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Remarks on L^p-CKN Interpolation Inequalities of Hardy and Rellich Types on the Sphere

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Abstract

We prove a new family of L^p-Caffarelli-Kohn-Nirenberg (CKN) interpolation inequalities with sharp constants on the unit sphere of dimension greater than or equal to two. Consequently, several sharp Hardy type inequalities are obtained as special cases. Moreover, the same method is adopted to establish a new class of Hardy-Rellich type inequalities on the sphere, though, uncertain whelther or not the constants in this case are optimal in general.

1. Introduction

The prevalence and wide applications of the Hardy type inequalities in several areas of mathematics like partial differential equations, differential geometry, geometric analysis, harmonic analysis, spectral theory, mathematical physics and so on cannot be overestimated [5, 6, 10, 14, 15, 17, 18, 20, 28]. The classical L^p-Hardy inequality [21] takes the form

$$\mathscr{A}_H \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} |\nabla f|^p dx \tag{1.1}$$

for every smooth compactly supported function f in $\mathbb{R}^n$, with optimal constant (not attained by a nontrivial function) $\mathscr{A}_H := ((n-p)/p)^p$, $1 < p < n$. The Hardy inequality (1.1) turns out to be a special case of a family of interpolation inequalities, popularly called Caffarelli-Kohn-Nirenberg (CKN) inequalities [11] which on its own states that for any $f \in C_0^\infty(\mathbb{R}^n)$, there exists a constant $\mathscr{A}_C > 0$ such that

$$\||x|^c f\|_{L^r(\mathbb{R}^n)} \leq \mathscr{A}_C \||x|^a \nabla f\|^{\delta}_{L^p(\mathbb{R}^n)} \||x|^b f\|^{(1-\delta)}_{L^q(\mathbb{R}^n)}, \tag{1.2}$$

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where $r > 0$, $p, q \geq 1$, $a, b, c \in \mathbb{R}$ and $\delta \in (0, 1]$ satisfying
$\frac{1}{r} + \frac{c}{n},\ \frac{1}{p} + \frac{a}{n},\ \frac{1}{q} + \frac{b}{n} > 0,\ c = \delta\sigma + (1-\delta)b$
if and only if

$$\frac{1}{r} + \frac{c}{n} = \delta\left(\frac{1}{p} + \frac{a-1}{n}\right) + (1-\delta)\left(\frac{1}{q} + \frac{b}{n}\right) \tag{1.3}$$

with $0 \leq \delta - \sigma$ if $\delta > 0$ and $a - \sigma \leq 1$ if $\frac{1}{r} + \frac{c}{n} = \frac{1}{p} + \frac{a-1}{n}$.
For instance, CKN inequality (1.2) yields Hardy inequality (1.1) if $a = 0$, $\delta = 1$ and $r = p$ are chosen, while it reduces to the L^2-Sobolev inequality by choosing $a = 0$, $\delta = 1$, $p = 2$ and $r = 2n/(n-2)$. It is therefore worth mentioning that the CKN inequalities also contain as special cases some other interesting and useful functional inequalities such as Gagliardo-Nirenberg inequalities, Sobolev-Hardy inequalities, Heisenberg-Pauli-Weyl (HPW) inequalities and Nash inequalities. This family of inequalities (1.2) (resp. Hardy inequalities (1.1)) have been generalized to some other settings like Riemannian manifolds, metric measure spaces, homogeneous groups, hyperbolic spaces and likes. We refer to [5, 9, 15, 22, 23, 24, 28] and the references therein for details.
We remark that inequalities (1.1) and (1.2) cannot be readily imported on to the unit n-sphere since the length $|x|$ is unity. In fact, they cannot also be transfered directly to Riemannian manifold since $|x|$ is not even defined for a point x in a Riemannian manifold. So different techniques have been engaged for the case of Riemannian manifold [9, 27, 28]. Since the unit n-sphere can be viewed as a submanifold of $\mathbb{R}^{n+1}$ with sectional curvature 1, it is then natural to consider the geodesic distance from a fixed point $q \in \mathbb{S}^n$ [1, 2, 30]. The interest in obtaining sharp Hardy type inequalities on the sphere began very recently with [29] where the inequalities were obtained for $n \geq 3$ in L^2-setting. The method in [29] made use of geodesic distance from the pole, thereby restricting the singularity to polar. This have been generalized to L^p-case in [1, 25] and [2, 3] where geodesic distance from an arbitrary point on the sphere was considered. The limiting case for $n = 2$ was considered in [7] while critical and subcritical exponents cases (with general geodesic distance) were considered for $n > 2$ in [8]. See also [4, 30, 31]. In literature, one can find some other interpolation inequalities [16] including Gagliardo-Nirenberg-Sobolev inequalities proved by the heat flow approach. It is observed that there is a dearth of literature on CKN inequalities on the sphere, a tleast to the best of our knowledge. This then prompts the consideration of the present paper.
The first goal of this paper therefore is to derive a more general family of L^p-interpolation inequalities ($1 < p < \infty$), which we refer to as the L^p-CKN inequalities of Hardy type, on the n-sphere. The method of proof is more direct and simpler when compared to some other methods, as it only makes use of elementary properties of Laplacian of the distance function on the sphere, the divergence theorem and Hölder's inequality. The application of Hölder's inequality here accounts for the optimality of the inequality as the equality can only be (possibly) achieved by constant functions, while the sharpness of the constant can be shown by considering extremal functions which achieve equality in the Hölder's inequality. The motivation for adopting this approach follows from [12, 13, 19]. Secondly, we will use the same approach (as used for Hardy type inequalities) to derive L^p-Rellich type interpolation inequalities, though, uncertain whelther or not the constant in this case is sharp. Note that Rellich type inequalities are usually obtained from Hardy inequalities as a consequence of Hölder's inequality. Further discussion on interesting consequences of these inequalities are made. Specifically, sharp L^p-Hardy type inequalities are derived to complement existing literature.

Preliminaries on the sphere. Denote $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ as the unit n-sphere $(n \geq 2)$ equipped with canonical Riemann surface measure ν_x in $\mathbb{R}^{n+1}$. Let $r_q := d(\cdot, q) : \mathbb{S}^n \to [0, \pi]$ be the geodesic distance from a fixed point $q \in \mathbb{S}^n$.
Througout, we denote by ∇_x, the gradient operator and by $\Delta_x := \mathrm{div}\nabla_x$, the Laplace-Beltrami operator on $\mathbb{S}^n$. Consider the functions $u, v \in W_0^{1,p}(\mathbb{S}^n)$, where $W_0^{1,p}(\mathbb{S}^n)$ is the completion of $C_0^\infty(\mathbb{S}^n)$ in the norm

$$\|u\|_{W_0^{1,p}(\mathbb{S}^n)} = \|\nabla_x u\|^p_{L^p(\mathbb{S}^n)}.$$

By the divergence theorem

$$\int_{\mathbb{S}^n} u\mathrm{div}V d\nu_x = -\int_{\mathbb{S}^n} \langle \nabla_x u, V\rangle d\nu_x, \tag{1.4}$$

where V is any smooth vector field on $\mathbb{S}^n$ and $\langle\cdot,\cdot\rangle$ is the inner product induced by the Riemann metric from $\mathbb{S}^n \in \mathbb{R}^{n+1}$. Whilst integration by parts yields

$$\int_{\mathbb{S}^n} \Delta_x uv d\nu_x = -\int_{\mathbb{S}^n} \langle \nabla_x u, \nabla_x v\rangle d\nu_x = \int_{\mathbb{S}^n} u\Delta_x v d\nu_x. \tag{1.5}$$

The geodesic distance r_q has the gradient and Laplacian (in distribution sense) which respectively satisfy

$$|\nabla_x r_q| = 1 \;\text{ and } \Delta_x r_q = (n-1)\cot r_q, \;(n > 1). \tag{1.6}$$

In the next section (Section 2) a new family of L^p-interpolation inequalities of Caffareli-Kohn-Nirenberg type is established on the unit n-sphere $\mathbb{S}^n$, and as a consequence sharp Hardy type inequalities are derived. The last section (Section 3) presents a new class of Hardy-Rellich type inequalities which relates first derivative of a function to its second derivatives.

2. L^p-CKN interpolation inequalities of Hardy type

In this section we derive a new family of L^p-interpolation inequalities of Caffareli-Kohn-Nirenberg type on the unit n-sphere $\mathbb{S}^n$, $n \geq 2$ and then present some of its consequences. Specifically, sharp Hardy type inequalities are obtained as special cases. We adopt a simple and direct method of proof using elementary properties of Laplacian of the distance function, divergence theorem and Hölder's inequality. The main result is stated as follows:

Theorem 2.1. *Let $\mathbb{S}^n, n \geq 2$ be the unit sphere in $\mathbb{R}^{n+1}$. For all $a, b \in \mathbb{R}$ and for all $f \in C_0^\infty(\mathbb{S}^n)$, there holds*

$$\begin{aligned}\mathscr{C}_p \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{\delta+1}} d\nu_x \leq &\left(\int_{\mathbb{S}^n} \frac{|\nabla_x f|^p}{|\tan r_q|^{ap}} d\nu_x\right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{b\frac{p}{p-1}}} d\nu_x\right)^{\frac{p-1}{p}} \\ &+ \frac{\delta}{p}\int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{\delta-1}} d\nu_x,\end{aligned} \tag{2.1}$$

where $1 < p < \infty$, $n \neq a+b+1$, $\delta = a+b$ and $\mathscr{C}_p := |(n-\delta-1)/p|$ is sharp.

Proof. In the sequel we write $\nabla := \nabla_x$, $\Delta := \Delta_x$ and $d\nu := d\nu_x$. By a simple computation for $\delta \in \mathbb{R}$

$$\begin{aligned}\mathrm{div}\left(\frac{\nabla r_q}{|\tan r_q|^\delta}\right) &= \frac{\Delta r}{|\tan r_q|^\delta} - \frac{\delta \sec^2 r_q}{|\tan r_q|^{\delta+1}} \\ &= \frac{n-1-\delta}{|\tan r_q|^{\delta+1}} - \frac{\delta}{|\tan r_q|^{\delta-1}},\end{aligned}$$

where we have used the two equations in (1.6) and trigonometric identity $\tan^2 r+1 = \sec^2 r$, $r \in [0,\pi]$. Then for $f \in C_0^\infty(\mathbb{S}^n)\setminus\{0\}$ and $1<p<\infty$ we have

$$\begin{aligned}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\delta+1}}d\nu &= \frac{1}{n-1-\delta}\int_{\mathbb{S}^n}|f|^p\mathrm{div}\left(\frac{\nabla r_q}{|\tan r_q|^{\delta}}\right)d\nu\\ &+\frac{\delta}{n-1-\delta}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\delta-1}}d\nu,\end{aligned} \tag{2.2}$$

By the application of divergence theorem and Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{n-1-\delta}\int_{\mathbb{S}^n}|f|^p\mathrm{div}\left(\frac{\nabla r_q}{|\tan r_q|^{\delta}}\right)d\nu &= \frac{-p}{n-1-\delta}\int_{\mathbb{S}^n}|f|^{p-2}f\frac{\langle\nabla f,\nabla r_q\rangle}{|\tan r_q|^{\delta}}d\nu\\ &\leq \left|\frac{-p}{n-1-\delta}\right|\int_{\mathbb{S}^n}|f|^{p-1}\frac{|\nabla f|}{|\tan r_q|^{\delta}}d\nu\\ &\leq \left|\frac{p}{n-1-\delta}\right|\left(\int_{\mathbb{S}^n}\frac{|\nabla_x f|^p}{|\tan r_q|^{ap}}d\nu\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{(\delta-a)\frac{p}{p-1}}}d\nu\right)^{\frac{p-1}{p}}.\end{aligned} \tag{2.3}$$

Inserting (2.3) into (2.2), rearranging and reverting to $\delta = a+b$ yields the desired result (2.1).

This approach clearly gives optimal constant. Thus, the the sharpness of the constant $\mathscr{C}_p$ can be proved by considering the condition for equality in the Hölder's inequality used as in [12, 19, 23].

□

Next we consider special cases of (2.1) for different values of a and b.

Corollary 2.2. *The inequality (2.1) implies:*

(i) When $a=0$ and $b=p-1$ for $1<p<n$

$$\begin{aligned}\frac{n-p}{p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^p}d\nu_x &\leq \left(\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^p}d\nu_x\right)^{\frac{p-1}{p}}\\ &+\frac{p-1}{p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{p-2}}d\nu_x.\end{aligned}$$

(ii) When $a=0$ and $b=p$ for $1<p<n-1$

$$\begin{aligned}\frac{n-p-1}{p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{p+1}}d\nu_x &\leq \left(\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\frac{p^2}{p-1}}}d\nu_x\right)^{\frac{p-1}{p}}\\ &+\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{p-2}}d\nu_x.\end{aligned}$$

(iii) When $a+b-1=0$ and $p=2$ for $n\geq 3$

$$\frac{n-2}{2}\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 r_q}d\nu_x \leq \left(\int_{\mathbb{S}^n}\frac{|\nabla_x f|^2}{|\tan r_q|^{2a}}d\nu_x\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^n}\frac{f^2}{|\tan r_q|^{2b}}d\nu_x\right)^{\frac{1}{2}}+\frac{1}{2}\int_{\mathbb{S}^n}f^2 d\nu_x.$$

(iv) When $a=0$, $b=1$ and $p=2$ for $n\geq 3$

$$\frac{n-2}{2}\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 r_q}d\nu_x \leq \left(\int_{\mathbb{S}^n}|\nabla_x f|^2 d\nu_x\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 r_q}d\nu_x\right)^{\frac{1}{2}}+\frac{1}{2}\int_{\mathbb{S}^n}f^2 d\nu_x.$$

(v) *When* $a+b=0$ *and* $a=-p$ *for* $n\geq 2$

$$\frac{n-1}{p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|}d\nu_x\leq\left(\int_{\mathbb{S}^n}|\nabla_x f|^p|\tan r_q|^{p^2}d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\frac{p^2}{p-1}}}d\nu_x\right)^{\frac{p-1}{p}}.$$

(vi) *When* $a+b+1=ap$

$$\left|\frac{n-ap}{p}\right|\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{ap}}d\nu_x\leq\left(\int_{\mathbb{S}^n}\frac{|\nabla_x f|^p}{|\tan r_q|^{ap}}d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{ap-\frac{p}{p-1}}}d\nu_x\right)^{\frac{p-1}{p}}$$
$$+\frac{ap-1}{p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\delta-1}}d\nu_x.$$

(vii) *When* $a+b+1=0$ *and* $a=-2$

$$\frac{n}{p}\int_{\mathbb{S}^n}|f|^p d\nu_x+\frac{1}{p}\int_{\mathbb{S}^n}|f|^p\tan^2 r d\nu_x\leq\left(\int_{\mathbb{S}^n}|\nabla_x f|^p|\tan r_q|^{2p}d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{\frac{p}{p-1}}}d\nu_x\right)^{\frac{p-1}{p}}.$$

(viii) *When* $a+b+1=0$ *and* $a=0$

$$\frac{n}{p}\int_{\mathbb{S}^n}|f|^p d\nu_x+\frac{1}{p}\int_{\mathbb{S}^n}|f|^p\tan^2 r d\nu_x\leq\left(\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}|f|^p|\tan r_q|^q d\nu_x\right)^{\frac{1}{q}}$$

with $q=p/(p-1)$.

(ix) *When* $a+b+1=0$ *and* $a=1$

$$\frac{n}{p}\int_{\mathbb{S}^n}|f|^p d\nu_x+\frac{1}{p}\int_{\mathbb{S}^n}|f|^p\tan^2 r d\nu_x\leq\left(\int_{\mathbb{S}^n}\frac{|\nabla_x f|^p}{|\tan r|^p}d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}|f|^p|\tan r_q|^{2q}d\nu_x\right)^{\frac{1}{q}}$$

with $q=p/(p-1)$.

(x) *When* $a+b+1=0$ *and* $a=-p$

$$\frac{n}{p}\int_{\mathbb{S}^n}|f|^p d\nu_x+\frac{1}{p}\int_{\mathbb{S}^n}|f|^p\tan^2 r d\nu_x\leq\left(\int_{\mathbb{S}^n}|\nabla_x f|^p|\tan r|^{p^2}d\nu_x\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r|^p}d\nu_x\right)^{\frac{p-1}{p}}.$$

□

Remark. (a) The cases (i) and (iii) in Corollary 3.2 are respectively L^p and L^2 versions of Hardy type inequalities on the sphere. They extend the n-Euclidean cases obtained in [19] and [12], respectively.

(b) The case (iv) is a special case of (iii), that is, when $a=0$ in (iii) one obtains (iv) at once. The case (iv) has been obtained in [31] and the authors showed that the constant $(n-2)/2$ is sharp.

(c) The case (v) is a version of HPW uncertainty principle with important consequences in harmonic analysis. See [23] for homogeneous group version.

(d) The cases $a+b+1=0$ (that is, cases (vii)–(x)) are not equivalent but each could be interesting on its own as they also illustrate HPW uncertainty principle.

(e) Two special cases of (vi), when $a=1$ and when $a=0$, are interesting. They can be compared with cases (i) and (viii), respectively.

Going by the first case in Corollary 3.2 we can state the following.

Proposition 2.3. *For $2 \le p < n$, $n \ge 3$ and $f \in C_0^\infty(\mathbb{S}^n) \setminus \{0\}$, it holds that*

$$C_p \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x \le \left(\int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x \right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x \right)^{\frac{p-1}{p}} + \frac{p-1}{p} \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{p-2}} d\nu_x, \tag{2.4}$$

where $r_q := d(q,x)$ with q being a fixed point in $\mathbb{S}^n$. Moreover, the constant $C_p := (n-p)/p$ is sharp.

When $p = 2$, (2.4) is the case (iii) of Corollary 3.2, which has been proved in [31] with sharp constant $C_2 := (n-2)/2$. Note that inequality (2.4) can be proved in the same way to the proof of Theorem 2.1 but using the following quantity instead

$$\mathrm{div}\left(\frac{\nabla r_q}{|\tan r_q|^{p-2} \tan r_q} \right) = \frac{n-p}{|\tan r_q|^p} - \frac{p-1}{|\tan r_q|^{p-2}}.$$

Proceeding from here we prove a sharp L^p-Hardy type inequality, which has been derived in [2, 8] using different approach.

Theorem 2.4. *For $2 \le p < n$, $n \ge 3$ and $q \in \mathbb{S}^n$, then there exists a positive constant $\mathscr{A}_{n,p}$ such that for all $f \in C_0^\infty(\mathbb{S}^n) \setminus \{0\}$*

$$\mathscr{C}_{n,p} \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x \le \int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x + \mathscr{A}_{n,p} \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{p-2}} d\nu_x, \tag{2.5}$$

where $\mathscr{C}_{n,p} := ((n-p)/p)^p$, $\mathscr{A}_{n,p} := (p-1)((n-p)/p)^{p-1}$ and $r_q := d(q,x)$ with q being a fixed point in $\mathbb{S}^n$. Moreover, the constant $\mathscr{C}_{n,p}$ is sharp.

Proof. We will proceed from (2.4) of Proposition 2.3. By the Young's inequality of the form $AB \le (\alpha A)^p/p + (p-1)(\alpha^{-1}B)^{p/(p-1)}/p$ for a positive nunmber α and $A, B > 0$. Taking $A = (\int |\nabla_x f|^p)^{1/p}$ and $B = (\int \frac{|f|^p}{|\tan r_q|^p})^{p-1/p}$ we obtain

$$\left(\int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x \right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x \right)^{\frac{p-1}{p}} \le \frac{\alpha^p}{p} \int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x + \frac{p-1}{p} \alpha^{-\frac{p}{p-1}} \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x. \tag{2.6}$$

Using (2.6) in (2.4) leads to

$$\Psi(\alpha) \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x \le \int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x + \alpha^{-p}(p-1) \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{p-2}} d\nu_x, \tag{2.7}$$

where $\Psi(\alpha) := \alpha^{-1}(n-p-(p-1)\alpha^{-p/(p-1)}$.
By the maximization procedure, it is obvious that the quantity $\alpha \mapsto \Psi(\alpha)$ attains its maximum at the point $\alpha_\star = (p/(n-p))^{(p-1)/p}$. Consequently, the maximum value attained is $\max_{\alpha>0} \Psi(\alpha) = \Psi(\alpha_\star) = ((n-p)/p)^p$. Hence, using $\alpha_\star$ and $\Psi(\alpha_\star)$ in (2.7) gives the required inequality (2.5).
Moreover, the constant $\mathscr{C}_{n,p} = ((n-p)/p)^p$ is sharp in the sense that

$$\inf_{f \in C_0^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{\int_{\mathbb{S}^n} |\nabla_x f|^p d\nu_x + \mathscr{A}_{n,p} \int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^{p-2}} d\nu_x}{\int_{\mathbb{S}^n} \frac{|f|^p}{|\tan r_q|^p} d\nu_x} = \left(\frac{n-p}{p} \right)^p. \tag{2.8}$$

The proof of (2.8) can be found in [2] and [8]. In a similar vein, the constant $\mathscr{A}_{n,p}$ can also be shown to be sharp (see [8]) in the sense that

$$\sup_{f\in C_0^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{\mathscr{C}_{n,p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^p}d\nu_x - \int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x}{\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{p-2}}d\nu_x} = (p-1)\left(\frac{n-p}{p}\right)^{p-1}.$$

Remark. (a). Letting $p=2$ in (2.5) we obtain for $n\geq 3$

$$\mathscr{C}_{n,2}\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 r}d\nu_x \leq \int_{\mathbb{S}^n}|\nabla_x f|^2 + \frac{n-2}{2}\int_{\mathbb{S}^n} f^2 d\nu_x \tag{2.9}$$

which has been derived in [30] through a different method and the constant $\mathscr{C}_{n,2}:=(n-2)^2/4$ was shown to be sharp.

(b). The Hardy inequality (2.5) implies the following HPW type uncertainty principle on the unit n-sphere for all $f\in C_0^\infty(\mathbb{S}^n)\setminus\{0\}$

(2.10)

$$\mathscr{C}_{n,p}\left(\int_{\mathbb{S}^n}|f|d\nu\right)^p \leq \left(\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu + \mathscr{A}_{n,p}\int_{\mathbb{S}^n}\frac{|f|^p}{|\tan r_q|^{p-2}}d\nu\right)\left(\int_{\mathbb{S}^n}|f||\tan r_q|^{\frac{p}{p-1}}d\nu\right)^{p-1}.$$

This is a consequence of Hölder's inequality, (see [2]).

The last remark in this section concerns the critical L^n Hardy type inequality on $\mathbb{S}^n$. Observe that in the limiting case $p=n$ the inequality (1.1) (also inequality (2.5)) makes no sense for any constant as the weight $\frac{1}{|x|^n}$ becomes too singular for the inequality to hold for a function in $W_0^{1,p}$. The following Hardy inequality (instead of (1.1))

$$\left(\frac{n-1}{n}\right)^n\int_\Omega\frac{|f|^n}{|x|^n(\log\frac{R}{|x|})^n}dx \leq \int_\Omega|\nabla f|^n dx$$

holds in the critical case for $W_0^{1,p}(\Omega)$, $\Omega\subset\mathbb{R}^n$ and $R=\sup_{x\in\Omega}|x|$, $|x|\neq 0, R$, again with the constant $(\frac{n-1}{n})^n$ known to be optimal. See for example [26] and the references therein.

Remark. Combining the argument in [8, Section 5] with the method of proof used in Theorem 2.1 for the quantity $\operatorname{div}\left(\frac{\nabla r_q}{|\tan r_q|^{p-2}\tan r_q(\log\frac{e}{\sin r_q})}\right)$, we remark that the following critical Hardy type inequality holds for $f\in C_0^\infty(\mathbb{S}^n)$, $n\geq 2$,

$$\begin{aligned}\mathscr{C}_n\int_{\mathbb{S}^n}\frac{|f|^n}{|\tan r_q|^n(\log\frac{e}{\sin r_q})^n}d\nu_x &\leq \left(\int_{\mathbb{S}^n}|\nabla_x f|^n d\nu_x\right)^{\frac{1}{n}}\left(\int_{\mathbb{S}^n}\frac{|f|^n}{|\tan r_q|^n(\log\frac{e}{\sin r_q})^n}d\nu_x\right)^{\frac{n-1}{n}}\\ &\quad + \mathscr{C}_n\int_{\mathbb{S}^n}\frac{|f|^n}{|\tan r_q|^{n-2}(\log\frac{e}{\sin r_q})^{n-2}}d\nu_x,\end{aligned} \tag{2.11}$$

where $\mathscr{C}_n:=(n-1)/n$.

Indeed, following the similar steps for proving (2.5) in Theorem 2.4 one sees that (2.11) implies

$$\begin{aligned}\mathscr{C}_n^n\int_{\mathbb{S}^n}\frac{|f|^n}{|\tan r_q|^n(\log\frac{e}{\sin r_q})^n}d\nu_x &\leq \int_{\mathbb{S}^n}|\nabla_x f|^n d\nu_x\\ &\quad + (n-1)\mathscr{C}_n^{n-1}\int_{\mathbb{S}^n}\frac{|f|^n}{|\tan r_q|^{n-2}(\log\frac{e}{\sin r_q})^{n-2}}d\nu_x,\end{aligned}$$

which is exactly the inequality (53) in [8] with optimal constants $\mathscr{C}_n^n$ and $(n-1)\mathscr{C}_n^{n-1}$.

3. A new class of L^p-Hardy-Rellich type inequalities

Here we discuss a new class of L^p-Hardy-Rellich type inequalities on the unit sphere and some of its consequences. The method of proof is similar to the one in previous section.

Theorem 3.1. *Let $\mathbb{S}^n, n \geq 2$ be the unit sphere in $\mathbb{R}^{n+1}$. For $a, b \in \mathbb{R}$ and for all $f \in C_0^\infty(\mathbb{S}^n)$, there holds for $1 < p < n$*

$$C_{p,\delta} \int_{\mathbb{S}^n} \frac{|\nabla_x f|^p}{|\tan r_q|^{\delta+1}} d\nu_x \leq \left(\int_{\mathbb{S}^n} \frac{|\Delta_{x,p} f|^p}{|\tan r_q|^{ap}} d\nu_x \right)^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} \frac{|\nabla_x f|^q}{|\tan r_q|^{bq}} d\nu_x \right)^{\frac{1}{q}} + \frac{\delta(1-p)}{p} \int_{\mathbb{S}^n} \frac{|\nabla_x f|^p}{|\tan r_q|^{\delta-1}} d\nu_x, \tag{3.1}$$

where $p = \frac{p}{p-1}$, $\delta = a + b$, $\frac{p-n}{p-1} \leq \delta + 1 \leq 0$, $C_{p,\delta} := \frac{n-p+(\delta+1)(p-1)}{p} \geq 0$ and $\Delta_{x,p} f := div(|\nabla_x f|^{p-2} \nabla_x f)$ is the p-Laplacian on $\mathbb{S}^n$.

Proof. Here we write $\nabla := \nabla_x$, $\Delta_p := \Delta_{x,p}$ and $d\nu_x := d\nu$. Clearly, a simple computation as before using the quantity $\mathrm{div}(\frac{\nabla r_q}{|\tan r_q|^\delta})$ for $\delta \in \mathbb{R}$ and application of the divergence theorem yields for any $f \in C_0^\infty(\mathbb{S}^n)$ and $1 < p < \infty$,

$$\begin{aligned} \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}} d\nu &= \frac{1}{n-1-\delta} \int_{\mathbb{S}^n} |\nabla f|^p \mathrm{div} \left(\frac{\nabla r_q}{|\tan r_q|^\delta} \right) d\nu \\ &\quad + \frac{\delta}{n-1-\delta} \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}} d\nu \\ &= -\frac{p}{n-1-\delta} \int_{\mathbb{S}^n} |\nabla f|^{p-2} \nabla^2 f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} d\nu \\ &\quad + \frac{\delta}{n-1-\delta} \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}} d\nu. \end{aligned} \tag{3.2}$$

On the other hand, we have also by applying divergencce theorem

$$\begin{aligned} \int_{\mathbb{S}^n} \Delta_p f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} d\nu &= -\int_{\mathbb{S}^n} |\nabla f|^{p-2} \nabla f \nabla \left(\frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} \right) d\nu \\ &= -\int_{\mathbb{S}^n} |\nabla f|^{p-2} \left(\nabla^2 f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} + \frac{|\nabla f|^2 (\nabla^2 r_q)}{|\tan r_q|^\delta} - \frac{\delta \langle \nabla f, \nabla r_q \rangle^2 \sec^2 r_q}{|\tan r_q|^{\delta+1}} \right) d\nu \\ &= -\int_{\mathbb{S}^n} |\nabla f|^{p-2} \left(\nabla^2 f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} - \delta \frac{|\nabla f|^2}{|\tan r_q|^{\delta+1}} - \delta \frac{|\nabla f|^2}{|\tan r_q|^{\delta-1}} \right) d\nu, \end{aligned}$$

where we have used the first equation in (1.6),trigonometric identity $\tan^2 r + 1 = \sec^2 r$ and the fact that $\nabla^2 r_q = 0$ (Hessian of distance function). Thus

$$\begin{aligned} \int_{\mathbb{S}^n} |\nabla f|^{p-2} \nabla^2 f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} d\nu &= \delta \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}} + \delta \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}} d\nu \\ &\quad - \int_{\mathbb{S}^n} \Delta_p f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} d\nu. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) we have

$$\begin{aligned} \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}} d\nu = -\frac{p}{n-1-\delta} \Big(\delta \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}} + \delta \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}} d\nu \\ - \int_{\mathbb{S}^n} \Delta_p f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^\delta} d\nu \Big) + \frac{\delta}{n-1-\delta} \int_{\mathbb{S}^n} \frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}} d\nu. \end{aligned}$$

Simplyfying the last equation further we arrive at

$$C_{p,\delta}\int_{\mathbb{S}^n}\frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}}d\nu=\int_{\mathbb{S}^n}\Delta_p f\frac{\langle\nabla f,\nabla r_q\rangle}{|\tan r_q|^{\delta}}d\nu+\frac{\delta(1-p)}{p}\int_{\mathbb{S}^n}\frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}}d\nu, \tag{3.4}$$

where $C_{p,\delta}:=\frac{n-1+\delta(p-1)}{p}$. Then by the Hölder's inequality

$$\begin{aligned}\int_{\mathbb{S}^n}\Delta_p f\frac{\langle\nabla f,\nabla r_q\rangle}{|\tan r_q|^{\delta}}d\nu&\le\int_{\mathbb{S}^n}\frac{|\Delta_p f||\nabla f|}{|\tan r_q|^{\delta}}d\nu\\&\le\Big(\int_{\mathbb{S}^n}\frac{|\Delta_p f|^p}{|\tan r_q|^{ap}}d\nu\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}\frac{|\nabla f|^q}{|\tan r_q|^{(\delta-a)q}}d\nu\Big)^{\frac{1}{q}}\end{aligned} \tag{3.5}$$

Inserting (3.5) into (3.4), then

$$\begin{aligned}C_{p,\delta}\int_{\mathbb{S}^n}\frac{|\nabla f|^p}{|\tan r_q|^{\delta+1}}d\nu&\le\Big(\int_{\mathbb{S}^n}\frac{|\Delta_p f|^p}{|\tan r_q|^{ap}}d\nu\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}\frac{|\nabla f|^q}{|\tan r_q|^{(\delta-a)q}}d\nu\Big)^{\frac{1}{q}}\\&\quad+\frac{\delta(1-p)}{p}\int_{\mathbb{S}^n}\frac{|\nabla f|^p}{|\tan r_q|^{\delta-1}}d\nu.\end{aligned}$$

Finally, reverting to $\delta=a+b$ and the condition $\frac{p-n}{p-1}\le\delta+1\le 0$, we note that $C_{p,\delta}:=\frac{n-1+\delta(p-1)}{p}=\frac{n-p+(\delta+1)(p-1)}{p}\ge 0$ and then arrive at (3.1) which is the desired inequality. □

Next we consider special cases of (3.1) for different values of a and b.

Corollary 3.2. *The inequality (3.1) implies:*

(i) If $a=0$ and $b=0$ then for $n\ge 2$

$$\frac{n-1}{p}\int_{\mathbb{S}^n}\frac{|\nabla_x f|^p}{\tan r_q}d\nu_x\le\Big(\int_{\mathbb{S}^n}|\Delta_{x,p}f|^p d\nu_x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}|\nabla_x f|^q d\nu_x\Big)^{\frac{1}{q}}.$$

(ii) If $a+b+1=0$, $a=-1$ and $b=0$ then for $1<p<n$

$$\begin{aligned}\frac{n-p}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x&\le\Big(\int_{\mathbb{S}^n}|\Delta_{x,p}f|^p|\tan r_q|^p d\nu_x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}|\nabla_x f|^q d\nu_x\Big)^{\frac{1}{q}}\\&\quad+\frac{p-1}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p\tan^2 r_q d\nu_x.\end{aligned}$$

(iii) If $a+b+1=0$, $a=0$ and $b=-1$then for $1<p<n$

$$\begin{aligned}\frac{n-p}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x&\le\Big(\int_{\mathbb{S}^n}|\Delta_{x,p}f|^p d\nu_x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}|\nabla_x f|^q|\tan r_q|^q d\nu_x\Big)^{\frac{1}{q}}\\&\quad+\frac{p-1}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p\tan^2 r_q d\nu_x.\end{aligned}$$

(iv) If $a+b+1=0$, $a=1$ and $b=-2$ then for $1<p<n$

$$\begin{aligned}\frac{n-p}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p d\nu_x&\le\Big(\int_{\mathbb{S}^n}\frac{|\Delta_{x,p}f|^p}{|\tan r_q|^p}d\nu_x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^n}|\nabla_x f|^q|\tan r_q|^{2q}d\nu_x\Big)^{\frac{1}{q}}\\&\quad+\frac{p-1}{p}\int_{\mathbb{S}^n}|\nabla_x f|^p\tan^2 r_q d\nu_x.\end{aligned}$$

□

Theorem A1 (L^2-CKN interpolation inequality of Hardy type). For $f \in C_0^\infty(\mathbb{S}^n \setminus \{0\})$ and all $a, b \in \mathbb{R}$. It holds that

$$\begin{aligned} C_H(a,b)\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b+1}} d\nu_x \leq \left(\int_{\mathbb{S}^n} \frac{|\nabla_x f|^2}{|\tan r_q|^{2a}} d\nu_x\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{2b}} d\nu_x\right)^{\frac{1}{2}} \\ + \frac{a+b}{2}\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b-1}} d\nu_x, \end{aligned} \tag{3.6}$$

with sharp constant $C_H(a,b) := \frac{|n-(a+b+1)|}{2}$, where $n \neq a+b+1$.

Proof. For all $a, b \in \mathbb{R}$, $f \in C_0^\infty(\mathbb{S}^n \setminus \{0\})$ and $s \in \mathbb{R}$ we have

$$\int_{\mathbb{S}^n} \left| \frac{\nabla f}{|\tan r_q|^a} + \frac{sf}{|\tan r_q|^b} \nabla r_q \right|^2 d\nu \geq 0,$$

that is,

$$\int_{\mathbb{S}^n} \frac{|\nabla f|^2}{|\tan r_q|^{2a}} d\nu + s^2 \int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{2b}} d\nu + 2s \int_{\mathbb{S}^n} f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^{a+b}} d\nu \geq 0,$$

which can be compared with the quadratic inequality $As^2 + Bs + C \geq 0$ with

$$A := \int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{2b}} d\nu, \; B := 2\int_{\mathbb{S}^n} f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^{a+b}} d\nu \text{ and } C := \int_{\mathbb{S}^n} \frac{|\nabla f|^2}{|\tan r_q|^{2a}} d\nu.$$

Simplifying B further by the application of divergence theorem:

$$\begin{aligned} \int_{\mathbb{S}^n} f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^{a+b}} d\nu &= -\int_{\mathbb{S}^n} f \text{div}\left(\frac{f \nabla r_q}{|\tan r_q|^{a+b}}\right) d\nu \\ &= -\int_{\mathbb{S}^n} f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^{a+b}} d\nu - \int_{\mathbb{S}^n} |f|^2 \frac{\Delta r_q}{|\tan r_q|^{a+b}} d\nu + \\ &+ (a+b)\int_{\mathbb{S}^n} \frac{|f|^2 |\nabla r_q|^2 \sec^2 r_q}{|\tan r_q|^{a+b+1}} d\nu. \end{aligned}$$

Hence,

$$\begin{aligned} B &:= 2\int_{\mathbb{S}^n} f \frac{\langle \nabla f, \nabla r_q \rangle}{|\tan r_q|^{a+b}} d\nu \\ &= -(n-1)\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b+1}} d\nu + (a+b)\int_{\mathbb{S}^n} \frac{|f|^2(1+\tan^2 r_q)}{|\tan r_q|^{a+b+1}} d\nu \\ &= -(n-(a+b+1))\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b+1}} d\nu + (a+b)\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b-1}} d\nu. \end{aligned}$$

Note that $As^2 + Bs + C \geq 0$ is equivalent to $B^2 - 4AC \leq 0$. Thus,

$$\begin{aligned} \left[(n-(a+b+1))\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b+1}} d\nu - (a+b)\int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{a+b-1}} d\nu\right]^2 \\ \leq 4\int_{\mathbb{S}^n} \frac{|\nabla f|^2}{|\tan r_q|^{2a}} d\nu \int_{\mathbb{S}^n} \frac{|f|^2}{|\tan r_q|^{2b}} d\nu, \end{aligned}$$

which implies (3.6). □

Theorem A2 (L^2-CKN interpolation inequality of Hardy-Rellich type). For $f \in C_0^\infty(\mathbb{S}^n \setminus \{0\})$ and all $a, b \in \mathbb{R}$. It holds that

$$C_R(a,b)\int_{\mathbb{S}^n}\frac{|\nabla_x f|^2}{|\tan r_q|^{a+b+1}}d\nu_x \le \left(\int_{\mathbb{S}^n}\frac{|\Delta_x f|^2}{|\tan r_q|^{2a}}d\nu_x\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^n}\frac{|\nabla_x f|^2}{|\tan r_q|^{2b}}d\nu_x\right)^{\frac{1}{2}}$$
$$-\frac{a+b}{2}\int_{\mathbb{S}^n}\frac{|\nabla_x f|^2}{|\tan r_q|^{a+b-1}}d\nu_x, \tag{3.7}$$

where $a+b+1 \le 0$ and $C_R(a,b) := \frac{n+a+b-1}{2}$.

Proof. For all $a, b \in \mathbb{R}$, $f \in C_0^\infty(\mathbb{S}^n \setminus \{0\})$ and $s \in \mathbb{R}$ we have

$$\int_{\mathbb{S}^n}\left|\frac{\Delta f}{|\tan r_q|^a} + \frac{s\nabla f}{|\tan r_q|^b}\nabla r_q\right|^2 d\nu \ge 0,$$

that is,

$$\int_{\mathbb{S}^n}\frac{|\Delta f|^2}{|\tan r_q|^{2a}}d\nu + s^2\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{2b}}d\nu + 2s\int_{\mathbb{S}^n}\Delta f\frac{\langle\nabla f, \nabla r_q\rangle}{|\tan r_q|^{a+b}}d\nu \ge 0,$$

and when compared with the quadratic inequality $As^2 + Bs + C \ge 0$ we see that

$$B := 2\int_{\mathbb{S}^n}\Delta f\frac{\langle\nabla f, \nabla r_q\rangle}{|\tan r_q|^{a+b}}d\nu.$$

Applying divergence theorem:

$$\begin{aligned}\int_{\mathbb{S}^n}\Delta f\frac{\langle\nabla f, \nabla r_q\rangle}{|\tan r_q|^{a+b}}d\nu &= -\int_{\mathbb{S}^n}\nabla f\nabla\left(\frac{f\nabla r_q}{|\tan r_q|^{a+b}}\right)d\nu\\ &= -\frac{1}{2}\int_{\mathbb{S}^n}\nabla(|\nabla f|^2)\frac{\nabla r_q}{|\tan r_q|^{a+b}}d\nu - \int_{\mathbb{S}^n}\frac{|\nabla f|^2\nabla(|\nabla r_q|)}{|\tan r_q|^{a+b}}d\nu +\\ &+ (a+b)\int_{\mathbb{S}^n}\frac{|\nabla f|^2\sec^2 r_q}{|\tan r_q|^{a+b+1}}d\nu\\ &= -\frac{1}{2}\int_{\mathbb{S}^n}\nabla(|\nabla f|^2)\frac{\nabla r_q}{|\tan r_q|^{a+b}}d\nu + (a+b)\int_{\mathbb{S}^n}\frac{|\nabla f|^2\sec^2 r_q}{|\tan r_q|^{a+b+1}}d\nu.\end{aligned}$$

By divergence theorem again

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{S}^n}\nabla(|\nabla f|^2)\frac{\nabla r_q}{|\tan r_q|^{a+b}}d\nu &= \frac{1}{2}\int_{\mathbb{S}^n}|\nabla f|^2\mathrm{div}\left(\frac{\nabla r_q}{|\tan r_q|^{a+b}}\right)d\nu\\ &= \frac{1}{2}\int_{\mathbb{S}^n}|\nabla f|^2\frac{\Delta r_q}{|\tan r_q|^{a+b}}d\nu - \frac{a+b}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2\sec^2 r_q}{|\tan r_q|^{a+b+1}}d\nu\\ &= \frac{n-1}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b+1}}d\nu - \frac{a+b}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2\sec^2 r_q}{|\tan r_q|^{a+b+1}}d\nu.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{S}^n}\Delta f\frac{\langle\nabla f, \nabla r_q\rangle}{|\tan r_q|^{a+b}}d\nu &= \frac{n-1}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b+1}}d\nu + \frac{a+b}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2(1+\tan^2 r_q)}{|\tan r_q|^{a+b+1}}d\nu\\ &= \frac{n+a+b-1}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b+1}}d\nu + \frac{a+b}{2}\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b-1}}d\nu.\end{aligned}$$

That is,

$$B = (n+a+b-1)\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b}}d\nu + (a+b)\int_{\mathbb{S}^n}\frac{|\nabla f|^2}{|\tan r_q|^{a+b-1}}d\nu.$$

Reverting to the fact that $B^2 - 4AC \leq 0$ holds as in the previous case shows that the expected inequality (3.7) holds true. □

As it has been done in Sections 2 and 3, the interested reader can check different values of a and b in (3.6) and (3.7) to see several interesting inequalities that have been discussed for general p.

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