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The Stability of Solution of a Multifactor Capital Asset Management Model

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Abstract

This work focuses on the presentation of a multifactor asset management model based on the uncertainty theory, and the stability of the multifactor model solution to the formulated state design of a real-life situation of management of capital assets. The model is designed as an optimization problem based on the assumption of the Hyperbolic Absolute Risk Aversion utility function. The formulated model is expanded from a single factor investment model to a model based on the extension to the multifactor uncertain differential equations. The stability in measure and stability in mean of the formulated model is examined whereby the multifactor model is characterized to be stable.

1. Introduction

Liu founded uncertainty theory and since 2007 [5], it has become an essential tool to be utilised while dealing with some mathematical formulations of real-life situations involving a level of uncertainty. This has been applied to different fields such as engineering, economics, finance, biology, etc. The choice of uncertainty theory over the conventional probability theory exists when there exists a small sample size to estimate a probability distribution, thus degree beliefs which are mostly based on recommendations from experts are ascertained to work instead of frequency since human beings always over-weigh unlikely events.
The theorems of existence and uniqueness of the solution of the uncertain differential equation were proved by [1]. [7] established the stability in measure of the uncertain differential equation while [13] proved some stability theorems of the uncertain differential equation. [15] presented and proved the stability theorems of multifactor uncertain systems or differential equations while providing the relationship between stability in mean

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and stability in measure. Other researchers have also worked on different approaches to the stability of the uncertain differential equations and problems formulated using the uncertain differential equations. These include the works of [11], [14], [9], [12] and so on. In this paper, a multifactor model of asset management formulated based on the uncertain theory is presented. Thus, following [15], the stability of the solution to the multifactor model of asset management is discussed by presenting and proving some stability theorems.

1.1. **Preliminaries.** For easy interpretation of the uncertainty theory formulated and refined by [8], some concepts and assumptions are given below.
Let Γ be a nonempty set and L be a σ- algebra over Γ such that (Γ, L) is a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 1.1. [5]: A set of function M defined on the σ-algebra over L is called an uncertain measure if it satisfies the following axioms:
Axiom 1 *(Normality Axiom)*: $M\{\Lambda\} = 1$ *for the universal set* Γ.
Axiom 2 *(Duality Axiom)*: $M\{\Lambda\} + M\{\Lambda^c\} = 1$ *for any event* Λ.
Axiom 3 *(Subadditivity Axiom)*: For every countable sequence of events, $\Lambda_1, \Lambda_2, \ldots,$ we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{1.1}$$

Axiom 4 *(Product Axiom)*: Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \ldots$
The product uncertain measure M is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \mathtt{min}_{1\leq k\leq\infty} M_k\{\Lambda_k\} \tag{1.2}$$

where $\Lambda_k (k = 1, 2, ..)$ are arbitrarily chosen events from L_k for $k = 1, 2, \cdots$
Definition 1.2. [6]: Let (Γ, L, M) be an uncertainty space and let T be a totally ordered set. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $X_t \in B$ is an event for any Borel set B of real numbers at each time t.
Definition 1.3. [12]: An uncertain process C_σ is said to be a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_σ has stationary and independent increments,
(iii) every increment $C_{t+\sigma} - C_t$ is a normal uncertain variable with expected value 0 and variance σ^2. The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \mathtt{exp}\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right]^{-1}, \quad x \in \mathbb{R} \tag{1.3}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi}\mathtt{ln}\frac{y}{1-y}, \quad y \in \mathbb{R} \tag{1.4}$$

Definition 1.4. [5]: Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx \tag{1.5}$$

provided that at least one of the two integrals is finite

Definition 1.5. [6]: An uncertain process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where $t_1, t_2, \cdots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$.

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. It is noted that the increments are also independent of the initial state.

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Definition 1.6. [6]: Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{1.6}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (1.3) and (1.4) identically in t.

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Definition 1.7. [6]: Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$is called a sample path of X_t.

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Definition 1.8. [8]: An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t.

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Definition 1.9. Uncertainty Distribution of Solution. [12] Let α be a number such that $0 < \alpha < 1$. An uncertain differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dC(t) \tag{1.7}$$

is said to have an α-path $X(t)^\alpha$ if it solves the corresponding ordinary differential equation

$$dX(t)^\alpha = f(t, X(t)^\alpha)dt + |g(t, X(t))|\Phi^{-1}(\alpha)dt \tag{1.8}$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of a standard normal uncertain variable, that is,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\texttt{ln}\frac{\alpha}{1-\alpha}, \quad \alpha \in \mathbb{R} \tag{1.9}$$

Definition 1.10. [4] Suppose $C_{1t}, C_{2t}, ..., C_{nt}$ are independent canonical processes, and $f, g_1, g_2, ..., g_n$ are some given functions. Then

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} g_i(t, X_t)dC_{it} \tag{1.10}$$

is called an uncertain differential equation with respect to $C_{1t}, C_{2t}, ..., C_{nt}$. A solution is an uncertain process X_t that satisfies (1.10) identically in t.

The uncertain differential equation (1.10) is equivalent to the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)dt + \sum_{i=1}^{n} \int_0^s g_i(t, X_t)dC_{it} \tag{1.11}$$

Definition 1.11. [15] A multifactor uncertain differential equation (1.10) is said to be stable in measure if for any two solutions X_t and Y_t with different initial values X_0 and Y_0, we have

$$\lim_{|X_0-Y_0|\to 0} M\left\{\sup_{t\geq 0} \mid X_t - Y_t \mid\leq \epsilon\right\} = 1 \tag{1.12}$$

for any given number $\epsilon > 0$.

Definition 1.12. [15] A multifactor uncertain differential equation (1.10) is said to be stable in mean if for any two solutions X_t and Y_t with different initial values X_0 and Y_0, we have

$$\lim_{|X_0-Y_0|\to 0} E\left[\sup_{t\geq 0} \mid X_t - Y_t \mid\right] = 0 \tag{1.13}$$

Theorem 1.1. [13]*: Let C_t be a canonical process. Then, there exists an uncertain variable K such that for each γ, $K(\gamma)$ is a Lipschitz constant of the sample path $C_t(\gamma)$,*

$$lim_{x\to+\infty} M\{K \leq x\} = 1 \tag{1.14}$$

and

$$M\{\gamma \in \Gamma \mid K(\gamma) \leq x\} \geq 2\left(1 + exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1} - 1 \tag{1.15}$$

2. The asset management model

Asset management problem is mainly based on decision making and the understanding of probable asset degradation and trading-off capital investments, maintenance costs, risks and other uncertainties to optimize decisions made by investors.
However, it is assumed that an investor invests her wealth in an asset (say, capital asset) $A(t)$ at a time t, within the period t_0 to t_n. Assuming she starts with an initial net worth $X_0(t)$. At a time t, what ratio of her net worth, ψ, must she choose to utilize on the asset and what ratio of her net worth, τ, must she choose to be incurred on the liability of the investment to maximize the expected present value of the utility of asset, $J(X)$?
The parameters used in the formulation of the model are presented in Table 1.
An optimal control model of the expected present value of an asset over a given period of time, based on uncertainty theory is presented following the study of portfolio selection by [10]. We assume that the capital asset manager aims to decide the optimal utilization and asset allocation for maximizing a value function which discounts exponentially future uncertain values of the Hyperbolic Absolute Risk Aversion (HARA) utility function over a given time horizon with the net worth of capital assets as the state variable.
Let the risky asset earn an uncertain return, an uncertain gain with the mean rate of return and capital gain. The change in liability is calculated as the sum of liability service with an assumption of uncertainty, consumptions, investment and net foreign supply, less taxation, depreciation and revenue over a period of time as proposed by [3] to have:

$$J(X) = \mathtt{max}_{\psi} E_C\left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t}(\psi X(t))^{\lambda} dt\right] \tag{2.1}$$

subject to

$$dX(t) = [(\kappa+\beta)\psi-(\omega(\psi-1)+\mu+h-j-g)]X(t)dt+[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]X(t)dC(t) \tag{2.2}$$

TABLE 1. Definition of Parameters to the model.

Parameter	Description
$X(t)$	Net worth at time t (state variable)
$\tau(t)$	Liability ratio (control) at time t, $\tau \in \mathbb{R}$
$\sigma_r(t)$	Diffusion volatility of liability (with variance σ_r^2 per unit time)
$\psi(t)$	Capital asset ratio at time t (control) $\psi \in \mathbb{R}$
$\sigma_b(t)$	Diffusion volatility of asset (with variance σ_b^2 per unit time)
$\kappa(t)$	Capital gain on asset due to inflation at time t
$\sigma_p(t)$	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
$\beta(t)$	Mean rate of return on the asset at time t
$\omega(t)$	Mean interest rate of liability at time t
$C(t)$	Liu canonical process at time t
$\mu(t)$	Consumption level at time t
$j(t)$	Tax ratio at time t
$g(t)$	Depreciation ratio at time t
$h(t)$	Asset supplies ratio at time t
η	subjective discount rate, e.g., $\frac{A}{\eta+1} = \texttt{Present value}$
λ	degree of relative risk, where $(1-\lambda)$ is the risk aversion
U	Utility function

2.1. **Optimality of the Solution.** The following principles are used in deriving the optimality of the formulated model.

Definition 2.1. (Principle of Optimality) [16]: For any $(t, x) \in [0, T) \times \mathbb{R}$ and $\Delta t > 0$ with $t + \Delta t < T$, we have

$$J(t, x) = \texttt{sup}_D E\left[\int_t^{t+\Delta t} f(X_s, D, S)ds + J(t + \Delta t, x + \Delta X_t)\right] \tag{2.3}$$

where $x + \Delta X_t = X_{t+\Delta t}$

.

Theorem 2.1. *(Equation of Optimality [16]) Let $J(t, x)$ be twice differentiable on $[0, T) \times \mathbb{R}$, Then we have*

$$-J_t(t, x) = \textit{sup}_D\left[f(x, D, t)) + J_x(t, x)V(x, D)\right] \tag{2.4}$$

where $J_t(t, x)$ and $J_x(t, x)$ are the partial derivatives of the function $J(t, x)$ in t and x respectively.

2.2. **Optimal control of the model.** The equation of optimality presented above is applied herein to the uncertain control problem to obtain the optimal controls analytically. Applying equation (2.4), we obtain Let

$$H = \frac{1}{\lambda}e^{-\eta t}(\psi X)^\lambda - \psi(\kappa + \beta)XJ_X + (\mu + j + g + h - (\psi - 1)\omega)XJ_X \tag{2.5}$$

$$-J_t = \texttt{max}_\psi H \tag{2.6}$$

Thus,

$$\frac{\partial H}{\partial \psi} = 0 \tag{2.7}$$

(condition the optimal ψ satisfies)

$$\frac{\partial H}{\partial \psi} = e^{-\eta t}(\psi X)^{\lambda-1}X - (\kappa+\beta-\omega)XJ_x = 0 \tag{2.8}$$

$$(\psi X)^{\lambda-1}X = (\omega-\kappa-\beta)J_x e^{\eta t} \tag{2.9}$$

$$(\psi X) = \left[(\omega-\kappa-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}} \tag{2.10}$$

$$\psi = \frac{1}{X}\left[(\omega-\kappa-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}} \tag{2.11}$$

Hence, by solving the above equations, we obtained the optimal ratio of the net worth in capital assets as

$$\psi^* = \frac{(\mu+j+g+\omega-h)\lambda-\eta}{(1-\lambda)(\kappa+\beta-\omega)} \tag{2.12}$$

However, the optimal liability ratio, τ^* can also be obtained as a control to the system. Since $\tau = \psi - 1$

$$\tau^* = \left[\frac{(\mu+j+g+\omega-h)\lambda-\eta}{(1-\lambda)(\kappa+\beta-\omega)}\right] - 1 \tag{2.13}$$

or

$$\tau^* = \frac{(\mu+j+g-h)\lambda-(1-\lambda)(\kappa+\beta)+\omega-\eta}{(1-\lambda)(\kappa+\beta-\omega)} \tag{2.14}$$

2.3. **Solution to the Model.** Here, the analytical and numerical solutions are derived for the single-factored problem.

For the analytic solution, the required problem under consideration is

$$J(\psi) = \mathtt{min}_{\psi} E_C\left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t}(\psi X(t))^{\lambda} dt\right] \tag{2.15}$$

subject to

(2.16)

$$dX(t) = [(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]X(t)dt + [\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi-1)]X(t)dC(t)$$

with α-path equation

(2.17)

$$dX(t)^{\alpha} = [(\kappa+\beta)\psi-(\omega(\psi-1)+\mu+h-j-g)]X(t)^{\alpha}dt + |[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]X(t)^{\alpha}|\Phi^{-1}(\alpha)dt.$$

The analytical solution to the constraint is

(2.18)

$$X(t) = X_0\mathtt{exp}\left([(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]t + [\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]C(t)\right)$$

and its inverse uncertainty distribution is

(2.19)

$$\Psi(t)^{-1}(\alpha) = X_0\mathtt{exp}([(\kappa+\beta)\psi-(\omega(\psi-1)+\mu+h-j-g)]t + \frac{[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]t\sqrt{3}}{\pi}\mathtt{ln}\frac{\alpha}{1-\alpha})$$

Hence,

$$\Psi(t)^{-1}(\alpha) = E(X(t)^{\alpha}) \tag{2.20}$$

2.4. **Multifactor Model.** The multifactor model can be expressed as:

$$J(X) = \mathtt{max}_{\psi} E_C \left[\sum_{i=1}^{n} \frac{1}{\lambda} \int_{t_0}^{t_f} e^{-\eta t} \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix}^{\lambda} \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix}^{1-\lambda} dt \right] \tag{2.21}$$

subject to

$$\begin{pmatrix} dX_{1t} \\ dX_{2t} \\ \vdots \\ dX_{nt} \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix} \begin{pmatrix} \kappa_1 + \beta_1 - \omega_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 + \beta_2 - \omega_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \kappa_n + \beta_n - \omega_n \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix} dt$$

$$+ \begin{pmatrix} \mu_1 + h_1 - j_1 - g_1 - \omega_1 & 0 & \cdots & 0 \\ 0 & \mu_2 + h_2 - j_2 - g_2 - \omega_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_n + h_n - j_n - g_n - \omega_n \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix} dt$$

(2.22)

$$+ \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix} \begin{pmatrix} \sigma_{1p} + \sigma_{1b} - \sigma_{1r} + 1 & 0 & \cdots & 0 \\ 0 & \sigma_{2p} + \sigma_{2b} - \sigma_{2r} + 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{np} + \sigma_{nb} - \sigma_{nr} + 1 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix} dC(t)$$

which can be rewritten as presented by [2]

$$J(X) = \mathtt{max}_{\psi} E_C \left[\int_{t_0}^{t_f} \frac{1}{\lambda} e^{-\eta t} (U^{\lambda})^T X^{1-\lambda} dt \right] \tag{2.23}$$

subject to

$$dX = FXdt + UPXdt + UQXdC(t) \tag{2.24}$$

where

$$X = \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix}, \tag{2.25}$$

$$U = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix}, \tag{2.26}$$

$$F = \begin{pmatrix} \mu_1 + h_1 - j_1 - g_1 - \omega_1 & 0 & \cdots & 0 \\ 0 & \mu_2 + h_2 - j_2 - g_2 - \omega_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_n + h_n - j_n - g_n - \omega_m \end{pmatrix}, \tag{2.27}$$

$$P = \begin{pmatrix} \kappa_1 + \beta_1 - \omega_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 + \beta_2 - \omega_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \kappa_n + \beta_n - \omega_n \end{pmatrix} \text{ and} \tag{2.28}$$

$$Q = \begin{pmatrix} \sigma_{1p} + \sigma_{1b} - \sigma_{1r} + 1 & 0 & \cdots & 0 \\ 0 & \sigma_{2p} + \sigma_{2b} - \sigma_{2r} + 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{np} + \sigma_{nb} - \sigma_{nr} + 1 \end{pmatrix}. \tag{2.29}$$

Equations (2.23) and (2.24) which is the model of risky capital assets is an uncertain optimal control system whereby $X(t)$ is the state, U is the control, F is $n \times n$ dimensional constant matrix while P and Q are $n \times m$ dimensional constant matrices, $C(t)$ is the Liu process and $J(X)$ is the objective functional.

3. Stability of the model

Here, we examine the stability in measure and stability in measure of the model as proposed in [15].

Theorem 3.1. *Suppose the multifactor system (2.24) has a unique solution for each initial value, then it is stable in measure if the coefficients F, U, P and Q satisfy the strong Lipschitz condition*

$$\begin{aligned} (3.1)\quad & | F(t,x) - F(t,y) | + | U(t,x)P(t,x) - U(t,y)P(t,y) | + | U(t,x)Q(t,x) \\ & \qquad - U(t,y)Q(t,y) | \leq L | x - y | \text{ for all } x, y \in \mathbb{R},\ t \geq 0 \end{aligned}$$

where L_t is some positive function satisfying

$$\int_0^{+\infty} L_t dt < +\infty \tag{3.2}$$

Proof. Assume X_t and Y_t to be the solutions of the multifactor system (2.24) with different initial values X_0 and Y_0 respectively. Then for Lipschitz continuous sample paths $C_t(\gamma)$, then we have

$$\begin{aligned} X_t(\gamma) = X_0 + \int_0^t F(s, X_s(\gamma))ds + \int_0^t U(s, X_s(\gamma))P(s, X_s(\gamma))ds \\ + \int_0^t U(s, X_s(\gamma))Q(s, X_s(\gamma))dC_s(\gamma) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} Y_t(\gamma) = Y_0 + \int_0^t F(s, Y_s(\gamma))ds + \int_0^t U(s, Y_s(\gamma))P(s, Y_s(\gamma))ds \\ + \int_0^t U(s, Y_s(\gamma))Q(s, Y_s(\gamma))dC_s(\gamma) \end{aligned} \tag{3.4}$$

Following the strong Lipschitz condition, we obtain

$$| X_t(\gamma) - Y_t(\gamma) |$$

$$\begin{aligned} \leq & | X_0 - Y_0 | + \int_0^t | F(s, X_s(\gamma)) - F(s, Y_s(\gamma)) | ds \\ & + \int_0^t | U(s, X_s(\gamma))P(s, X_s(\gamma)) - U(s, Y_s(\gamma))P(s, Y_s(\gamma)) | ds \\ & + \int_0^t | U(s, X_s(\gamma))Q(s, X_s(\gamma)) - U(s, Y_s(\gamma))Q(s, Y_s(\gamma)) | dC_s(\gamma) \end{aligned}$$

$$\leq\mid X_0 - Y_0 \mid + \int_0^t L_s \mid X_s(\gamma) - Y_s(\gamma) \mid ds + \int_0^t L_s \mid X_s(\gamma)X_s(\gamma) - Y_s(\gamma)Y_s(\gamma) \mid ds$$
$$+ \int_0^t K(\gamma) L_s \mid X_s(\gamma)X_s(\gamma) - Y_s(\gamma)Y_s(\gamma) \mid ds$$

$$\leq\mid X_0 - Y_0 \mid + \int_0^t L_s \mid X_s(\gamma) - Y_s(\gamma) \mid ds + \int_0^t K(\gamma) L_s \mid X_s(\gamma) - Y_s(\gamma) \mid ds$$

$$=\mid X_0 - Y_0 \mid + (1 + K(\gamma)) \int_0^t L_s \mid X_s(\gamma) - Y_s(\gamma) \mid ds \tag{3.5}$$

where $K(\gamma)$ is the Lipschitz constant od $C_t(\gamma)$.
Following the Gronwall's inequality, we have

$$\mid X_t(\gamma) - Y_t(\gamma) \mid$$
$$\leq\mid X_0 - Y_0 \mid \exp\left((1 + K(\gamma)) \int_0^t L_s ds\right)$$

For any $t \geq 0$,

$$\leq\mid X_0 - Y_0 \mid \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right).$$

Thus,

$$\sup \mid X_t - Y_t \mid\leq\mid X_0 - Y_0 \mid \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right).$$

Let $K(\gamma)$ be a nonnegative uncertain variable. Then, by Theorem 1.1, we have

$$\lim_{x\to\infty} M\{\gamma \in \Gamma \mid K(\gamma) \leq x\} = 1. \tag{3.6}$$

Thus, there exists a real number T such that

$$M\{\gamma \mid K(\gamma) \leq T\} \geq 1 - \epsilon \tag{3.7}$$

for each $\epsilon > 0$.
Taking

$$\delta = \exp\left(-(1 + T) \int_0^{+\infty} L_s ds\right) \epsilon \tag{3.8}$$

we have $\mid X_t(\gamma) - Y_t(\gamma) \mid\leq \epsilon$, $\forall t \geq 0$ where $\mid X_0 - Y_0 \mid\geq \delta$ and $K(\gamma) \leq T$.
If $\mid X_0 - Y_0 \mid\leq \delta$, we have

$$M\left\{\sup_{t\geq 0} \mid X_t - Y_t \mid\leq \epsilon\right\} > 1 - \epsilon \tag{3.9}$$

which implies

$$\lim_{\mid X_0 - Y_0\mid\to 0} M\left\{\sup_{t\geq 0} \mid X_t - Y_t \mid\leq \epsilon\right\} = 1. \tag{3.10}$$

Hence, the system (2.24) is stable in measure. □

Remark. Theorem 3.1 shows the sufficient condition for multifactor model being stable in measure.

Theorem 3.2. *The multifactor system (2.24) is said to be stable in mean if the coefficients F, U, P and Q satisfy the strong Lipschitz condition*

$$\begin{aligned} &\mid F(t,x) - F(t,y) \mid \leq L_{1t} \mid x-y \mid \\ &\mid U(t,x)P(t,x) - U(t,y)P(t,y) \mid \leq L_{2t} \mid x-y \mid \\ &\mid Q(t,x) - Q(t,y) \mid \leq L_{3t} \mid x-y \mid, \ \forall x,y \in \mathbb{R}, t \geq 0 \end{aligned} \tag{3.11}$$

where L_{1t}, L_{2t} and L_{3t} are functions satisfying

$$\int_0^{+\infty} L_{1t}dt < +\infty, \quad \int_0^{+\infty} L_{2t}dt < +\infty, \quad \int_0^{+\infty} L_{3t}dt < \frac{\pi}{\sqrt{3}}. \tag{3.12}$$

Proof. Suppose X_t and Y_t are the solutions of the multifactor system (2.24) with corresponding initial values of X_0 and Y_0. Then, we have the following for the Lipschitz continuous sample paths $C_t(\gamma)$.

$$X_t(\gamma) = X_0 + \int_0^t F(s, X_s(\gamma))ds + \int_0^t U(s, X_s(\gamma))P(s, X_s(\gamma))ds + \int_0^t Q(s, X_s(\gamma))dC_s(\gamma) \tag{3.13}$$

and

$$Y_t(\gamma) = Y_0 + \int_0^t F(s, Y_s(\gamma))ds + \int_0^t U(s, Y_s(\gamma))P(s, Y_s(\gamma))ds + \int_0^t Q(s, Y_s(\gamma))dC_s(\gamma). \tag{3.14}$$

By utilising strong the Lipschitz condition, we have

$$\begin{aligned} &\mid X_t(\gamma) - Y_t(\gamma) \mid \\ &\leq \mid X_0 - Y_0 \mid + \int_0^t \mid F(s, X_s(\gamma)) - F(s, Y_s(\gamma)) \mid ds \\ &+ \int_0^t \mid U(s, X_s(\gamma))P(s, X_s(\gamma)) - U(s, Y_s(\gamma))P(s, Y_s(\gamma)) \mid ds \\ &+ \int_0^t \mid Q(s, X_s(\gamma)) - Q(s, Y_s(\gamma)) \mid\mid dC_s(\gamma) \mid \\ &\leq \mid X_0 - Y_0 \mid + \int_0^t L_{1s} \mid X_s(\gamma) - Y_s(\gamma) \mid ds \\ &+ \int_0^t L_{2s} \mid X_s(\gamma)X_s(\gamma) - Y_s(\gamma)Y_s(\gamma) \mid ds \\ &+ \int_0^t K(\gamma)L_{3s} \mid X_s(\gamma) - Y_s(\gamma) \mid ds \end{aligned} \tag{3.15}$$

$$\text{(3.16)} \quad \leq \mid X_0 - Y_0 \mid + \int_0^t L_{1s} \mid X_s(\gamma) - Y_s(\gamma) \mid ds + \int_0^t L_{2s} \mid X_s(\gamma) - Y_s(\gamma) \mid ds + \int_0^t K(\gamma) L_{3s} \mid X_s(\gamma) - Y_s(\gamma) \mid ds$$

where $K(\gamma)$ is the Lipschitz constants of $C_t(\gamma)$.
Following the Gronwall's inequality, we have

(3.17)

$$\mid X_t(\gamma) - Y_t(\gamma) \mid \leq \mid X_0 - Y_0 \mid \exp\left(\int_0^t L_{1s} ds\right) \exp\left(\int_0^t L_{2s} ds\right) \exp\left(K(\gamma) \int_0^t L_{3s} ds\right)$$
$$\leq \mid X_0 - Y_0 \mid \exp\left(\int_0^{+\infty} L_{1s} ds\right) \exp\left(\int_0^{+\infty} L_{2s} ds\right) \exp\left(K(\gamma) \int_0^{+\infty} L_{3s} ds\right), \forall t \geq 0.$$

Thus,

$$\text{(3.18)} \quad \sup_{t \geq 0} \mid X_t - Y_t \mid \leq \mid X_0 - Y_0 \mid \exp\left(\int_0^{+\infty} L_{1s} ds\right) \exp\left(\int_0^{+\infty} L_{2s} ds\right) \exp\left(K \int_0^{+\infty} L_{3s} ds\right)$$

K represents the nonnegative uncertain varible.
Since $\{K(\gamma) \leq x\}$, by Theorem 1.1 and independence of $K(\gamma)$,we have

$$\text{(3.19)} \qquad M\{\gamma \in \Gamma \mid K(\gamma) \leq x\} \geq 2\left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1} - 1$$

By taking the expected value of both sides of (3.18), we derive

$$\text{(3.20)} \quad E\left[\sup_{t \geq 0} \mid X_t - Y_t \mid\right] \leq \mid X_0 - Y_0 \mid \exp\left(\int_0^{+\infty} L_{1s} ds\right) \exp\left(\int_0^{+\infty} L_{2s} ds\right) E\left[\exp\left(K \int_0^{+\infty} L_{3s} ds\right)\right]$$

Observing that

$$\int_0^{+\infty} L_{1s} ds < +\infty,$$
$$\int_0^{+\infty} L_{2s} ds < +\infty,$$

we have

$$\exp\left(\int_0^{+\infty} L_{1s} ds\right) < +\infty,$$
$$\exp\left(\int_0^{+\infty} L_{2s} ds\right) < +\infty$$

and since

$$\int_0^{+\infty} L_{3s} ds < +\frac{\pi}{\sqrt{3}},$$

by definition 1.4, we have

$$E\left[\exp\left(K\int_0^{+\infty} L_{3s}ds\right)\right]$$

$$(3.21) \quad = \int_0^{+\infty} M\left\{\exp\left(K\int_0^{+\infty} L_{3s}ds\right) \geq x\right\} dx$$

$$\leq 1 + \int_1^{+\infty} M\left\{\exp\left(K\int_0^{+\infty} L_{3s}ds\right) \geq x\right\} dx$$

$$= 1 + \left(\int_0^{+\infty} L_{3s}\right)\int_0^{+\infty} \exp\left(y\int_0^{+\infty} L_{3s}ds\right) M\{K \geq y\}dy$$

$$\leq 1 + \left(\int_0^{+\infty} L_{3s}\right)\int_0^{+\infty} \exp\left(y\int_0^{+\infty} L_{3s}ds\right)\left\{1 - \left(2\left(1+\exp\left(\frac{-\pi y}{\sqrt{3}}\right)\right)^{-1} - 1\right)\right\} dy$$

$$= 1 + 2\int_1^{+\infty} \frac{1}{1 + x^{\pi/\left(\sqrt{3}\int_0^{+\infty} L_{3s}ds\right)}} dx < +\infty$$

Hence, the proof of stability in mean for the multifactor system. □

Remark. Theorem 3.2 shows the sufficient condition for the multifactor model being stable in mean. Generally, the stability in measure of the multifactor model does not result to the stability in mean.

Conclusions: In this work, the multifactor capital asset management model formulated based on some uncertain conditions have been presented. The measure and mean stability theorems of the multifactor uncertain system representing the problem formulation were proved. This served as sufficient conditions for the multifactor asset management model to be stable.

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References

[1] Chen X. W. & Liu B. (2010). Existence and uniqueness theorem for uncertain differential equations. *Fuzzy Optimization and Decision Making* **9** (1), 69–81.

[2] Latunde T. (2020). Multifactor modelling in asset management. International Journal of Mathematics in Operational Research. *International Journal of Mathematics in Operational Research (IJMOR)* **17** (3), 333-352.

[3] Latunde T. & Bamigbola O. M. (2016). Uncertain optimal control model for management of the net risky capital asset. *IOSR Journal of Mathematics.* **12** (3), 22-30.

[4] Li S., Peng Y. & Zhang B. (2015). Multifactor uncertain differential equation. *Journal of Uncertainty Analysis and Applications.* **3** (7), 1-19.

[5] Liu B. (2007). *Uncertainty theory (2nd ed.).*, Berlin: Springer.

[6] Liu B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain Systems.* **2** (1), 3-16.

[7] Liu B. (2009). Some research problems in uncertainty theory. *Journal of Uncertain Systems.* **3** (1), 3–10.

[8] Liu B. (2016) *Uncertain theory* (5th ed.), *Uncertainty Theory Laboratory*, China.
[9] Liu H. J., Ke H. & Fei W. Y. (2014). Almost sure stability for uncertain differential equation. *Fuzzy Optimization and Decision Making.* **13** (4), 463–473.
[10] Merton R. C. (1971). Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory.* **42** (3), 373-413.
[11] Sheng Y. H. & Wang C. G. (2014). Stability in the p-th moment for the uncertain differential equation. *Journal of Intelligent and Fuzzy Systems.* **26** (3), 1263–1271.
[12] Yao K. & Chen X. (2013). A numerical method for solving uncertain differential equations. *Journal of Intelligent and Fuzzy Systems.* **25** (3), 825-832.
[13] Yao K., Gao J. & Gao Y. (2013). Some stability theorems of uncertain differential equation. *Fuzzy Optimization and Decision Making.* **12** (1), 3–13.
[14] Yao K., Ke H. & Sheng Y. H. (2015). Stability in mean for uncertain differential equation. *Fuzzy Optimization and Decision Making* **14** (3), 365–379.
[15] Zhang Z., Gao R. & Yang X. (2016). The stability of multifactor uncertain differential equation. *Journal of Intelligent and Fuzzy Systems*, **30**, 3281–3290.
[16] Zhu Y. (2010). Uncertain optimal control with application to a portfolio selection model. . *Cybernetics and Systems: An International Journal.* **41**, 535-547.