Nigerian Journal of Mathematics and Applications
Volume **32** (**II**), (2022), 129 − 137. *Printed by Unilorin press*
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Special Issue in Honor of Prof. J. A. Gbadeyan's Retirement

Tripled Fixed Point Theorems in Partially Ordered G-Metric Space

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Abstract

Tripled fixed point theorem is an improvement to coupled fixed point. In this manuscript, we prove the existence and uniqueness of tripled fixed point for mixed monotone mapping satisfying nonlinear contractions in the framework of partially ordered G-metric space. Our results generalize and improve some results in the literature.

1. Introduction

Fixed point theory plays significant roles in numerous areas of pure and applied sciences such as control theory, economic theory, global analysis, nonlinear analysis and in the fields of engineering. Several researches have been conducted on the applications, generalizations and extensions of the Banach contraction principle in different ways by either weakening the contractive conditions or considering different mappings [1]-[14].

Recently, there has been an increasing interest in the study of the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. The first fixed point result on a partially ordered metric space was given by Turinici [14] where he extended the Banach contraction principle in partially ordered sets. Furthermore, Ran and Reurings [15] presented some applications of Turinici's theorem to matrices. Subsequently, Nieto and Lopez [16] extended the result of Ran and Reurings for nondecreasing mappings and use the results to obtain a unique solution for a first order differential equation.

Bhaskar and Lakshmikantham [1] introduced the ideology of coupled fixed point for contractive mappings $F : X \times X \to X$ satisfying the mixed monotone property and proved

Received: 11/06/2022, Accepted: 18/07/2022, Revised: 29/07/2022. * Corresponding author.
2015 *Mathematics Subject Classification.* 55M20 & 54F05.
Keywords and phrases. Fixed point theorems, Partially ordered set, Nonlinear contractions and Tripled fixed point
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some interesting results. Also, Berinde and Borcut [13] introduced the concept of tripled fixed point and proved some related theorems.
Hence, in this work, the aim is to prove the existence and uniqueness of tripled fixed point for mixed monotone mapping satisfying nonlinear contractions in the context of partially ordered G-metric space.

2. Preliminaries

Some basic definitions and list of results that motivated our tripled fixed point theorems are enlisted here.

Definition 2.1. (see [2]). A G-metric space is a pair (X, G) where X is nonempty set and $G : X \times X \times X \longrightarrow [0, \infty)$ is a function such that, for all $x, y, z, a \in X$, the following conditions are fulfilled:
$(G_1)\, G(x, y, z) = 0$ if $x = y = z$;
$(G_2)\, G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
$(G_3)\, G(x, x, y) = G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
$(G_4)\, G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (Symmetry in 3);
$(G_5)\, G(x, y, z) = G(x, a, a) + G(a, y, z)$ (rectangle inequality).

The function G is called a G-metric on X. The properties may be easily interpreted in the setting of metric spaces. Let (X, d) be a metric space and define $G : X \times X \times X \longrightarrow [0, \infty.)$ by $G(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then (X, G) is a G-metric space.

Definition 2.2. (see [2]). Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,\ m \to \infty} G(x_n,\ x_m,\ x) = 0$, and one says the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in G-metric space (X, G), then, for any $\varepsilon > 0$, there exists a positive integer $\mathbb{N}$ such that $G(x,\ x_n,\ x_m) < \varepsilon$ for all $n, m > N$.

Proposition 2.1. *(see [2]). If (X, G) is a G-metric space, then the following are equivalent:*

- *(1) $\{x_n\}$ is G-convergent to x;*
- *(2) $G(x_n,\ x_n,\ x) \to 0$ as $n \to \infty$;*
- *(3) $G(x_n,\ x,\ x) \to 0$ as $n \to \infty$; and*
- *(4) $G(x_m,\ x_n,\ x) \to 0$ as $m, n \to \infty$.*

Definition 2.3. (see [2]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is said to be G-Cauchy if every $\varepsilon > 0$, there exists a positive integer $\mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l > N$, that is, if $G(x_n,\ x_m,\ x_l) \to 0$, as $n, m, l \to \infty$.

Proposition 2.2. *(see [2]). If (X, G) is a G-metric space, then the following are equivalent:*

- *(1) The sequence $\{x_n\}$ is G-Cauchy; and*
- *(2) For every $\epsilon > 0$, there exists a positive integer $\mathbb{N}$ such that $G(x_n,\ x_m,\ x_m) < \epsilon$ for all $n, m > N$.*

Proposition 2.3. *(see [2]). If (X, G) is a G-metric space, then $G(x,\ y,\ y) = 2G(y,\ x,\ x)$ for all $x, y \in X$.*

Proposition 2.4. *(see [2]). If (X, G) is a G-metric space, then $G(x,\ x,\ y) = G(x,\ x,\ z) + G(z,\ z,\ y)$ for all $x, y,\ z \in X$.*

Definition 2.4. (see [2]). Let $(X,G),(X^{'},G^{'})$ be two G-metric spaces. Then a function $f:X\rightarrow X^{'}$ is G-continuous at a point $x\in X$ if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to $x,\{f(x_n)\}$ is $G^{'}$-convergent to $f(x)$.

Definition 2.5. (see [2]). A G-metric space (X,G) is said to be G-complete or a complete G-metric space if every G-Cauchy sequence in (X,G) is convergent in X.

Proposition 2.5. *(see [2]). Let (X,G) be a G-metric space. Then the function $G\,(x,\ y,\ z)$ is jointly continuous in all three of its variables.*

Definition 2.6. (see [2]). Let $(X,=)$ be a partially ordered set and $F:X^3\rightarrow X$ be a mapping. F is said to have the mixed monotone property if $F(x,\ y,\ z)$ is non-decreasing in x and z and is non-increasing in y, that is, for any $x,\ y,\ z\in X$,

$$\begin{aligned} x_1,\ x_2\in X,\quad x_1\leq x_2 &\implies F(x_1,y,z)\leq F(x_2,\ y,\ z),\\ y_1,\ y_2\in X,\quad y_1\leq y_2 &\implies F(x,y_1,\ z)\geq F(x,\ y_2,\ z,\\ z_1,\ z_2\in X,\quad z_1\leq z_2 &\implies F(x,y,z_1)\leq F(x,\ y,\ z_2). \end{aligned}$$

Definition 2.7. (see [2]). Let $(X,\leq)$ be a partially ordered set and $F:X^3\rightarrow X$ be a mapping. F is said to have the mixed monotone property if $F\,(x,\ y,\ z)$ is non-decreasing in x and z and is non-increasing in y that is, for any $x,\ y,\ z\in X$,

$$\begin{aligned} x_1,\ x_2\in X,\quad x_1\leq x_2 &\implies F\,(x_1,y,z)\leq F\,(x_2,\ y,\ z)\,,\\ y_1,\ y_2\in X,\quad y_1\leq y_2 &\implies F\,(x,y_1,\ z,\ w)\geq F\,(x,\ y_2,\ z)\,,\\ z_1,\ z_2\in X,\quad z_1\leq z_2 &\implies F\,(x,y,z_1,\ w)\leq F\,(x,\ y,\ z_2)\,. \end{aligned}$$

Theorem 2.8. *(see [1]). Let $F:X^2\rightarrow X$ be a continuous mapping having the mixed monotone propertyon X. Assume that there exists a $k\in[0,\ 1)$ with*
$d(F(x,\ y),F(u,\ v))\leq\frac{k}{2}[d(x,\ u)+d(y,\ v)]$, *for all $u\leq x,\ y\leq v$.*
If there exist $x_0,y_0\in X$ such that $x_0\leq F(x_0,y_0)$ and $F(y_0,\ x_0)\leq y_0$, then, there exist $x,\ y\in X$ such that $F(x,\ y)=x$ and $F(y,\ x)=y$.

Theorem 2.9. *(see [1]). Let $F:X^2\rightarrow X$ be a mapping having the mixed monotone property on X. Suppose that X has the following properties:*
(i) if a non-decreasing sequence $\{x_n\}$ tends to x, then $x_n\leq x,\forall n$;
(ii) if a non-increasing sequence $\{y_n\}$ tends to y, then $x\leq y_n,\forall n$.
Assume that there exists a $k\in[0,\ 1)$ with
$d\,(F\,(x,\ y)\,,F\,(u,\ v))\leq\frac{k}{2}\,[d\,(x,\ u)+d\,(y,\ v)]$ *,for all $u\leq x,\ y\leq v$.*
If there exist $x_0,y_0\in X$ such that $x_0\leq F\,(x_0,\ y_0)$ and $F\,(y_0,\ x_0)\leq y_0$, then, there exist $x,\ y\in X$ such that $F\,(x,\ y)=x$ and $F\,(y,\ x)=y$.

Definition 2.10. (see [1]).Let $(X,\leq)$ be a partially ordered set and $F:X^3\rightarrow X$ be a mapping. F is said to have the mixed monotone property if $F\,(x,\ y,\ z)$ is non-decreasing in x and z and is non-increasing in y, that is, for any $x,\ y,\ z\in X$,

$$\begin{aligned} x_1,\ x_2\in X,\quad x_1\leq x_2 &\implies F\,(x_1,y,z)\leq F\,(x_2,\ y,\ z)\,,\\ y_1,\ y_2\in X,\quad y_1\leq y_2 &\implies F\,(x,y_1,\ z)\geq F\,(x,\ y_2,\ z)\,,\\ z_1,\ z_2\in X,\quad z_1\leq z_2 &\implies F\,(x,y,z_1)\leq F\,(x,\ y,\ z_2) \end{aligned}$$

Definition 2.11. (see [1]). An element $(x,\ y,\ z)\in X^3$ is said to be a tripled fixed point of mapping $F:X^3\rightarrow X$ if $F\,(x,\ y,\ z)=x$, $F\,(y,\ x,\ y)=y$ and $F\,(z,\ y,\ x)=z$.
For a metric space $(X,\ d)$, the function $\rho:X^3\times X^3\longrightarrow[0,\ \infty)$, given by

$$\rho\,((x,\ y,\ z)\,,\ (u,\ v,\ w))\ :=d\,(x,\ u)+d\,(y,\ v)+d\,(z,\ w)$$

.

Is a metric on X^3, that is, the pair $\left(X^3,\ \rho\right)$ is a metric space induced by d.

Theorem 2.12. *(see [3]). Let $(X,\ \leq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that $F:\ X^3 \longrightarrow X$ is a continuous mapping having the mixed monotone property on X. Assume there exists $\phi \in \Phi$ such that*

$$G(F(x,\ y,\ z),\ F(u,\ v,\ s),\ F(a,\ b,\ c) + G(F(y,\ z,\ x),\ F(v,\ s,u),\ F(b,\ c,a))$$
$$+G(F(z,x,\ y),\ F(s,u,\ v),\ F(c,a,\ b)) \leq \phi(G(x,\ u,\ a) + G(y,\ v,\ b)$$
$$+G(z,\ s,\ c))(G(x,\ u,\ a) + G(y,\ v,\ b) + G(z,\ s,\ c))$$

for all $x,\ y,\ z,\ u,\ v,\ s,\ a,\ b,\ c \in X$ with $x \succcurlyeq u \succcurlyeq a, y \preccurlyeq v \preccurlyeq b$, and $z \succcurlyeq s \succcurlyeq c$, where either $u \neq a$ or $v \neq b$ or $s \neq c$. If there exists $x_0, y_0, z_0 \in X$ such that $x \geq u,\ y \leq v,\ \ z \geq w$,$x_0 \leq F(x_0,\ y_0,\ z_0),\ y_0 \geq F(y_0,\ z_0,\ x_0)\ , z_0 \leq F(z_0,\ y_0,\ x_0)$, then F has a tripled fixed point; that is, there exist $x, y, z \in X$ such that$F(x,\ y,\ z) = x$,$F(y,\ x,\ y) = y$,$F(z,\ y,\ x) = z$.

3. Main Result

In this section, we establish some tripled fixed point results by considering maps on generalized metric spaces endowed with partial order. Before going further, we define the following function which will be used in our results.

Let sequences $\{x_n\},\ \{y_n\}$ and $\{z_n\}$ be nonnegative real numbers. Let Φ denote all the functions $\phi\ :\ [0,\ \infty)^3 \longrightarrow [0,\ 1)$ which satisfy that $\phi(x_n,\ y_n,\ z_n) \longrightarrow 1$, implies $x_n,\ y_n,\ z_n \longrightarrow 0$. An example of such function is as follows:

$$\phi(x,\ y,\ z) = \begin{cases} \frac{In(1+k_1x+k_2y+k_3z)}{k_1x+k_2y+k_3z}; at\ least\ one\ of\ x,y,z > 0\ and\ k_1,k_2,k_3 \in [0,1) \\ \qquad t \in [0,\ 1); \qquad x = 0 = y = z \end{cases}$$

Theorem 3.1. *Let $(X,\ \leq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that $F:\ X^3 \longrightarrow X$ is a continuous mapping having the mixed monotone property on X. Assume there exists $\phi \in \Phi$ such that*

$$\left(\frac{1}{3}\left[G(H,\ J,\ K) + G(L,\ M,\ N) + G(P,\ Q,\ R)\right]\right)$$

(3.1)

$$\leq \phi(G(x,\ u,\ a) + G(y,\ v,\ b) + G(z,\ s,\ c)) \times \frac{(G(x,\ u,\ a) + G(y,\ v,\ b) + G(z,\ s,\ c))}{3}$$

where, $H = F(x,\ y,\ z), J = F(u,\ v,\ s), K = F(a,\ b,\ c), L = F(y,\ z,\ x), M = F(v,\ s,u), N = F(b,\ c,a), P = F(z,x,\ y), Q = F(s,u,\ v)$ and $R = F(c,a,\ b)$ for all $x,\ y,\ z,\ u,\ v,\ s,\ a,\ b,\ c \in X$ with $x \succcurlyeq u \succcurlyeq a, y \preccurlyeq v \preccurlyeq b$, and $z \succcurlyeq s \succcurlyeq c$, where either $u \neq a$ or $v \neq b$ or $s \neq c$. If there exists $x_0, y_0, z_0 \in X$ such that $x_0 \preccurlyeq F(x_0,\ y_0,\ z_0),\ y_0 \succcurlyeq F(y_0,\ z_0,\ x_0)\ , z_0 \preccurlyeq F(z_0,\ y_0,\ x_0)$, then F has a tripled fixed point; that is, there exist $x, y, z \in X$ such that

$$F(x,\ y,\ z) = x, F(y,\ x,\ y) = y, F(z,\ y,\ x) = z.$$

Proof. Let $x_0, y_0, z_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0,\ y_0,\ z_0),\quad y_0 \succcurlyeq F(y_0,\ z_0,\ x_0)\ ,\ z_0 \preccurlyeq F(z_0,\ y_0,\ x_0).$$

We construct the sequences $\{x_n\},\ \{y_n\}$ and $\{z_n\}$ as

$$x_{n+1} = F(x_n,\ y_n,\ z_n),$$
$$y_{n+1} = F(y_n,\ x_n,\ y_n),$$

$$z_{n+1} = F(z_n,\ y_n,\ x_n)\,, \tag{3.2}$$

for $n = 1, 2, 3 \ldots$. By the mixed monotone property, we have

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \leq x_{n+1} \preccurlyeq \ldots,$$
$$y_0 \succcurlyeq y_1 \succcurlyeq y_2 \succcurlyeq \cdots \succcurlyeq y_{n+1} \succcurlyeq \ldots,$$
$$z_0 \preccurlyeq z_1 \preccurlyeq z_2 \preccurlyeq \cdots \preccurlyeq z_{n+1} \preccurlyeq \ldots.$$

Assume that there exists a nonnegative integer n such that

$$G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n) = 0.$$

It follows that

$$G(x_{n+1}, x_{n+1}, x_n) = 0 = G(y_{n+1}, y_{n+1}, y_n) = G(z_{n+1}, z_{n+1}, z_n)\,.$$

By property $(G1)$ of G-metric space, we have $x_{n+1} = x_n,\ y_{n+1} = y_n$ and $z_{n+1} = z_n$.
Now, suppose that for all nonnegative integer$n$

$$G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n) \neq 0.$$

Using 3.1 and 3.2 we have

$$\begin{aligned} &\left(\tfrac{1}{3}\left[G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n)\right]\right) \\ &= \left(\tfrac{1}{3}\left[G(A, B, C) + G(L, M, N) + G(U, V, W)\right]\right) \\ &\leq \phi\left(G(x_n,\ x_n,\ x_{n-1}), G(y_n,\ y_n,\ y_{n-1}), G(z_n,\ z_n,\ z_{n-1})\right) \\ &\times \tfrac{(G(x_n,\ x_n,\ x_{n-1}) + G(y_n,\ y_n,\ y_{n-1}) + G(z_n,\ z_n,\ z_{n-1}))}{3}. \end{aligned} \tag{3.3}$$

where $A = F(x_n,\ y_n,\ z_n), B = F(x_n,\ y_n,\ z_n), C = F(x_{n-1},\ y_{n-1},\ z_{n-1}), L = F(y_n,\ x_n,\ y_n)$,
$M = F(y_n,\ x_n,\ y_n), N = F(y_{n-1},\ x_{n-1},\ y_{n-1}), U = F(z_n,\ y_n,\ x_n), V = F(z_n,\ y_n,\ x_n)$,
$W = F(z_{n-1},\ y_{n-1},\ x_{n-1})$,

$$\begin{aligned} &G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n) \\ &< G(x_n,\ x_n,\ x_{n-1}) + G(y_n,\ y_n,\ y_{n-1}) + G(z_n,\ z_n,\ z_{n-1})\,. \end{aligned} \tag{3.4}$$

For all $n \in \mathbb{N}$, we let

$$\delta_n = \frac{1}{3}\left[G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n)\right].$$

Then the sequence (δ_n) is decreasing; therefore, there is some $\delta \geq 0$ such that

$$\begin{aligned} \lim_{n\to\infty} \delta_n &= \frac{1}{3}\lim_{n\to\infty}\left[G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n)\right] \\ &= \delta \end{aligned} \tag{3.5}$$

We will show that $\delta = 0$. Suppose to the contrary that $\delta > 0$, we have from 3.3.

$$\begin{aligned} &\tfrac{G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n)}{G(x_n,\ x_n,\ x_{n-1}) + G(y_n,\ y_n,\ y_{n-1}) + G(z_n,\ z_n,\ z_{n-1})} \\ &\leq \phi\left(G(x_n,\ x_n,\ x_{n-1}), G(y_n,\ y_n,\ y_{n-1}), G(z_n,\ z_n,\ z_{n-1})\right) < 1. \end{aligned} \tag{3.6}$$

Letting $n \to \infty$, we have

$$\phi\left(G(x_n,\ x_n,\ x_{n-1}), G(y_n,\ y_n,\ y_{n-1}), G(z_n,\ z_n,\ z_{n-1})\right) \to 1. \tag{3.7}$$

Using the property of the function ϕ, we have
$G(x_n,\ x_n,\ x_{n-1}), G(y_n,\ y_n,\ y_{n-1}), G(z_n,\ z_n,\ z_{n-1}) \to 0$ as $n \to \infty$.
So, we have

$G(x_n,\ x_n,\ x_{n-1}) + G(y_n,\ y_n,\ y_{n-1}) + G(z_n,\ z_n,\ z_{n-1}) \to 0 \text{as} n \to \infty.$
a contradiction to 3.6. Thus, $\delta = 0$. From 3.4, we have

$$G(x_{n+1}, x_{n+1}, x_n) + G(y_{n+1}, y_{n+1}, y_n) + G(z_{n+1}, z_{n+1}, z_n) \to 0. \tag{3.8}$$

To prove that the sequence (x_n), (y_n), and (z_n) are G-Cauchy in the G-metric space (X, G). Suppose on the contrary that at least one of the sequences (x_n), (y_n), and (z_n) is not a G-Cauchy sequence in (X, G). So there exist $\epsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ with $m(k) > l(k) \geq k$ for every natural numbers k, such that

$$t_k = \frac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right) + G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right] \geq \epsilon. \tag{3.9}$$

We choose $m(k)$ such that it is the smallest integer satisfying 3.9. Thus,

$$\frac{1}{3}\left[G\left(x_{m(k)-1}, x_{m(k)-1}, x_{l(k)}\right) + G\left(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)}\right) + G\left(z_{m(k)-1}, z_{m(k)-1}, z_{l(k)}\right)\right] < \epsilon. \tag{3.10}$$

Using rectangle inequality property of a G-metric, with 3.9 and 3.10 in mind, we get

$$\begin{aligned} t_k &= \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right) + G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right] \\ &\leq \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{m(k)-1}\right) + G\left(x_{m(k)-1}, x_{m(k)-1}, x_{l(k)}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(y_{m(k)}, y_{m(k)}, y_{m(k)-1}\right) + G\left(y_{m(k)-1}, y_{m(k)-1}, y_{l(k)}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(z_{m(k)}, z_{m(k)}, z_{m(k)-1}\right) + G\left(z_{m(k)-1}, z_{m(k)-1}, z_{l(k)}\right)\right] \\ &< \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{m(k)-1}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{m(k)-1}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(z_{m(k)}, z_{m(k)}, z_{m(k)-1}\right)\right] + \epsilon. \end{aligned} \tag{3.11}$$

By letting $n \to \infty$ in 3 and using 3.7 we have

$$t_k = \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right) + G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right] \to \epsilon \tag{3.12}$$

Using rectangle inequality property, we get

$$\begin{aligned} t_k &= \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right) + G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right] \\ &\leq \tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{m(k)+1}\right) + G\left(x_{m(k)+1}, x_{m(k)+1}, x_{l(k)+1}\right) + G\left(x_{l(k)+1}, x_{l(k)+1}, x_{l(k)}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(y_{m(k)}, y_{m(k)}, y_{m(k)+1}\right) + G\left(y_{m(k)+1}, y_{m(k)+1}, y_{l(k)+1}\right) + G\left(y_{l(k)+1}, y_{l(k)+1}, y_{l(k)}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(z_{m(k)}, z_{m(k)}, z_{m(k)+1}\right) + G\left(z_{m(k)+1}, z_{m(k)+1}, z_{l(k)+1}\right) + G\left(z_{l(k)+1}, z_{l(k)+1}, z_{l(k)}\right)\right] \\ &= \delta_{l(k)} + \tfrac{1}{3}\left[\left[G\left(x_{m(k)}, x_{m(k)}, x_{m(k)+1}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{m(k)+1}\right) + G\left(z_{m(k)}, z_{m(k)}, z_{m(k)+1}\right)\right]\right] \\ &\quad + \tfrac{1}{3}\left[\left[G\left(x_{m(k)+1}, x_{m(k)+1}, x_{l(k)+1}\right) + G\left(y_{m(k)+1}, y_{m(k)+1}, y_{l(k)+1}\right) + G\left(z_{m(k)+1}, z_{m(k)+1}, z_{l(k)+1}\right)\right]\right]. \end{aligned}$$

Using the fact that $G(x,\ x,\ y) \leq 2G(x,\ y,\ y)$ in the above inequality for any $x,\ y \in X$, we obtain

$$\begin{aligned} t_k &\leq \delta_{l(k)} + \tfrac{2}{3}\delta_{m(k)} \\ &\quad + \tfrac{1}{3}\left[G\left(x_{m(k)+1}, x_{m(k)+1}, x_{l(k)+1}\right) + G\left(y_{m(k)+1}, y_{m(k)+1}, y_{l(k)+1}\right)\right] \\ &\quad + \tfrac{1}{3}\left[G\left(z_{m(k)+1}, z_{m(k)+1}, z_{l(k)+1}\right)\right] \end{aligned} \tag{3.13}$$

Using 3.1, 3.2 and 3.13 we have

$$
\begin{aligned}
&t_k \leq \delta_{l(k)} + \tfrac{2}{3}\delta_{m(k)} \\
&+\tfrac{1}{3}\left[G\left(F\left(x_{m(k)}, y_{m(k)}, z_{m(k)}\right),\ F\left(x_{m(k)}, y_{m(k)}, z_{m(k)}\right),\ F\left(x_{l(k)}, y_{l(k)}, z_{l(k)}\right)\right)\right] \\
&+\tfrac{1}{3}\left[G\left(F\left(y_{m(k)}, x_{m(k)}, y_{m(k)}\right),\ F\left(y_{m(k)}, x_{m(k)}, y_{m(k)}\right),\ F\left(y_{l(k)}, x_{l(k)}, y_{l(k)}\right)\right)\right] \\
&+\tfrac{1}{3}\left[G\left(F\left(z_{m(k)}, y_{m(k)}, x_{m(k)}\right),\ F\left(z_{m(k)}, y_{m(k)}, x_{m(k)}\right),\ F\left(z_{l(k)}, y_{l(k)}, x_{l(k)}\right)\right)\right] \\
&\leq \phi\left(G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right), G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right),\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right) \\
&\times\left(\tfrac{1}{3}\left[G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right) + G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right) +\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right]\right) \\
&+\delta_{l(k)} + \tfrac{2}{3}\delta_{m(k)} \\
&= \phi\left(G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right), G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right),\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right) t_k + \delta_{l(k)} + \tfrac{2}{3}\delta_{m(k)}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{t_k - \delta_{l(k)} - \frac{2}{3}\delta_{m(k)}}{t_k} \\
&\leq \phi\left(G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right), G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right),\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right) < 1.
\end{aligned} \tag{3.14}
$$

Let $k \to \infty$ in 3.14, we have

$$\phi\left(G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right), G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right),\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right)\right) \to 1.$$

As $\phi\left(x_n, y_n, z_n\right) \to 1$it implies that $x_n, y_n, z_n \to 0$, we have

$$G\left(x_{m(k)}, x_{m(k)}, x_{l(k)}\right), G\left(y_{m(k)}, y_{m(k)}, y_{l(k)}\right),\ G\left(z_{m(k)}, z_{m(k)}, z_{l(k)}\right) \to 0.$$

a contradiction. Thus, the sequence$\left(x_n\right), \left(y_n\right)$, and $\left(z_n\right)$ are G-Cauchy in the G-metric space (X, G). Since (X, G) is complete G-metric space, hence $\left(x_n\right), \left(y_n\right)$, and $\left(z_n\right)$ are G-convergent. Then there exists $x, y, z \in X$ such that $\left(x_n\right), \left(y_n\right)$, and $\left(z_n\right)$ are G-convergent to x, y and z respectively. Since F is continuous. Letting $n \to \infty$ in 3.2, we have
$F\left(x,\ y,\ z\right) = x$, $F\left(y,\ x,\ y\right) = y$, and $F\left(z,\ y,\ x\right) = z$. Thus, we conclude that F has a tripled fixed point. □

3.1. **Uniqueness of tripled fixed point.** In this section we shall prove the uniqueness of tripled fixed point. For a tripled space X^3 of partial ordered set $(X,\ \preccurlyeq)$. We define a partial ordering as follows: for all $\left(x,\ y,\ z\right), \left(u,\ v,\ w\right) \in X^3$

$$(x,\ y,\ z) \preccurlyeq (u,\ v,\ w) \Leftrightarrow x \preccurlyeq u,\ y \succcurlyeq v,\ z \preccurlyeq w. \tag{4.1}$$

We say that $(x,\ y,\ z)$ is equal to $(u,\ v,\ w)$ if and only if $x = u,\ y = v,$ and $z = w$.

Theorem 3.2. *Adding the following condition to the hypothesis of theorem 8, suppose that for all $\left(x,\ y,\ z\right), \left(u,\ v,\ w\right) \in X^3$, there exists $(a,\ b,\ c) \in X^3$ that is comparable to $(x,\ y,\ z)$ and $\left(u,\ v,\ w\right)$, then F has a unique tripled fixed point.*

Proof. The set of tripled fixed point of F is not empty due to theorem 8. Assume, now, $(x,\ y,\ z)$ and $(u,\ v,\ w)$ are the tripled fixed point of F, that is,

$$
\begin{aligned}
F\left(x,\ y,\ z\right) &= x, & F\left(u,\ v,\ w\right) &= u, \\
F\left(y,\ x,\ y\right) &= y, & F\left(v,\ u,\ v\right) &= v, \\
F\left(z,\ y,\ x\right) &= z, & F\left(w,\ v,\ u\right) &= w,
\end{aligned}
$$

We shall show that $(x,\ y,\ z)$ and $(u,\ v,\ w)$ are equal. By assumption, there exists $(a,\ b,\ c) \in X^3$ that is comparable to $(x,\ y,\ z)$ and $\left(u,\ v,\ w\right)$. Define sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ such that

$$a = a_0, \quad b = b_0,\ c = c_0,$$

$$a_n = F(a_{n-1},\ b_{n-1},\ c_{n-1}),\ b_n = F(b_{n-1},\ a_{n-1},\ b_{n-1}),$$

and

$$c_n = F(c_{n-1}, b_{n-1}, a_{n-1}). \tag{3.15}$$

Since $(x,\ y,\ z)$ is comparable with $(a,\ b,\ c)$, we may assume that $(x,\ y,\ z) \geq (a,\ b,\ c) = (a_0,\ b_0,\ c_0)$. Recursively, we get that

$$(x, y, z) \geq (a_n, b_n, c_n) \ for\ all\ n. \tag{3.16}$$

Thus, from 3.1 we have

$$\begin{aligned} &\left(\tfrac{1}{3}\left[G\left(a_n, x, x\right) + G\left(b_n, y, y\right) + G\left(c_n, z, z\right)\right]\right) \\ &= \left(\tfrac{1}{3}\left[G\left(F\left(a_{n-1},\ b_{n-1},\ c_{n-1}\right), F\left(x,\ y,\ z\right), F\left(x,\ y,\ z\right)\right)\right]\right) \\ &+ \left(\tfrac{1}{3}\left[G\left(F\left(b_{n-1},\ a_{n-1},\ b_{n-1}\right), F\left(y,\ x,\ y\right), F\left(y,\ x,\ y\right)\right)\right]\right) \\ &+ \left(\tfrac{1}{3}\left[G\left(F\left(c_{n-1},\ b_{n-1},\ a_{n-1}\right), F\left(z,\ y,\ x\right), F\left(z,\ y,\ x\right)\right)\right]\right) \\ &\leq \phi\left(G\left(a_{n-1}, x, x\right) + G\left(b_{n-1}, y, y\right) + G\left(c_{n-1}, z, z\right)\right) \\ &\times \tfrac{1}{3}\left(G\left(a_{n-1}, x, x\right) + G\left(b_{n-1}, y, y\right) + G\left(c_{n-1}, z, z\right)\right) \end{aligned} \tag{3.17}$$

which implies

$$G(a_n, x, x) + G(b_n, y, y) + G(c_n, z, z) < G(a_{n-1}, x, x) + G(b_{n-1}, y, y) + G(c_{n-1}, z, z) \tag{3.18}$$

We see that the sequence $\left(G\left(a_n, x, x\right) + G\left(b_n, y, y\right) + G\left(c_n, z, z\right)\right)$ is decreasing, there exists some $\epsilon \geq 0$ such that

$$G(a_n, x, x) + G(b_n, y, y) + G(c_n, z, z) \to \epsilon \ as\ n \to \infty. \tag{3.19}$$

Now, we show that $\epsilon = 0$. On the contrary, suppose that $\epsilon > 0$. Following the same arguments as in the proof of Theorem 3.1, we have

$$\phi\left(G\left(a_{n-1}, x, x\right), G\left(b_{n-1}, y, y\right), G\left(c_{n-1}, z, z\right)\right) \to 1.$$

Implies,

$$G\left(a_{n-1}, x, x\right), G\left(b_{n-1}, y, y\right), G\left(c_{n-1}, z, z\right) \to 0.$$

Consequently, we have

$$G\left(a_n, x, x\right) + G\left(b_n, y, y\right) + G\left(c_n, z, z\right) \to 0,$$

a contradiction in virtue of (4.4). Hence, $\epsilon = 0$. Therefore, (4.6) becomes

$$G(a_n, x, x) + G(b_n, y, y) + G(c_n, z, z) \to 0 \ as\ n \to \infty. \tag{3.20}$$

Similarly, we can show that

$$G\left(a_n, u, u\right) + G\left(b_n, v, v\right) + G\left(c_n, w, w\right) \to 0 \ as\ n \to \infty \tag{3.21}$$

$$G(a_n, a_n, x) + G(b_n, b_n, y) + G(c_n, c_n, z) \to 0 \ as\ n \to \infty \tag{3.22}$$

$$G(a_n, a_n, u) + G(b_n, b_n, v) + G(c_n, c_n, w) \to 0 \ as\ n \to \infty. \tag{3.23}$$

Using $(4.7) - (4.10)$, the rectangle inequality and taking $n \to \infty$, we obtain $G\left(u, x, x\right) + G\left(v, y, y\right) + G\left(w, z, z\right) = 0$. Thus, we conclude that $x = u, y = v$ and $z = w$. Hence, F has a unique tripled fixed point. □

Conclusion: The existence and uniqueness of tripled fixed point for continuous mapping with mixed monotone property satisfying nonlinear contractions were proved. It was done in the framework of partial ordered metric space.

Acknowledgement: The authors would like to extend their sincere thanks to all that have contributed in one way or the other for the actualization of the review of this work.

Competing interests: The content of the manuscript was approved by all authors. Therefore, no competing interest between authors.

Funding: Authors received no financial support for the research, authorship, and/or publication of this article.

References

[1] Gnana-Bhaskar T., Lakshmikantham V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis.* **65**, 1379-1393.
[2] Mustafa Z., Sims B. (2006). A new approach to generalized metric spaces. *Journal of Nonlinear Convex Analysis.* **7**, 289-297.
[3] Mohluddine S. A., Alotaibi A. (2012).Some results on a tripled fixed points for nonlinear contractions in partially ordered G-metric spaces. *Fixed Point Theory Application.* 22 pages.
[4] Mustafa Z., Obiedat H., Awawdeh F. (2008). Some common fixed point theorems for mapping on complete G-metric spaces. *Fixed Point Theory Application.* Article **ID 189870**.
[5] Mustafa Z. (2005). *A New Structure for Generalized Metric spaces with Applications to Fixed Point Theory.* Ph. D. Thesis, The University of Newcastle, Callaghan, Australia,
[6] Mustafa Z., Sims B. (2003). Some results concerning D-metric spaces. *Proc. Internat. Conf. Fixed Point Theory and Applications.* 189-198, Valencia, Spain.
[7] Sedghi S., Shobe N., Zhou H. (2007). A common fixed point theorem in D^*-metric spaces. *Fixed Point Theory Applications.* Article **ID 27906**, 13 pages.
[8] Shatanawi W. (2010). Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces. *Fixed Point Theory Application.* Article ID 181650.
[9] Arab R., Rabbani M. (2014). Coupled coincidence and common fixed point theorems for mappings in partially ordered metric spaces. *Math. Sci. Lett.* **3** (2), 81-87.
[10] Arab R. (2014). Coupled fixed point theorems for two pairs of w-compactible mappings in G-metric spaces. *Sohag Journal of Mathematics.* **1** (1), 37-43.
[11] Berinde V. (2011). Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. *Nonlinear Analysis TMA.* **74** (18), 7347-7355.
[12] Shatanawi W., Samet B. (2012). Coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. *Math. Comput. Model.* **55** (34), 680-687.
[13] Berinde V., Borcut, M. (2011).Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Analysis.* **74** (15), 4889-4897.
[14] Turinici M. (1986). Abstract comparison principles and multivariable Gronwali-Bellman inequalities. *J. Math. Anal. Appl.* **117**, 100-127.
[15] Ran A. C., Reurings M. C. B. (2004).A fixed point theorems in partially ordered sets and some application to matric equations. *Proc. Amer. Math. Soc.* **132**, 1435-1443.
[16] Nieto J. J., Lopez R. R. (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order.* **22**, 223-239.