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Development of an Order (k+3) Block-Hybrid Linear Multistep Method for the Direct Solution of General Second Order Initial Value Problems

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Abstract

Block hybrid linear multistep method was proposed to overcome the Dahlquist order barrier for linear multistep methods. We are interested in answering questions relating to the convergence, accuracy and effectiveness of block hybrid method when utilized to solve Initial Value Problems. In this research work, we presented an order (k+3) block hybrid method for the direct solution of initial value problems of ordinary differential equations. The zero stability, consi stency, convergence and the accuracy of the method are improved by collocating and interpolating the power series at finely selected off-grid points. To illustrate the accuracy and efficiency of the proposed method, linear and system of initial value problems are considered and the results obtained are compared with the existing methods in literature.

1. Introduction

In this paper, we consider an approximate solution of general second order initial value problem (IVP) of the form:

$$y^{''}(x) = f(x, y, y^{'}); y(x_0) = y_0, y^{'}x_0 = y_0^{'} \tag{1.1}$$

where f is continuously differentiable on the given interval $[a, b]$ Ordinary Differential Equation is an equation in which the dependent variable is a function of a single independent variable [8]. Equation (1.1) has a wide range of application because many problems that are encountered in sciences, real life, control theory and engineering are modeled into Differential Equations. This is why the numerical solution of (1.1) is of great interest

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to researchers. [13] and only few can be solved analytically. Hence, the need to study numerical methods and their solution [6].
Conventionally, we often reduce (1.1) to system of first order ordinary differential equations and then use appropriate numerical methods such as Euler method to solve the resultant system [9]. The reduction process and the setbacks of this approach has been discussed by numerous author among them is [10].
In order to speed up computation, achieve better accuracy, reduce computational time and eliminate overlapping of solution model, Block methods for approximating the numerical solution of (1.1) has been vastly explored in literature [2]. Block-Hybrid methods were first introduced according to [5] and later by [14], while hybrid methods were initially introduced to overcome zero stability barrier occurred in block methods mentioned by [4]. The method of interpolation and collocation of the power series approximation to generate continuous LMM has been adopted by many scholars [3]. Meanwhile, some scholars such as, [1] proposed a single-step hybrid block method of order five for the direct solution of second order ordinary differential equation. We were motivated to develop an order (k+3) block hybrid method for the direct solution of general second order initial value problems which can solve general second order initial value problem more accurately and efficiently.

2. Derivation of the Method

We consider power series of a single variable as an approximate solution to the general second order initial value problem of the form (1.1) to be

$$y(x) = \sum_{j=0}^{r+s-1} \alpha_j x^j \tag{2.1}$$

where $\alpha_j$are the real unknown parameters to be determined and $r + s$ is the sum of the number of interpolation and number of collocation points.
The first and second derivatives of (2.1) are given as;

$$y^{'}(x) = \sum_{j=1}^{r+s-1} j\alpha_j x^{j-1} \tag{2.2}$$

$$y^{''}(x) = \sum_{j=2}^{r+s-1} j(j-1)\alpha_j x^{j-2} \tag{2.3}$$

The comparison of (2.3) and (1.1) gives rise to below expression

$$f(x, y, y^{'}) = \sum_{j=2}^{r+s-1} j(j-1)\alpha_j x^{j-2} \tag{2.4}$$

Interpolating (2.1) at $x_{n+j}, j = 1, \frac{5}{3}$ and collocating (2.4) at $x_{n+j}, j = 0, \frac{2}{3}, 1, \frac{5}{3},\ 2,\ 3, 4$ give rise to a system of nonlinear equation $Ax = b$ as given below:

(2.5)

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+\frac{5}{3}} & x_{n+\frac{5}{3}}^2 & x_{n+\frac{5}{3}}^3 & x_{n+\frac{5}{3}}^4 & x_{n+\frac{5}{3}}^5 & x_{n+\frac{5}{3}}^6 & x_{n+\frac{5}{3}}^7 & x_{n+\frac{5}{3}}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x_{n+\frac{2}{3}}^2 & 20x_{n+\frac{2}{3}}^3 & 30x_{n+\frac{2}{3}}^4 & 42x_{n+\frac{2}{3}}^5 & 56x_{n+\frac{2}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{5}{3}} & 12x_{n+\frac{5}{3}}^2 & 20x_{n+\frac{5}{3}}^3 & 30x_{n+\frac{5}{3}}^4 & 42x_{n+\frac{5}{3}}^5 & 56x_{n+\frac{5}{3}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 & 56x_{n+4}^6 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ y_{n+\frac{5}{3}} \\ f_n \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}$$

By solving for $\alpha_j,\ j = 018$ in equation (2.5) above using the matrix inversion and then substituting into the proposed formulae from (2.1) gives the continuous formulae;

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x)\, y_{n+j} + h^2 \left(\sum_{j=0}^{k} \beta_j(x)\, f_{n+j} + \beta_v(x)\, f_{n+v} \right) \tag{2.6}$$

where $y(x)$ is the approximate solution of the initial value problem and $v = \frac{2}{3}, \frac{5}{3}$. α_j and β_j are coefficients that are continuously differentiable. Since (2.6) is continuous and differentiable, then α_0 and β_0 are not both zero.

Given the block method which is presented a single r-point multistep method of the form:

$$A^{(0)} Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^i F_{m-i} \tag{2.7}$$

where $Y_m = [y_{n+1}, y_{n+2}, \ldots, y_{n+r}]^T$, $Y_{m-1} = [y_{n-1}, y_{n-2}, \ldots, y_n]^T$, $F_m = [f_{n+1}, f_{n+2}, \ldots, f_{n+k}]^T$, $Y_{m-1} = [f_{n-1}, f_{n-2}, \ldots, f_n]^T$.

After obtaining the coefficients of y_{n+j} and f_{n+j}, i.e. $\alpha_1,\ \alpha_{\frac{5}{3}}$ and $\beta_0, \beta_{\frac{2}{3}}, \beta_1, \beta_{\frac{5}{3}}, \beta_2, \beta_3, \beta_4$ respectively. The parameters obtained are therefore substituted into the continuous scheme as in equation (2.6) and evaluated at non-interpolating points i.e. $x_n, x_{n+\frac{2}{3}}, x_{n+2},\ x_{n+3}, x_{n+4}$ yields the following scheme:

$$\begin{aligned} y_n + \tfrac{3}{2} y_{n+\frac{5}{3}} &= \tfrac{1}{1524096} h^2 \left(45801 f_n + 320180 f_{n+1} - 160293 f_{n+2}\right) + \\ &\tfrac{1}{1524096} h^2 \left(8154 f_{n+3} - 569 f_{n+4} + 654165 f_{n+\frac{2}{3}} + 402642 f_{n+\frac{5}{3}}\right) + \tfrac{5}{2} y_{n+1} \end{aligned} \tag{2.8}$$

$$y_{n+\frac{2}{3}} + \tfrac{1}{2}y_{n+\frac{5}{3}} = \tfrac{1}{68584320}h^2\left(26089f_n + 9132452f_{n+1} - 944013f_{n+2}\right) + \tfrac{1}{68584320}h^2\left(34802f_{n+3} - 2153f_{n+4} - 54675f_{n+\frac{2}{3}} + 3238218f_{n+\frac{5}{3}}\right) + \tfrac{3}{2}y_{n+1} \tag{2.9}$$

$$y_{n+2} - \tfrac{3}{2}y_{n+\frac{5}{3}} = \tfrac{1}{7620480}h^2\left(3465f_n + 317828f_{n+1} - 29757f_{n+2} + 3618f_{n+3}\right) + \tfrac{1}{7620480}h^2\left(-233f_{n+4} - 82179f_{n+\frac{2}{3}} + 1057338f_{n+\frac{5}{3}}\right) - \tfrac{1}{2}y_{n+1} \tag{2.10}$$

$$y_{n+3} - 3y_{n+\frac{5}{3}} = \tfrac{1}{59535}h^2\left(630f_n + 27398f_{n+1} + 60018f_{n+2} + 4833f_{n+3}\right) + \tfrac{1}{59535}h^2\left(-134f_{n+4} - 10818f_{n+\frac{2}{3}} + 2547f_{n+\frac{5}{3}}\right) - 2y_{n+1} \tag{2.11}$$

$$y_{n+4} - \tfrac{9}{2}y_{n+\frac{5}{3}} = \tfrac{-1}{362880}h^2\left(7119f_n + 73612f_{n+1} - 353283f_{n+2} - 415098f_{n+3}\right) + \tfrac{-1}{362880}h^2\left(-21343f_{n+4} - 76365f_{n+\frac{2}{3}} - 484722f_{n+\frac{5}{3}}\right) - \tfrac{7}{2}y_{n+1} \tag{2.12}$$

The continuous scheme in equation (2.6) is differentiated with respect to x to obtain the first derivative which is evaluated at all the points i.e both interpolation points $(x_{n+1}, x_{n+\frac{5}{3}})$ and collocation points $x_n, x_{n+\frac{2}{3}}, x_{n+2}, x_{n+3}, x_{n+4}$ which gives;

$$hy'_n = \tfrac{-1}{3810240}h^2\left[724689f_n - 1348480f_{n+1} - 1100127f_{n+2} + 62856f_{n+3}\right) + \tfrac{-1}{3810240}h^2\left[-4541f_{n+4} + 4407435f_{n+\frac{2}{3}} + 2338488f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.13}$$

$$hy'_{n+\frac{2}{3}} = \tfrac{-1}{34292160}h^2\left[9737f_n + 16336768f_{n+1} - 1127175f_{n+2} + 37192f_{n+3}\right) + \tfrac{-1}{34292160}h^2\left[-2197f_{n+4} + 3367251f_{n+\frac{2}{3}} + 4239864f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.14}$$

$$hy'_{n+1} = \tfrac{-1}{238140}h^2\left[441f_n + 63518f_{n+1} - 11739f_{n+2} + 459f_{n+3}\right) + \tfrac{-1}{238140}h^2\left[-29f_{n+4} - 11169f_{n+\frac{2}{3}} + 37899f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.15}$$

$$hy'_{n+\frac{5}{3}} = \tfrac{1}{2143260}h^2\left[3514f_n + 294770f_{n+1} - 125202f_{n+2} + 4331f_{n+3}\right) + \tfrac{1}{2143260}h^2\left[-266f_{n+4} - 80190f_{n+\frac{2}{3}} + 617463f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.16}$$

$$hy'_{n+2} = \tfrac{1}{3810240}h^2\left[4095f_n + 429184f_{n+1} + 347151f_{n+2} + 2808f_{n+3}\right) + \tfrac{1}{3810240}h^2\left[-211f_{n+4} - 103419f_{n+\frac{2}{3}} + 1860552f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.17}$$

$$hy'_{n+3} = \tfrac{1}{238140}h^2\left[5418f_n + 196546f_{n+1} + 440958f_{n+2} + 77355f_{n+3}\right) + \tfrac{1}{238140}h^2\left[-1402f_{n+4} - 87246f_{n+\frac{2}{3}} - 234729f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.18}$$

$$hy'_{n+4} = \tfrac{-1}{3810240}h^2\left[410193f_n + 10650752f_{n+1} + 9931425f_{n+2} - 6600312f_{n+3}\right) + \tfrac{-1}{3810240}h^2\left[-1065533f_{n+4} - 5670261f_{n+\frac{2}{3}} - 17816904f_{n+\frac{5}{3}}\right] - \tfrac{3}{2}y_{n+1} + \tfrac{3}{2}y_{n+\frac{5}{3}} \tag{2.19}$$

The proposed Block-Hybrid method is given as

$$y_{n+\frac{2}{3}} = \frac{1}{1071630}h^2\left(104083f_n - 335272f_{n+1} - 108318f_{n+2} + 6596f_{n+3}\right) + \frac{1}{1071630}h^2\left(-485f_{n+4} + 365580f_{n+\frac{2}{3}} + 205956f_{n+\frac{5}{3}}\right) + \frac{2}{3}hy_n^{'} + y_n$$

$$y_{n+1} = \frac{1}{282240}h^2\left(45199f_n - 159180f_{n+1} - 51807f_{n+2} + 3146f_{n+3}\right) + \frac{1}{282240}h^2\left(-231f_{n+4} + 205335f_{n+\frac{2}{3}} + 98658f_{n+\frac{5}{3}}\right) + hy_n^{'} + y_n$$

$$y_{n+\frac{5}{3}} = \frac{25}{4572288}h^2\left(52479f_n - 146300f_{n+1} - 68775f_{n+2} + 4050f_{n+3}\right) + \frac{25}{4572288}h^2\left(-295f_{n+4} + 274095f_{n+\frac{2}{3}} + 138762f_{n+\frac{5}{3}}\right) + \frac{5}{3}hy_n^{'} + y_n$$

$$y_{n+2} = \frac{1}{4410}h^2\left(1547f_n - 3864f_{n+1} - 2100f_{n+2} + 124f_{n+3}\right) + \frac{1}{4410}h^2\left(-9f_{n+4} + 8262f_{n+\frac{2}{3}} + 4860f_{n+\frac{5}{3}}\right) + 2hy_n^{'} + y_n$$

(2.20)

$$y_{n+3} = \frac{3}{31360}h^2\left(5761f_n - 8484f_{n+1} + 2583f_{n+2} + 1310f_{n+3} - 57f_{n+4} + 29889f_{n+\frac{2}{3}} + 16038f_{n+\frac{5}{3}}\right) + 3hy_n^{'} + y_n$$

$$y_{n+4} = \frac{8}{2205}h^2\left(196f_n - 504f_{n+1} - 21f_{n+2} + 332f_{n+3} + 15f_{n+4} + 1215f_{n+\frac{2}{3}} + 972f_{n+\frac{5}{3}}\right) + 4hy_n^{'} + y_n$$

$$y_{n+\frac{2}{3}}^{'} = \frac{1}{2143260}h\left(407029f_n - 1779568f_{n+1} - 548373f_{n+2} + 33032f_{n+3} - 2417f_{n+4} + 2268729f_{n+\frac{2}{3}} + 1050408f_{n+\frac{5}{3}}\right) + y_n^{'}$$

$$y_{n+1}^{'} = \frac{1}{141120}h\left(26579f_n - 87584f_{n+1} - 33789f_{n+2} + 2056f_{n+3} - 151f_{n+4} + 169857f_{n+\frac{2}{3}} + 64152f_{n+\frac{5}{3}}\right) + y_n^{'}$$

$$y_{n+\frac{5}{3}}^{'} = \frac{5}{6858432}h\left(263137f_n - 296800f_{n+1} - 476175f_{n+2} + 25400f_{n+3} - 1805f_{n+4} + 1535355f_{n+\frac{2}{3}} + 1237032f_{n+\frac{5}{3}}\right) + y_n^{'}$$

$$y_{n+2}^{'} = \frac{1}{8820}h\left(1687f_n - 2128f_{n+1} - 1743f_{n+2} + 152f_{n+3} - 11f_{n+4} + 9963f_{n+\frac{2}{3}} + 9720f_{n+\frac{5}{3}}\right) + y_n^{'}$$

$$y_{n+3}^{'} = \frac{3}{15680}h\left(1113f_n + 2464f_{n+1} + 8169f_{n+2} + 1784f_{n+3} - 37f_{n+4} + 4131f_{n+\frac{2}{3}} - 1944f_{n+\frac{5}{3}}\right) + y_n^{'}$$

$$y_{n+4}^{'} = \frac{2}{2205}h\left(91f_n - 3472f_{n+1} - 3192f_{n+2} + 1928f_{n+3} + 307f_{n+4} + 2916f_{n+\frac{2}{3}} + 5832f_{n+\frac{5}{3}}\right) + y_n^{'}$$

3. Analysis of the Block

3.1. Order and Error constant of the Block. Let the Linear Difference Operator L defined on the method be given by:

$$L\left[y\left(x\right);h\right]=\sum_{i=0}^{k}\left[\alpha_i\ \ y\left(x+ih\right)-h^2\beta_i\ y^{''}\left(x+ih\right)\right] \tag{3.1}$$

where $y\left(x\right)$ is an arbitrary function that is continuously differentiable many times on closed interval $[a,b]$. Expanding (3.1) using Taylor series about $y(x)$ and if the coefficients of power of h are gathered we have:

$$L[y(x);h]=c_0y(x)+c_1hy^{'}(x)+c_2h^2y^{''}(x)+\cdots+c_qh^qy^q(x)+0(h^{q+1}) \tag{3.2}$$

whose coefficients $c_q\ \forall\ q=0,1,2,\ldots$ are constants and given as:

$$c_0\ =\sum_{i=0}^{k}\alpha_i\ \ =\alpha_0\ +\alpha_1\ \ +\alpha_2\ \ +\ \ldots\cdots+\alpha_k$$

$$c_1\ =\sum_{i=0}^{k}i\alpha_i\ \ =(\alpha_1\ +2\alpha_2\ \ +3\alpha_3\ \ +\cdots+k\alpha_k\ \ )-(\beta_0\ +\beta_1\ +\beta_2\ +\cdots+\beta_k)$$

$$c_2\ =\sum_{i=0}^{k}\frac{1}{2!}i^2\alpha_i\ \ -\sum_{i=0}^{k}\beta_i\ =\left\{\begin{array}{c}\frac{1}{2!}\left(\alpha_1\ +2^2\alpha_2\ \ +3^2\alpha_3\ \ +\cdots+k^2\alpha_k\ \ \right)\\ -(\beta_1\ +2\beta_2\ +3\beta_3+\cdots+k\beta_k)\end{array}\right\}$$

$$c_q\ =\sum_{i=0}^{k}\{\frac{1}{q!}i^q\alpha_i\ \ -\ \frac{1}{(q-2)!}i^{q-2}\beta_i\ \}$$

$$c_q=\left\{\begin{array}{c}\frac{1}{q!}\left(\alpha_1\ +2^q\alpha_2\ \ +3^q\alpha_3\ \ +\cdots+k^q\alpha_k\ \ \right)\\ -\frac{1}{(q-2)!}(\beta_1\ +2^{(q-2)}\beta_2\ +3^{(q-2)}\beta_3+\cdots+k^{(q-2)}\beta_k)\end{array}\right\} \tag{3.3}$$

Thus (3.1) is said to be order p if and only if $c_0=c_1=c_2=\ldots c_{p+1}=0$ and $c_{p+2}\neq 0$. c_{p+2} is called the error constant. It implies that the local truncation error is given as $T_{n+k}=c_{p+2}h^{p+2}y^{p+2}\left(x\right)+0\left(h^{p+3}\right)$.

Comparing the coefficients of h, the order of the block is $p=7$ with the error constants

$$C_{p+2}=\left[-\frac{50473}{16665989760},-\frac{10369}{3333197952},-\frac{3340}{26040609},\frac{432493}{793618560}\right]^T$$

3.2. Consistency. A linear Multistep method is said to be consistent if the order $p\geq 1$ and obeys the following axioms;

(1) $\sum_{i=0}^{k}\alpha_i=0$
(2) $\rho(r)=\rho^{'}(r)=0$
(3) $\rho^{''}(r)=2!\sigma(r)$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomial of our method respectively.

According to [12], the sufficient condition for associated block method to be consistent is that $p \geq 1$. Since the proposed method is of order $p = 7$. Hence the proposed method is consistent.

3.3. **Zero Stability.** Given block method as a single block r-point multistep method of the form:

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^i F_{m-i} \tag{3.4}$$

Applying the block in equation (2.20) we have:

$$\left|\det[?I - A_1^{(1)}]\right| = \left|\begin{bmatrix} \Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Omega^5\,(\Omega-1) = 0 \quad \rightarrow \quad ?_1 = 0,\ ?_2 = 0, ?_3 = 0,\ ?_4 = 0, ?_5 = 0, ?_6 = 1$$

Since no root has modulus greater than one and $|\Omega| = 1$ is simple. This implies zero-stability, That is the Block Hybrid Method derived is zero stable.

3.4. **Convergence.** According to Fatunla 1973, the necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero stable. Hence the proposed method is convergent.

4. Implementation of method

The performance of the method is tested on some linear problem, real life problem and system of equations of second order initial value problems. The absolute error of the approximate solutions is therefore compared with the existing methods. Specifically, we compared the proposed method with the method of [12], [11] and Abhulimen and Aigbiremhon (2018).

4.1. Numerical problems.

4.1.1. *Cooling of a Body.* The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$, or the otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6In\ 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute.

Formulation of the Problem

$$y'' = \frac{-y'}{3},\ y(0) = 60,\ y'(0) = \frac{-80}{9},\ h = 0.1 \tag{4.1}$$

Exact Solution

$$y(x) = \frac{80}{3}e^{-\frac{1}{3}x} + \frac{100}{3}$$

4.1.2. *System of equations.* Consider the Stiefel and Bettis Problem

$$y_1'' + y_1 = 0.001\cos x,\quad y_1(0) = 1,\quad y_1'(0) = 0 \quad h = \frac{1}{320}$$

.

$$y_2'' + y_2 = 0.001\sin(x)\ ,\ y_2(0) = 1,\quad y_2'(0) = 0.9995 \tag{4.2}$$

Exact solutions are given as;

$$y_1(x) = \cos(x)\ + 0.0005\,(x)\sin(x)\ ,$$
$$y_2(x) = \sin(x)\ - 0.0005\,(x)\cos(x)\ .$$

Table 1. The result of test problem 1 (Real-life Problem)

X	Exact-solution	Computed-solution	Error in our proposed method	Error in [12]
0	60	60	0	0
0.1	59.125762679520157388	59.125762679520157532	1.44E-16	3.55E-11
0.2	58.280186267509806339	58.280186267509806686	3.47E-16	4.58E-11
0.3	57.462331147625588618	57.462331147625589314	6.96E-16	7.00E-11
0.4	56.671288507811932107	56.671288507811932127	2.00E-17	6.50E-12
0.5	55.906179330416375308	55.906179330416372921	2.39E-15	3.33E-11
0.6	55.166153415412849564	55.166153415412844904	4.66E-15	4.20E-11
0.7	54.450388435647511050	54.450388435647504326	6.72E-15	4.38E-11
0.8	53.758089023057298472	53.758089023057288864	9.61E-15	1.07E-10
0.9	53.088485884845809762	53.088485884845795829	1.39E-14	6.58E-11
1.0	52.440834948634380011	52.440834948634361944	1.80E-14	1.69E-10

Table 2a. Shown the results for problem 4.1.2

X	y_1- Exact-Solution	y_1-Approximate-Solution	Error in the Proposed Method	Error in [11]
0	1	1	0	0
0.003125	0.99999512207427819441	0.99999512207427819441	5.66851E-22	1.64E-18
0.006250	0.99998048834470104865	0.99998048834470104865	9.22820E-22	2.87E-18
0.009375	0.99995609895403291149	0.99995609895403291149	2.05284E-22	1.26E-18
0.012500	0.99992195414021281668	0.99992195414021281668	3.01769E-21	5.73E-18
0.015625	0.99987805423635216164	0.99987805423635216164	2.63888E-21	4.10E-18
0.018750	0.99982439967073145770	0.99982439967073145770	1.88479E-22	8.60E-18
0.021875	0.99976099096679615186	0.99976099096679615186	4.59462E-21	6.97E-18
0.025000	0.99968782874315152015	0.99968782874315152015	3.80667E-21	1.14E-17
0.028125	0.99960491371355663261	0.99960491371355663261	3.92052E-21	9.83E-18
0.031250	0.99951224668691738996	0.99951224668691738996	1.20963E-21	1.43E-17

Table 2b. Shown the results for problem 4.1.2

X	y_2-Exact-Solution	y_2-Approximate-Solution	Error in the Proposed Method	Error in [11]
0	0	0	0	0
0.003125	0.00312343242136885101	0.0031234324213688510154	1.029194E-23	7.20E-21
0.006250	0.00624683437101026369	0.0062468343710102636872	2.885345E-23	2.10E-21
0.009375	0.00937017537749407687	0.0093701753774940768711	4.672538E-23	4.33E-20
0.012500	0.01249342496998468092	0.012493424969984680920	3.866963E-22	6.30E-20
0.015625	0.01561655267853828619	0.015616552678538286185	3.006767E-22	1.09E-19
0.018750	0.01873952803440182810	0.018739528034400182811	4.670196E-22	1.15E-19
0.021875	0.02186232057030198893	0.021862320570301988933	1.492241E-22	1.85E-19
0.025000	0.02498489982075888438	0.024984899820758884380	1.541913E-22	1.81E-19
0.028125	0.02810723532236682696	0.028107235322366826964	1.336198E-22	2.79E-19
0.031250	0.0312292966140997484	0.031229296614099748484	1.882721E-22	2.61E-19

Discussion of Results. The results of the proposed method with step number four and order of accuracy seven were compared with other methods. The accuracy of the method developed was tested with two test problems and their corresponding results are discussed below;

Table 1 shows the exact solution, approximate solution, error of proposed scheme and error of [12]. The proposed method is more accurate than that of [12].

From Table 2a It was observed that the maximum absolute error of the proposed method is 9.22820E-22 which is (smaller) and more accurate than 1.64E-18 of [11]. The proposed method performed better than [11]. Also, the accuracy comparison in table 2b shows that the proposed method is substantially more accurate than that of [11].

Conclusion: We explored an approach for solving second order ordinary differential equations by proposing an accurate implicit Block-Hybrid method that yields approximate solutions at suitable points when applied to solve Initial Value Problems (IVPs). The method is consistent, convergent ad zero stable. The proposed method performed efficiently when applied to solve second order Initial Value Problems as can be seen in the low error constant and hence better approximation when compared with the existing methods as can be seen in table 1-2.

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