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Optimization of two-step method with Bhaskara points for solving non-linear dynamical problems

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Abstract

We derive a two-step method for solving non-linear dynamical problems employing Bhaskara points as off grid points. Deterministic in nature, non-linear dynamical systems display a periodic behavior that is highly dependent on the beginning circumstances, making long-term predictions impossible. Four Bhaskara points are generated as hybrid points to optimized the system. The method is use to solve non-linear dynamical systems to demonstrate the accuracy and efficient of the method. The method is zero stable and consistent. Some examples from literature were considered to demonstrate the efficiency and the accuracy of the method.

1. Introduction

Ordinary differential equations (ODEs) arise science and technology, whose solutions are not only important but also necessary and compulsory for the advancement of science and technology. Obtaining solutions to ODEs is as important as modelling that gives rise to them.

A linear ODE is one that can be split down into components, solved individually, and then recombined to produce the result. Nonlinear interactions take place whenever components of a system interact, compete, or cooperate. This kind of system is deterministic, but its long-term predictions are erroneous because of the periodic behavior it displays in response to the starting circumstances. However, most non-linear dynamical systems do not have analytical solutions, thereby relying on numerical methods. On of such methods

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is the block method. Block methods were introduce by Milne [13]. The block methods proposed by Milne [13] has some draw backs, which led to the introduction of hybrid method. Hybrid methods has the advantage of easy change of step size and making use of data from off-set points that generally enhance the accuracy and effectiveness of the methods. The solution of non-linear dynamical problems is considered in this paper. Motsa [16] considered one-step solution of first order ODEs with equal space interval as hybrid point. Bothayna [10], considered single step method with equally three intra-step points for solving first order ODEs. Many researchers recently have applied different steps with different hybrid points for the development of numerical integrators for first order differential equation [3–7, 14, 20–23]. Recently, there have been discussions on how to generate intra-step points methodically, in order to yield optimal accuracy for block hybrid algorithms. By minimizing the local truncation errors (LTEs), Ramos [17] presented an optimized two-step block hybrid method for solving general first order IVPs. In [18], Ramos et al. expanded the concept of minimizing the LTE to derive optimized two-step block hybrid methods for the numerical solution of general second order IVPs. It is important to note that, in both [17] and [18], only two intra-step points were established. In light of the above we present a more direct method for the solution of first order non-linear ODEs, which will be applicable to special, stiff, non-linear and general forms of first order differential equations. The proposed method is a two-step hybrid method using four generated Bhaskara points as hybrid points to optimize the method. The proposed methods will be time efficient, have wider integration range and economically reliable. The aim of the study is to develop an optimized multi-derivative method with intra-step point for solving non-linear dynamical system and Partial Differential Equations.
We implement an implicit two-step method using Bhaskara points as hybrid points. The Gauss-Lobatto grid points are points generated from the algorithm $\{x\}_{i=0}^{M} = \frac{k}{2}\cos\frac{\pi i}{M} + \frac{k}{2}$, while Bhaskara points are points generated from approximating the cosine function $\{\cos\frac{\pi i}{M}\}_{i=0}^{M} \approx \frac{M^2-4i^2}{M^2+i^2}$ where $M = m+1$ is number of intra-steps and $M \in \mathbf{N} : m \geq 2$, and k is the step size.
We studied the order, zero stability, convergence and consistency of the method. Some numerical problem which are non-linear will be solved and compared to others in literature to show the efficiency and accuracy. The solution of first order initial value problem is of the form.

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \in [a, b] \tag{1.1}$$

The function $f(x, y)$ satisfies the existence and uniqueness theorem and is continuous [8]. The hybrid block technique has recently been extended by a number of authors [3–7, 14, 20–23] to numerically solve first order differential equations with varied numbers of steps and hybrids.

2. Derivation of the Method

To derive the method, the power series of the form below is considered:

$$y(x) = \sum_{j=0}^{I+C_1+C_2} \alpha_j x^j \tag{2.1}$$

where I is the interpolation point, C_1 is the number of collocation points for the first derivative and C_2 is the number of collocation points for the second derivative as an approximate solution to the general first order problems of the form:

$$y' = f(x, y), \quad y(a) = \alpha. \tag{2.2}$$

The first and second derivatives of (2.1) are

$$y'(x) = \sum_{j=1}^{I+C_1+C_2} j\alpha_j x^{j-1} \tag{2.3}$$

$$y''(x) = \sum_{j=2}^{I+C_1+C_2} j(j-1)\alpha_j x^{j-2} \tag{2.4}$$

(2.1) is interpolated at $x = x_{n+1}$ while (2.3) and (2.4) are collocated at

$$x = x_{n+i}, \quad i = 0, 5/26, 20/29, 1, 38/29, 47/26, 2,$$

where (2.3) satisfies (2.2). The points i are the Bhaskara hybrid points that optimized the method, n represent the number of iterations for *-++++ step number of 2.

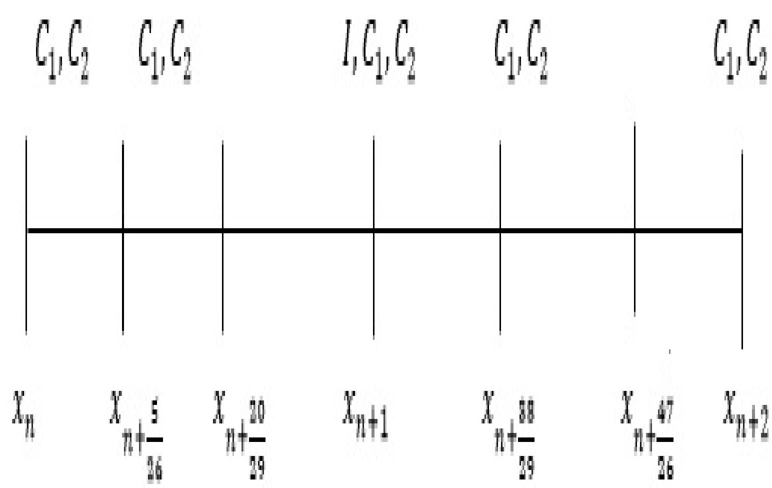

Figure 1: Two-step interpolation and collocation method for first order ordinary differential equations

$$
\begin{bmatrix}
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \dots & x_{n+1}^{13} & x_{n+1}^{14} \\
0 & 1 & 2x_n & 3x_n^2 & \dots & 13x_n^{12} & 14x_n^{13} \\
0 & 1 & 2x_{n+\frac{5}{26}} & 3x_{n+\frac{5}{26}}^2 & \dots & 13x_{n+\frac{5}{26}}^{12} & 14x_{n+\frac{5}{26}}^{13} \\
0 & 1 & 2x_{n+\frac{20}{29}} & 3x_{n+\frac{20}{39}}^2 & \dots & 13x_{n+\frac{20}{39}}^{12} & 14x_{n+\frac{20}{39}}^{13} \\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & \dots & 13x_{n+1}^{12} & 14x_{n+1}^{13} \\
0 & 1 & 2x_{n+\frac{47}{26}} & 3x_{n+\frac{47}{26}}^2 & \dots & 13x_{n+\frac{47}{26}}^{12} & 14x_{n+\frac{47}{26}}^{13} \\
0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & \dots & 13x_{n+2}^{12} & 14x_{n+2}^{13} \\
0 & 0 & 2 & 6x_n & \dots & 156x_n^{11} & 182x_n^{12} \\
0 & 0 & 2 & 6x_{n+\frac{20}{29}} & \dots & 156x_{n+\frac{20}{29}}^{11} & 182x_{n+\frac{20}{29}}^{12} \\
0 & 0 & 2 & 6x_{n+1} & \dots & 156x_{n+1}^{11} & 182x_{n+1}^{12} \\
0 & 0 & 2 & 6x_{n+\frac{38}{29}} & \dots & 156x_{n+\frac{38}{29}}^{11} & 182x_{n+\frac{38}{29}}^{12} \\
0 & 0 & 2 & 6x_{n+\frac{47}{26}} & \dots & 156x_{n+\frac{47}{26}}^{11} & 182x_{n+\frac{47}{26}}^{12} \\
0 & 0 & 2 & 6x_{n+2} & \dots & 156x_{n+2}^{11} & 182x_{n+2}^{12}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \\ \alpha_{15}
\end{bmatrix}
=
\begin{bmatrix}
y_{n+1} \\ f_n \\ f_{n+\frac{5}{26}} \\ f_{n+\frac{20}{29}} \\ f_{n+1} \\ f_{n+\frac{38}{29}} \\ f_{n+\frac{47}{26}} \\ f_{n+2} \\ g_n \\ g_{n+\frac{5}{26}} \\ g_{n+\frac{20}{29}} \\ g_{n+1} \\ g_{n+\frac{38}{29}} \\ g_{n+\frac{47}{26}} \\ g_{n+2}
\end{bmatrix}
\tag{2.5}
$$

The system (6) was solved using the power series method, enhanced by Maple, to obtain the unknown coefficient α_j which are substituted into (1.1) to have the continuous form of the collocation method:

$$
\begin{aligned}
y(x) &= \alpha_0(x)y_{n+1} + h\sum_{j=0}^{k}\gamma_j(x)f_n + h^2\sum_{j=0}^{k}\beta_j(x)g_n \\
&+ \gamma_{\frac{5}{26}}(x)f_{n+\frac{5}{26}} + \gamma_{\frac{20}{29}}(x)f_{n+\frac{20}{29}} + \gamma_{\frac{38}{29}}(x)f_{n+\frac{38}{29}} + \gamma_{\frac{47}{26}}(x)f_{n+\frac{47}{26}} \\
&+ \beta_{\frac{5}{26}}(x)g_{n+\frac{5}{26}} + \beta_{\frac{20}{29}}(x)g_{n+\frac{20}{29}} + \beta_{\frac{38}{29}}(x)g_{n+\frac{38}{29}} + \beta_{\frac{47}{26}}(x)g_{n+\frac{47}{26}}
\end{aligned}
\tag{2.6}
$$

where $h = x_{n+1} - x_n$ and whose derivative is given by

$$
f(x) = \frac{dy}{dx}, \quad g(x) = \frac{d^2y}{dx^2}
\tag{2.7}
$$

where $\alpha_j(x)$, $\gamma_j(x)$ and $\beta_j(x)$ are continuous coefficients that are uniquely determined. The main methods are gotten by evaluating (2.6) at 0, 5/26, 20/29, 38/29, 47/26, 2 to give the following:

$$
\begin{aligned}
y_{n+\frac{5}{26}} = y_{n+1} &- \frac{1340805343721016667265 73\, h^2 g_n}{99064732150286353285120000} \\
&+ \frac{797053372151911\, h^2 g_{n+1}}{7623023165282304} \\
&+ \frac{3419073802024227174624 3\, h^2 g_{n+2}}{99064732150286353285120000} \\
&- \frac{1987403759042034403\, h^2 g_{n+\frac{5}{26}}}{1053831331445750000 00} \\
&+ \frac{166347507223791022202879798139 7\, h^2 g_{n+\frac{20}{29}}}{199185867015246850033152000000 00} \\
&+ \frac{1574473641884535170865428853007\, h^2 g_{n+\frac{38}{29}}}{5975576010457405500994560000000 0} \\
&+ \frac{257675377558397869\, h^2 g_{n+\frac{47}{26}}}{105383133144575000000} \\
&- \frac{9942519746690395448227142 31\, h f_n}{340249253116175821090816000 00} \\
&- \frac{7437477576455479\, h f_{n+1}}{23583727917592128} \\
&- \frac{119544008219271944528449509\, h f_{n+2}}{1769296116204114269672243 2000} \\
&- \frac{116388840770935608536418 1\, h f_{n+\frac{5}{26}}}{669132230500193281250000 0} \\
&- \frac{318336729168460636550734218095948641 01\, h f_{n+\frac{20}{29}}}{358915503598110989660362003200000000000 0} \\
&- \frac{6867800111496626928789321542163036872 3\, h f_{n+\frac{38}{29}}}{358915503598110989660362003200000000000 0} \\
&- \frac{1583802590975465659756 3\, h f_{n+\frac{47}{26}}}{669132230500193281250000 0}
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
y_{n+\frac{20}{29}} = y_{n+1} &- \frac{76943164634576627738300 1\, h^2 g_n}{33066146958818461463261440000} \\
&+ \frac{201236166015324651\, h^2 g_{n+1}}{14141652940152956414} \\
&+ \frac{5513389648608931694027 91\, h^2 g_{n+2}}{33066146958818461463261440000} \\
&- \frac{99718889493963361099196159251 2\, h^2 g_{n+\frac{5}{26}}}{52994555596996953229630559269531 25} \\
&- \frac{1482810138576009927\, h^2 g_{n+\frac{20}{29}}}{210959219336800000000}
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&+ \frac{2578074557010805647\, h^2 g_{n+\frac{38}{29}}}{1476714535357600000000} \\
&+ \frac{6576124785907113881736198341 76\, h^2 g_{n+\frac{47}{26}}}{529945555969969532296305592695 3125} \\
&- \frac{1383835117714029260576917976 97\, h f_n}{295280692342248860866924659200000} \\
&- \frac{414775413207763584\, h f_{n+1}}{268203762658073311 3} \\
&- \frac{3898585983635975383940973591\, h f_{n+2}}{11811227693689954434676986368000} \\
&- \frac{20849926617627333885543535033008682368\, h f_{n+\frac{5}{26}}}{6124117087983464161407644467386118164062 5} \\
&- \frac{9516977519857159758323397\, h f_{n+\frac{20}{29}}}{679670595196054000000000000} \\
&- \frac{1395815459982313597371 33\, h f_{n+\frac{38}{29}}}{97095799313722000000000000} \\
&- \frac{96300581117986803401844306512677136 64\, h f_{n+\frac{47}{26}}}{6124117087983464161407644467386118164062 5}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} = y_n \quad &+ \frac{31390770194016413\, h f_n}{300800314636800000} \\
&+ \frac{24459243465753795421107944216 32\, h f_{n+\frac{5}{26}}}{859559715889873680867055664062 5} \\
&+ \frac{631917825685573712572445538817\, h f_{n+\frac{20}{29}}}{1120980478287681104940000000000 0} \\
&+ \frac{290009389184\, h f_{n+1}}{884260959813} + \frac{269342114006369860597873651399\, h f_{n+\frac{38}{29}}}{1245533864764090116600000000000} \\
&+ \frac{22990550579431312246763678336\, h f_{n+\frac{47}{26}}}{859559715889873680867055664062 5} \\
&+ \frac{659138643026173\, h f_{n+2}}{84224088098304000} \\
&+ \frac{706332168403\, h^2 g_n}{235789720320000} \\
&+ \frac{968454076853861104832\, h^2 g_{n+\frac{5}{26}}}{826458999871279113281 25} \\
&- \frac{254369447333755655449333\, h^2 g_{n+\frac{20}{29}}}{27994767516705528000000000}
\end{aligned} \qquad (2.10)
$$

$$
\begin{aligned}
&- \frac{6220321673\, h^2 g_{n+1}}{53591573322} \\
&- \frac{83896869075739821207541\, h^2 g_{n+\frac{38}{29}}}{2799476751670552800000000} \\
&- \frac{70026639494338876 4608\, h^2 g_{n+\frac{47}{26}}}{247937699961383733984375} \\
&- \frac{94340553773\, h^2 g_{n+2}}{235789720320000}
\end{aligned}
$$

$$
\begin{aligned}
(2.11) \qquad y_{n+\frac{38}{29}} = y_{n+1} &+ \frac{5513389648608931694027 91\, h^2 g_n}{330661469588184614632614 40000} \\
&+ \frac{201236166015324651\, h^2 g_{n+1}}{141416529401529564 14} \\
&- \frac{7694316463457662773830 01\, h^2 g_{n+2}}{3306614695881846146326144 0000} \\
&+ \frac{6576124785907113881736198 34176\, h^2 g_{n+\frac{5}{26}}}{52994555596996953229630559269531 25} \\
&+ \frac{257807455701080564 7\, h^2 g_{n+\frac{20}{29}}}{1476714535357600000000} \\
&- \frac{148281013857600992 7\, h^2 g_{n+\frac{38}{29}}}{2109592193368000000000} \\
&- \frac{9971888949396336109919615925 12\, h^2 g_{n+\frac{47}{26}}}{5299455559699695322963055926953125} \\
&+ \frac{38985859836359753839409735 91\, h f_n}{11811227693689954434676986368000} \\
&+ \frac{414775413207763584\, h f_{n+1}}{2682037626580733113} \\
&+ \frac{13838351177140292605769179 7697\, h f_{n+2}}{29528069234224886086692465920 0000} \\
&+ \frac{963005811179868034018443065126 7713664\, h f_{n+\frac{5}{26}}}{612411708798346416140764446738611816 40625} \\
&+ \frac{1395815459982313597371 33\, h f_{n+\frac{20}{29}}}{970957993137220000000000 0} \\
&+ \frac{951697751985715975832339 7\, h f_{n+\frac{38}{29}}}{67967059519605400000000000} \\
&+ \frac{208499266176273338855435350330086823 68\, h f_{n+\frac{47}{26}}}{61241170879834641614076444673861181640625}
\end{aligned}
$$

$$
\begin{aligned}
(2.12)\qquad y_{n+\frac{47}{26}} = y_{n+1} \quad & + \frac{3419073802024227171746243\, h^2 g_n}{9906473215028635328512000 0} \\
& + \frac{797053372151911\, h^2 g_{n+1}}{7623023165282304} \\
& - \frac{1340805343721016667265 73\, h^2 g_{n+2}}{99064732150286353285120000} \\
& + \frac{257675377558397869\, h^2 g_{n+\frac{5}{26}}}{105383133144575000000} \\
& + \frac{157447364188453517086542885300 7\, h^2 g_{n+\frac{20}{29}}}{5975576010457405500994560000000 0} \\
& + \frac{166347507223791022202879798139 7\, h^2 g_{n+\frac{38}{29}}}{1991858670152468500331520000000 0} \\
& - \frac{1987403759042034403\, h^2 g_{n+\frac{47}{26}}}{105383133144575000000} \\
& + \frac{11954400821927194452844950 9\, h f_n}{1769296116204114269672243200 0} \\
& + \frac{7437477576455479\, h f_{n+1}}{23583727917592128} \\
& + \frac{99425197466903954482271423 1\, h f_{n+2}}{3402492531161758210908160000 00} \\
& + \frac{1583802590975465659756 3\, h f_{n+\frac{5}{26}}}{669132230500193281250000 0} \\
& + \frac{6867800111496626928789321542163036872 3\, h f_{n+\frac{20}{29}}}{3589155035981109896603620032000000000 00} \\
& + \frac{3183367291684606365507342180959486410 1\, h f_{n+\frac{38}{29}}}{3589155035981109896603620032000000000 00} \\
& + \frac{1163888407709356085364181\, h f_{n+\frac{47}{26}}}{6691322305001932812500000}
\end{aligned}
$$

$$
\begin{aligned}
(2.13)\qquad y_{n+2} = y_{n+1} &+ \frac{94340553773\, h^2 g_n}{235789720320000} \\
&+ \frac{6220321673\, h^2 g_{n+1}}{53591573322} \\
&- \frac{706332168403\, h^2 g_{n+2}}{235789720320000} \\
&+ \frac{70026639494338876 4608\, h^2 g_{n+\frac{5}{26}}}{247937699961383733984375} \\
&+ \frac{838968690757398212075 41\, h^2 g_{n+\frac{20}{29}}}{279947675167055280000000 0} \\
&+ \frac{2543694473337556554 49333\, h^2 g_{n+\frac{38}{29}}}{279947675167055280000000 0} \\
&- \frac{968454076853861104832\, h^2 g_{n+\frac{47}{26}}}{8264589998712791132812 5} \\
&+ \frac{659138643026173\, h f_n}{84224088098304000} \\
&+ \frac{290009389184\, h f_{n+1}}{884260959813} \\
&+ \frac{31390770194016413\, h f_{n+2}}{300800314636800000} \\
&+ \frac{22990550579431312246763678336\, h f_{n+\frac{5}{26}}}{859559715889873680867055664 0625} \\
&+ \frac{26934211400636986059787365 1399\, h f_{n+\frac{20}{29}}}{124553386476409011660000000000 0} \\
&+ \frac{6319178256855737125724455 38817\, h f_{n+\frac{38}{29}}}{11209804782876811049400000000000 0} \\
&+ \frac{24459243465753795421107944 21632\, h f_{n+\frac{47}{26}}}{859559715889873680867055664 0625}
\end{aligned}
$$

3. Analysis of the method

3.1. **Local truncation error.** The above two-step method is presented in this section. The linear operator is considered as:

$$
L[y(x_n), h] = \sum_{i=0}^{k} \alpha_i \cdot y(x_n + ih) - h\gamma_i f(x_n + ih) + h^2 \beta_i g(x_n + ih) \tag{3.1}
$$

The function $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + ih)$, $f(x_n + ih)$ and $g(x_n + ih)$ in Taylors series about x_n and factoring the coefficients of h to get

$$L[y(x_n),h] = C_0 y(x_n) + C_1 h y(x_n) + C_2 h^2 y^2(x_n) + \ldots + C_p h^p y^p(x_n) + \ldots \tag{3.2}$$

where the constant C_i, $i = 0, 1, \ldots$ are given as follows

$$\begin{cases} C_0 &= \alpha_0 + \alpha_1, \\ C_1 &= \alpha_1 - (\gamma_0 + \gamma_{\frac{5}{26}} + \gamma_{\frac{20}{29}} + \gamma_1 + \gamma_{\frac{38}{29}} + \gamma_{\frac{47}{26}} + \gamma_2), \\ C_2 &= \frac{1}{2!}\alpha_i - \left(\frac{5}{26}\gamma_{\frac{5}{26}} + \frac{20}{29}\gamma_{\frac{20}{29}} + \gamma_1 + \frac{38}{29}\gamma_{\frac{38}{29}} + \frac{47}{26}\gamma_{\frac{47}{26}} + 2\gamma_2\right) \\ &\quad - \left(\beta_0 + \beta_{\frac{5}{26}} + \beta_{\frac{20}{29}} + \beta_1 + \beta_{\frac{38}{29}} + \beta_{\frac{47}{26}} + \beta_2\right) \\ \ldots \\ C_p &= \frac{1}{p!}\alpha_1 - \frac{1}{(p-1)!}\left(\left(\frac{5}{26}\right)^{p-1}\gamma_{\frac{5}{26}} + \left(\frac{20}{29}\right)^{p-1}\gamma_{\frac{20}{29}} + \gamma_1 + \left(\frac{38}{29}\right)^{p-1}\gamma_{\frac{38}{29}}\right. \\ &\quad \left. + \left(\frac{47}{26}\right)^{p-1}\gamma_{\frac{47}{26}} + 2^{p-1}\gamma_2\right) \\ &\quad -\frac{1}{(p-2)!}\left(\left(\frac{5}{26}\right)^{p-2}\beta_{\frac{5}{26}} + \left(\frac{20}{29}\right)^{p-2}\beta_{\frac{20}{29}} + \gamma_1 + \left(\frac{38}{29}\right)^{p-2}\beta_{\frac{38}{29}}\right. \\ &\quad \left. + \left(\frac{47}{26}\right)^{p-2}\beta_{\frac{47}{26}} + 2^{p-2}\beta_2\right) \end{cases} \tag{3.3}$$

From (3.2), we can obtain $C_0 = C_1 = \ldots = C_{14} = 0$ and

$$\begin{aligned} C_{15} = &\left(\frac{742953457}{25425280337675685667200 0}, \right. \\ &\frac{1898244635071516210773}{776272612894431197030421294481408000}, \\ &\frac{373071477459644081757}{425516297744335528917977272629347200 0}, \\ &\frac{1898244635071516210773}{776272612894431197030421294481408000}, \\ &\left.\frac{742953457}{25425280337675685667200 0}\right)^\top. \end{aligned}$$

Therefore the proposed method is of order 14 [8].

3.2. **Stability Analysis.** (2.8) can be written in the matrix form

$$A_1 Y_{n+1} = A_0 Y_n + h(B_0 F_n + B_1 F_{n+1}) + h^2(C_0 G_n + C_1 G_{n+1}) \tag{3.4}$$

where A_0, A_1, B_0, B_1, C_0, C_1 are 6×6 matrix and

$Y_n = (y_n, y_{n-\frac{5}{26}}, y_{n-\frac{20}{29}}, y_{n-\frac{38}{29}}, y_{n-\frac{47}{26}}, y_{n-2})^\top$,
$Y_{n+1} = (y_{n+\frac{5}{26}}, y_{n+\frac{20}{29}}, y_{n+1}, y_{n+\frac{38}{29}}, y_{n+\frac{47}{26}}, y_{n+2})^\top$,
$F_n = (f_n, f_{n-\frac{5}{26}}, f_{n-\frac{20}{29}}, f_{n-\frac{38}{29}}, f_{n-\frac{47}{26}}, f_{n-2})^\top$,
$F_{n+1} = (f_{n+\frac{5}{26}}, f_{n+\frac{20}{29}}, f_{n+1}, f_{n+\frac{38}{29}}, f_{n+\frac{47}{26}}, f_{n+2})^\top$,
$G_n = (g_n, g_{n-\frac{5}{26}}, g_{n-\frac{20}{29}}, g_{n-\frac{38}{29}}, g_{n-\frac{47}{26}}, g_{n-2})^\top$,
$G_{n+1} = (g_{n+\frac{5}{26}}, g_{n+\frac{20}{29}}, g_{n+1}, g_{n+\frac{38}{29}}, g_{n+\frac{47}{26}}, g_{n+2})^\top$.

According to Lambert [11], a numerical technique is considered to be zero-stable if no root of the first characteristics polynomial $\rho(z)$ has a modulus more than one, and every root of modulus one has simplicity not greater than the order of the differential equation.

The characteristics equation of the hybrid block method in (3.3) is

$$\rho(z) = det(ZA_1 - A_0) = z^5(z+1) = 0. \tag{3.5}$$

This implies that $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ and $|z_6| = 1$. Therefore, the method is zero stable.

A method is consistency if it has order greater than or equal to 1. Therefore, the proposed method is consistency Jator [9].

In addition, the method is converging only if it is consistent and zero stable Henrici [8]. Because the proposed method meets the two requirements, it is convergent.

The absolute stability region was studied. The linear stability properties of the proposed method are studied by applying it to the test problem $y' = \lambda y$ with $\lambda < 0$ to get

$$Y_{n+1} = M(z)Y_n, \quad z = \lambda h \tag{3.6}$$

where

$$M(z) = (A_1 - zB_1 - z^2C_1)^{-1}(A_0 + zB_0 + z^2C_0). \tag{3.7}$$

The matrix $\sigma(z)$ has eigenvalues $\{0, 0, 0, ..., \lambda_k\}$, and the dominant eigenvalue $\lambda_k : \mathrm{C} \to \mathrm{C}$ is a rational function (called the stability function) with real coefficients given by

$$\lambda_k = \frac{P(z)}{P(-z)}. \tag{3.8}$$

It is clear from the stability functions that for $Re(z) < 0$, $|\lambda_k| \leq 1$.

The absolute stability region is given Figure 2. The stability region contains the entire left half complex plane and thus, the method is A-stable [2].

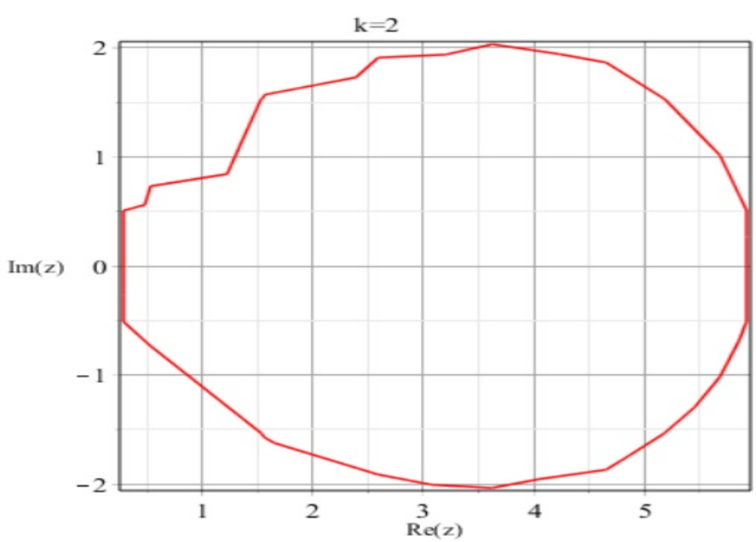

Figure 2: Absolute Stability Region of the Method

3.3. **Convergence of the method.** Consistency and zero stability are both required and sufficient for a linear multistep technique to reach convergence in the spirit of Lambert [11]. As a result, we infer that our technique is convergent since it has an order of accuracy (which implies consistency) and zero–stable.

4. Numerical experiments

The efficiency of the method will be demonstrated using some examples.

Example 1. Consider the following nonlinear equation:

$$y' + y = 2t + e^t, \quad y(0) = -1$$

with exact solution

$$y(t) = \cosh(t) + 2t - 2.$$

Table 1 shows the comparison between the proposed method and Motsa [16]. Comparison between the numerical solutions and the exact solutions is shown in Figure 3 when $h = 0.1$.

Table 1: Comparison between the absolute errors in the proposed method and the method in paper [16].

t	Error in our method	Error in [16], $M = 5$
0.0	0.000	0.0000
0.1	2.62E-31	1.40E-15
0.2	5.33E-31	5.44E-15
0.3	1.30E-30	1.19E-14
0.4	2.05E-30	2.07E-14
0.5	3.27E-30	3.17E-14
0.6	4.44E-30	4.47E-14
0.7	6.08E-30	5.97E-14
0.8	7.66E-30	7.69E-14
0.9	9.72E-30	9.61E-14
1.0	1.17E-29	1.17E-13
1.1	1.42E-29	1.41E-13
1.2	1.67E-29	1.67E-13
1.3	1.97E-29	1.96E-13
1.4	2.27E-29	2.27E-13
1.5	2.63E-29	2.61E-13
1.6	2.98E-29	2.99E-13
1.7	3.42E-29	3.40E-13
1.8	3.85E-29	3.85E-13
1.9	4.37E-29	4.34E-13
2.0	4.88E-29	4.88E-13

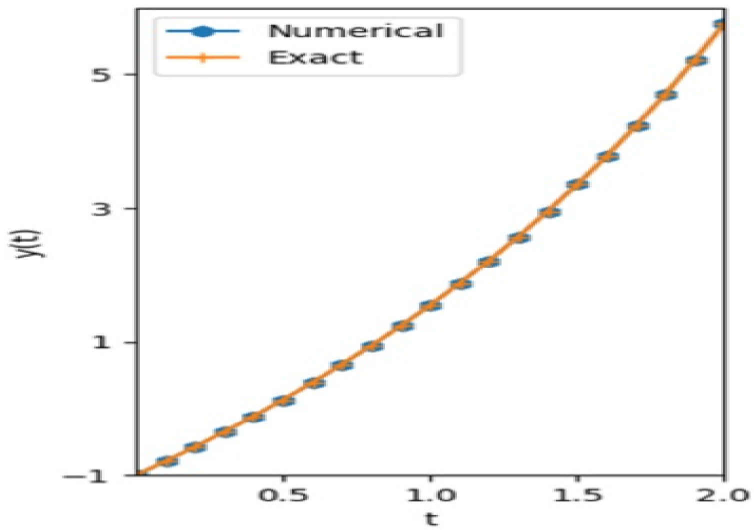

Figure 3: Exact solution and Numerical solution for Example 1

Example 2. Consider the following nonlinear equation

$$y' = y^2 - ty, \quad y(0) = \frac{1}{2}$$

with exact solution

$$y(t) = \frac{2\exp\{-t^2\}}{\sqrt{\pi}t^2 - 4}.$$

The exact and numerical solutions for Example 2 are displayed in Figure 4. The errors comparison with other methods are illustrated in Table 2.

Table 2: Numerical, exact solutions, and the absolute errors.

t	Solution by our method	Exact solution	Absolute Error
0.0	0.50000	0.50000	0.00000
1.0	0.530010126	0.530010126	2.04E-22
2.0	0.168387788	0.168387788	5.36E-22
3.0	1.48E-02	1.48E-02	1.40E-23
4.0	4.49E-04	4.49E-04	6.73E-26
5.0	4.99E-06	4.99E-06	1.05E-26

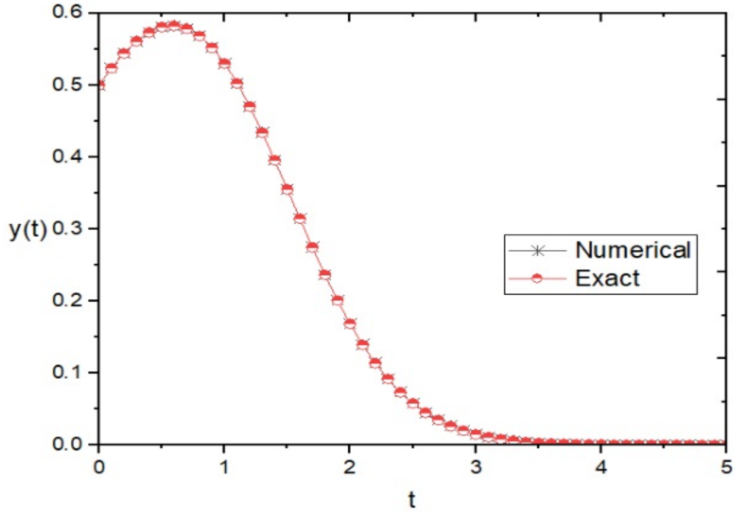

Figure 4: Exact solution and Numerical solution for Example 2

Example 3. Consider the Riccati equation. A matrix equation with an analogous quadratic term is referred to as Riccati equation, this matrix occurs in both discrete-time and continuous-time linear-quadratic-gaussian control.

$$y' = -y^2 + 2y + 1, \quad y(0) = 0,$$

with exact solution

$$y(t) = 1 + \sqrt{2}\tanh\sqrt{2}t + \frac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}.$$

The exact and numerical solutions are presented in the Table 3.

Table 3: Comparison between the absolute errors in our method and the methods in [10, 16].

t	Error in our method	Error in [16], $M = 5$	Error in [10]
0	0.0000	0.0000	0.0000
1	1.79E-23	3.54E-13	1.61E-10
2	2.21E-24	3.07E-14	2.14E-10
3	3.96E-24	8.08E-15	2.22E-10
4	3.73E-24	1.74E-15	2.14E-10
5	4.00E-24	1.81E-16	1.99E-10
6	4.72E-24	1.53E-17	1.82E-10
7	4.37E-24	1.18E-18	1.65E-10
8	3.96E-24	8.60E-20	1.50E-10
9	3.50E-24	6.04E-21	1.37E-10
10	3.75E-24	4.14E-22	1.25E-10

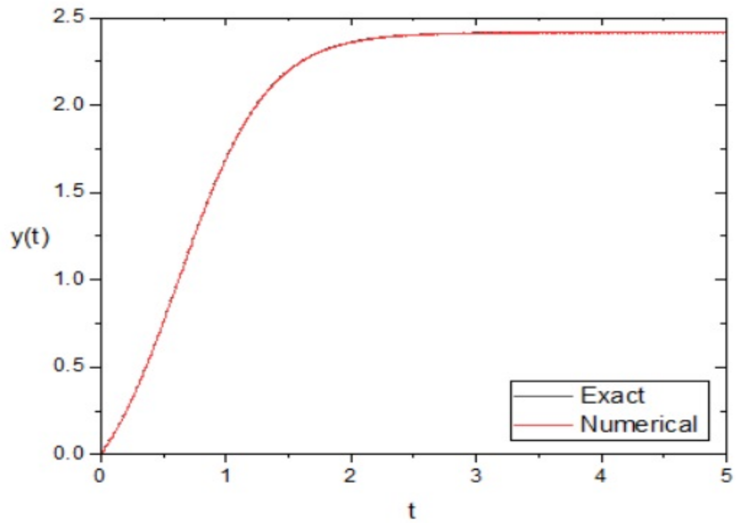

Figure 5: Exact solution and Numerical solution for Example 3

Example 4. Consider the Lorenz system equation are derived to model some unpredictable behavior of weather.

$$\begin{cases} y_1' = a(y_2 - y_1), & y_1(0) = 1, \\ y_2' = -y_1 y_3 + b y_1 - y_2, & y_2(0) = 5, \\ y_3' = y_1 y_3 - c y_3, & y_3(0) = 10. \end{cases}$$

The constants are $a = 10$, $b = 28$, and $c = 8/3$. The phase portraits for the Lorenz equation are displayed in Figure 6.

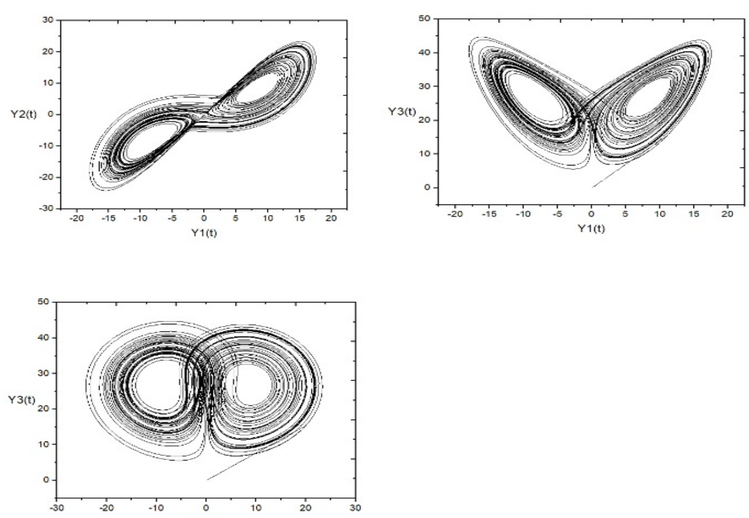

Figure 6: Phase portraits for the Lorenz system, $h = 0.01$

Example 5. Consider the chaotic the chaotic system called the Arnodo-Coullet which is system given by the following set of equations

$$\begin{cases} y_1' = y_2, & y_1(0) = 0.21, \\ y_2' = y_3, & y_2(0) = 0.22, \\ y_3' = ay_1 - by_2 - y_3 - y_1^3, & y_3(0) = 0.61. \end{cases}$$

The constants are $a = 5$, $b = 3.8$. The phase portraits for the Arnodo-Coullet are displayed in Figure 7.

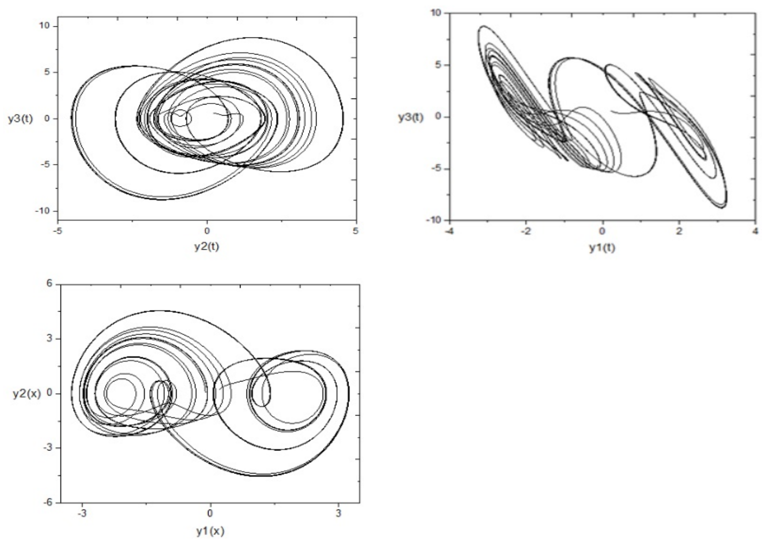

Figure 7: Phase portraits for the Arnodo-Coullet system, $h = 0.025$

Discussion of Results. Example 1 is a non-linear equation, the result from our method was compared to paper [16], our method performs better than that of [16] which has 4.88×10^{-29} as error compared to 1.40×10^{-15} of [16]. Example 2 is a non-linear equation, we compared the result with the exact solution which compare favorably well with the exact solution. Example 3 is a Riccati equation (this matrix occurs in both discrete-time and continuous-time linear-quadratic-gaussian control) as shown in Table 3, our method performs significantly better than papers [16] and [10]. The Lorenz system equation of Example 4 are derived to model some unpredictable behavior of weather, The phase portraits shows the graphical solution of the problem that arises from chaos theory. Lastly, we considered the chaotic system called the Arnodo-Coullet which is system given by a set of equations, The phase portraits also demonstrates the solution of the problem solved by our method. It is clearly demonstrated from the examples that the proposed method is able to provide solution to non-linear dynamical problems.

Conclusion. For the self-starting solution of general first initial value problems of ordinary differential equations, we provide an optimized two-step block technique using Bhaskara hybrid points for non-linear dynamical first order differential equations. The method's performance was assessed using four Bhaskara hybrid points. For orders up to 14, the suggested hybrid block technique is both zero stable and converging. Riccati equation, chaotic system dubbed Arnodo-Coullet and non-linear stiff issue are all solved by the suggested approach. The absolute inaccuracy falls as h decreases in these situations. The numerical examples demonstrate the method's effectiveness and accuracy in comparison to other approaches in the literature.

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