

$$y''' + y' = 0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2$$

whose analytic solution is $y(x) = 2(1 - \cos x) + \sin x$
was solved with step size $h = 0.1$.

Problem 2

Blasius Equation, a nonlinear application problem

$$2y''' + yy'' = 0$$

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$$

and sourced from [6] was solved here with $h = 0.1$.

Problem 3

A non-homogeneous problem with constant -coefficient

$$y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2, 0 \leq x \leq 1$$

$$y(0) = -1, y'(0) = 1, y''(0) = -3$$

sourced from [3] the exact solution is
$y(x) = x^2 - e^x + e^x \sin 2x$.
This was solved using step length $h = 0.1$.

Results for Problem 1.

	Exact solution	One-step with $v = \frac{1}{2}, \frac{2}{3}$	One-step with $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$
0.1	0.109825086090778	0.109825086080846	0.109825086130422
0.2	0.238536175112581	0.238536175046740	0.238529544748088
0.3	0.384847228410130	0.384847228197318	0.384812851867273
0.4	0.547296354302880	0.547296353801839	0.547196339977173
0.5	0.724260414823453	0.724260413838638	0.724038144715492
0.6	0.913971243575675	0.913971241854273	0.913549805754709
0.7	1.114533312668710	1.114533309898870	1.113814354050060
0.8	1.323942672205190	1.323942668015470	1.322805702527840
0.9	1.540106973086150	1.540106967046310	1.538409144072180
1.0	1.760866373071620	1.760866364694680	1.758442749431920

Results for Problem 2

	Analytical Solution	One-step with $v = \frac{1}{2}$ *and* $\frac{2}{3}$	One-step with $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ *and* 1
0.1	0.0049999551874560	0.004999958335190	0.004999958334025
0.2	0.0199986590802381	0.019998666851503	0.019998666841335
0.3	0.04498987410259470	0.044989879515865	0.044989862893084
0.4	0.0799573773516761	0.079957378107060	079957245467482
0.5	0.1248700476465370	0.124870057805781	0.124869495246206
0.6	0.179677126361217	0.179677141900617	0.179675425766641
0.7	0.2443036129003850	0.244303618122044	0.244299360051512
0.8	0.3186459794646740	0.318646011157456	0.318636849784214
0.9	0.4025686062131340	0.402568623380723	0.402550872052049
1.0	0.4959003376293370	0.495900386919863	0.495868650046730

Results for Problem 3

	Exact solution	One-step with $v = \frac{1}{2}, \frac{2}{3}$	One-step with $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$
0.1	-0.915407473756115	-0.915407471472194	-0.915407476365875
0.2	-0.862573985499430	-0.862573971547757	-0.862593072343530
0.3	-0.841561375114166	-0.841561334324264	-0.841640544951523
0.4	-0.850966529765556	-0.850966443450176	-0.851141944598394
0.5	-0.888343319155557	-0.888343167189773	-0.888627794689754
0.6	-0.950604904717256	-0.950604667310843	-0.950977249163137
0.7	-1.034392853933000	-1.034392513023200	-1.034794231460630
0.8	-1.136403556878910	-1.136403097109680	-1.136739716605010
0.9	-1.253666211231610	-1.253665620518850	-1.253814242400400
1.0	-1.383769999219790	-1.383769268929200	-1.383588221229960

Error for Problem 1

	One-step *with* $v = \frac{1}{2}, \frac{2}{3}$	One-step $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ *and* 1	[6]
0.1	$9.93200000 \times 10^{-12}$	$-8.1000000 \times 10^{-14}$	$1.60880000 \times 10^{-9}$
0.2	$6.5841000 \times 10^{-11}$	$-1.59900000 \times 10^{-12}$	$1.03870000 \times 10^{-8}$
0.3	$2.128120000 \times 10^{-10}$	$8.266651900 \times 10^{-8}$	$2.95720000 \times 10^{-8}$
0.4	$5.010410000 \times 10^{-10}$	$4.94372607 \times 10^{-7}$	$2.31470000 \times 10^{-7}$
0.5	$9.848150000 \times 10^{-10}$	1.6402909×10^{-6}	$4.54200000 \times 10^{-7}$
0.6	$1.72140200 \times 10^{-9}$	$4.076458420 \times 10^{-6}$	$1.47460000 \times 10^{-6}$
0.7	$2.76984000 \times 10^{-9}$	$8.499105670 \times 10^{-6}$	$2.87340000 \times 10^{-6}$
0.8	4.1897200×10^{-9}	$15731706160 \times 10^{-5}$	$4.68260000 \times 10^{-6}$
0.9	$6.03984000 \times 10^{-9}$	$2.6710264150 \times 10^{-5}$	$6.92170000 \times 10^{-6}$
1.0	$8.37694000 \times 10^{-9}$	$4.246740913 \times 10^{-5}$	$9.59740000 \times 10^{-6}$

Error for Problem 2

	One-step *with* $v = \frac{1}{2}, \frac{2}{3}$	One-step $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ *and* 1	Error in [6]
0.1	$3.1477340000 \times 10^{-9}$	$3.146569000 \times 10^{-9}$	4.2730000×10^{-8}
0.2	$7.7712649000 \times 10^{-9}$	$7.76109690 \times 10^{-9}$	1.2075900×10^{-6}
0.3	$5.41327030 \times 10^{-9}$	$1.120951070 \times 10^{-8}$	8.60719×10^{-6}
0.4	$7.55383900 \times 10^{-10}$	$1.3188419410 \times 10^{-7}$	$3.40900400 \times 10^{-5}$
0.5	$1.01592440 \times 10^{-8}$	$5.524003310 \times 10^{-7}$	9.7406800×10^{-5}
0.6	$1.55394000 \times 10^{-8}$	$1.700594576 \times 10^{-6}$	2.2571100×10^{-4}
0.7	5.221659×10^{-9}	$4.252848873 \times 10^{-6}$	4.5145470×10^{-4}
0.8	$3.169278200 \times 10^{-8}$	$9.129680460 \times 10^{-6}$	8.084729×10^{-4}
0.9	1.7167580×10^{-8}	$1.7734161085 \times 10^{-5}$	1.3262207×10^{-3}
1.0	$4.929052600 \times 10^{-8}$	$6.43844100 \times 10^{-5}$	2.0220546×10^{-3}

Error for Problem 3

	One-step with $v = \frac{1}{2}, \frac{2}{3}$	One-step $v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ *and* 1	Error in [8]
0.1	2.283921×10^{-9}	$2.60976001 \times 10^{-9}$	6.408641×10^{7}
0.2	1.3951673×10^{-8}	$1.90868441 \times 10^{-5}$	1.511330×10^{-5}
0.3	4.0789902×10^{-8}	$7.9169837357 \times 10^{-5}$	6.364443×10^{-5}
0.4	8.631538×10^{-8}	$1.75414832838 \times 10^{-5}$	1.675667×10^{-4}
0.5	$1.51965784 \times 10^{-7}$	$2.84475534197 \times 10^{-4}$	3.507709×10^{-4}
0.6	$2.37406413 \times 10^{-7}$	$3.72344445881 \times 10^{-4}$	6.410875×10^{-4}
0.7	3.409098×10^{-7}	$4.0137752763 \times 10^{-4}$	1.071642×10^{-3}
0.8	4.5976923×10^{-7}	$1.4803116879 \times 10^{-4}$	1.682213×10^{-3}
0.9	5.9071276×10^{-7}	$1.8177798983 \times 10^{-4}$	2.520603×10^{-3}
1.0	7.3029059×10^{-7}	$1.8177798983 \times 10^{-4}$	3.644014×10^{-3}

Conclusion: In this work, Orthogonal polynomial with weight function $w(x) = x^2 + x + 1$ in the interval $[-1, 1]$ was constructed and used as the basis function in developing our new schemes for third order initial problems. This was achieved by the process of interpolation and collocation. Different off-grid points were chosen as the hybrid points for the developed schemes.The three application problems which were solved show that our new scheme performed better when compared with the existing ones. The order, zero stability, consistency and the region of absolute stability have been investigated.The tables of results show that the developed schemes can be adopted for solving third order ODEs.

Acknowledgement: The author is grateful to Bingham University Karu, Nigeria for the supports received during the compilation of this work.

Competing interests: The content of the manuscript was approved by the author and agreed that there is no competing interest.

Funding: The Authors received no financial support for the research, authorship, and/or publication of this article.

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