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Stability of Solutions for a Class of Second Order Nonlinear Integro-Differential Equations with Delay

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Abstract

In this paper, we investigate the stability of solutions for a class of second order nonlinear delay integro-differential equations by using a suitable Lyapunov-Krasovskiǐ functional with sufficient conditions to establish some new results. An example is given to show the validity of the results obtained.

1. Introduction

In relevant literature, the mathematical model known as Volterra integro-differential equation which appeared after its establishment by Vito Volterra in 1926 is viable and its application is increasing to various fields ([25], [27]). Volterra integro-differential equations are important effective mathematical models used to describe many real world phenomena concerning atomic energy, biology, chemistry, control theory, economy, engineering technique fields, information theory, medicine, population dynamics and so on. For a survey on integro-differential equations, see the following references: Adeyanju et.al [3], Ahmad and Stamova (Eds.) [4], Burton [6], Burton and Mahfoud [7], Corduneanu [8], Corduneanu and

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Sandberg (Eds.) [9], Da Prato and Iannelli (Eds.) [11], Gripenberg et.al [14], Lakshmikantham and Mohana [15], Peschel and Mende [19], Staffans [23], Tunç and Tunç [25] and Wazwaz [26]. Importantly, second order integro-differential equations appear in stability problems of visco elastic shells (see [5], [12]).
There are various research articles that have been devoted to stability of the first and second order integro-differential equations which include but not limited to the following references: [3], [5], [10], [13], [16], [17], [18], [21], [22], [24] and [27] while there are numerous published papers which considered other aspects of differential equations using Lyapunov's direct method to establish their results, see [1], [2] and [20] to mention a few.
Furthermore, in 2018 Zhao and Meng [28] considered a kind of nonlinear delay integro-differential equations

$$x'' + f(t,x,x')x' + g(t,x,x') + h(x(t-\tau)) = p(t,x(t)) \int_0^t q(s,x'(s))ds \tag{1}$$

and

$$x'' + f(t,x,x')x' + g(t,x,x') + h(x) = p(x(t-\tau)) \int_0^t q(s,x'(s))ds, \tag{2}$$

where $\mathbb{R}^+ = [0,\infty)$, $\tau > 0$ is a constant while f, g, h, p and q are continuous functions with respect to the argument displayed explicitly. The authors by using different Lyapunov funtionals, established the zero solutions for each of the integro-differential equation with delay and also examined the stability of the generalized form for each of (1) and (2) with variable delay $\tau(t)$ respectively.
Motivated by the work of Zhao and Meng [28], our aim is to obtain sufficient conditions for the stability of solutions for a class of nonlinear delay integro-differential equations by using a suitable Lyapunov-Krasovskiǐ functional with sufficient criteria to achieve the new results.
In this work, we consider the nonlinear second order delay integro-differential equation of the form:

$$x'' + a(t)f(t,x,x')x' + g(t,x,x') + h(x(t-\tau)) = p(x(t-\tau)) \int_0^t q(s,x'(s))ds, \tag{3}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\tau > 0$ is a constant and $a : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ while $f,g : \mathbb{R}^+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are continuous with $h(0) = 0$, $p,q : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ with $p(0) = 0$ and $q(t,0) = 0$. Also, $'$ implies differentiation with respect to t. Indeed, the integro-differential equation (3) extends and improves that of [28] and it is different from those in the references. Also, we provide an example to show the validity of our results.

2. Preliminary Definitions

We give some basic definitions on the stability of solution of integro-differential equations (see [3] and [6]). Consider the system of first order non-linear and non-homogeneous Voltera integro-differential equation

$$X'(t) = A(t)X(t) + \int_{t-\tau}^{t} F(t,\phi,h(X(\phi)))g(X(\phi))d\phi + E(t,X(t)), \tag{4}$$

where $t \in [0,\infty)$, $X \in \mathbb{R}^n$, $A(t)$, $F(t,\phi,h(X(\phi)))$ and $E(t,X(t))$ are continuous functions in their respective arguments explicitly displayed, such that $0 \leq \phi \leq t < \infty$, $h(0) = 0$, $h(X) \neq 0$, $X \neq 0$, $F(t,\phi,0) = 0$; $h, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g(0) = 0$ are continuous functions and $\tau > 0$ is a constant delay.
Let $X(t,t_0,B)$, $t \geq t_0$ be a solution of (4) on $[t_0-\tau,\beta)$, $\beta > 0$ such that $X(t) = B(t)$ on $B \in [t_0-\tau, t_0]$ and $\|B(t)\| = sup_{t\in[t_0-\tau,t_0]}\|B(t)\|$, where $B : [t_0-\tau,t_0] \longrightarrow \mathbb{R}^n$ is a continuous initial function.
Then, we have the following definitions:

Definition 2.1. The zero solution of (4) is said to be stable if for each $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that if $\|B(t)\| < \delta$ on $[t_0-\tau,t_0]$, we have $\|X(t,B)\| < \epsilon$, for all $t \geq t_0$.

Definition 2.2. The zero solution of (4) is said to be uniformly stable if δ is independent of t_0.

Definition 2.3. The zero solution of the (4) is said to be asymptotically stable if it is stable and for each $t_0 \geq 0$, there is a $\delta > 0$ such that $t \geq t_0$, $\|B(t)\| < \delta$ on $[0,t_0]$ implies $\|X(t,B)\| \longrightarrow 0$ as $t \longrightarrow \infty$.

3. Main Results

In this section, we give our main results.
We rewrite (3) as the system

$$\begin{aligned} x' &= y \\ y' &= -a(t)f(t,x,y)y - g(t,x,y) - h(x) + \textstyle\int_{t-\tau}^{t} h'(x(s))y(s)ds \\ &\quad -p(x)\textstyle\int_0^t q(s,y(s))ds + \int_{t-\tau}^{t} p'(x(s))y(s)ds \int_0^t q(s,y(s))ds. \end{aligned} \tag{5}$$

Now, we have the following Theorem.

Theorem 3.1. *Consider the system (5) above. There exist positive constants K_1, K_2, M, N, Ω_1, Ω_2, Φ, δ, τ and functions $P(t), Q(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the following conditions hold:*

(c_1): *$a(t) \geq \alpha > 0$, $t \in \mathbb{R}^+$;*

(c_2): $\dfrac{h(x)}{x} \geq \delta$ *when* $x \neq 0$, *for all* $x \in \mathbb{R}$, δ *a constant;*

(c_3): $|h'(x)| \leq K_1$, $|p'(x)| \leq K_2$ *and* $|q(t,y)| \leq Q(t)|y|$ *for* $y \neq 0$, *and* $x, y \in \mathbb{R}$*;*

(c_4): $f(t,x,0) = 0$, $g(t,x,0) = 0$, $\dfrac{f(t,x,y)}{y^2} \geq M$, $\dfrac{g(t,x,y)}{y} \geq N$ *when* $y \neq 0$ *for all* $x, y \in \mathbb{R}$*;*

(c_5): $|p(x)| \leq P(t)$, $\Omega_1 \leq P(t) \leq \Omega_2$, $Q(t) \leq \Phi$, $t \in \mathbb{R}^+$*;*

(c_6): $\dfrac{K_2\tau}{\alpha}\displaystyle\int_0^\infty Q(s)ds \leq 2M$*;*

(c_7): $\Omega_2 \displaystyle\int_0^\infty Q(s)ds + \frac{(\Omega_2 + K_2\tau)}{\Omega_1}\Phi \int_0^\infty P(s)ds$

$+K_1\Big(\tau - y^{-2}\displaystyle\int_{t-\tau}^t y^2(s)ds\Big) \leq 2N$ *when* $y \neq 0$*;.*

Then, the zero solution of system (5) is stable.

Proof. We define the Lyapunov-Krasovskiǐ functional $V(t,x,y) = V(t)$ as follows:

$$V(t,x,y) = \frac{1}{2}y^2 + \int_0^x h(\phi)d\phi + \lambda \int_{-\tau}^0 ds \int_{t+s}^t y^4(\theta)d\theta + \mu \int_0^t ds \int_t^\infty |p(x(\theta))|Q(s)y^2(s)d\theta,$$

where λ and μ are two positive constants to be determined. So, since we have

$$\int_0^x \frac{h(\phi)}{\phi}\ \phi d\phi \geq \int_0^x \delta\ \phi d\phi = \frac{\delta}{2}x^2.$$

Then,

$$V(t,x,y) \geq \frac{\delta}{2}x^2 + \frac{1}{2}y^2 > 0$$

which shows that the functional is positive definite.
Differentiating $V(t)$ along the solution path of (5), we have

$$\begin{aligned}\frac{dV(t)}{dt} &= yy' + h(x)y + \lambda\tau y^4 - \lambda\int_{t-\tau}^t y^4(s)ds + \mu Q(t)y^2(t)\int_t^\infty |p(x(\theta))|d\theta \\ &\qquad -\mu|p(x(t))|\int_0^t Q(s)y^2(s)ds \\ &= -a(t)f(t,x,y)y^2 - g(t,x,y)y + \lambda\tau y^4 + y\int_{t-\tau}^t h'(x(s))y(s)ds \\ &\quad -p(x)y\int_0^t q(s,y(s))ds + y\int_{t-\tau}^t p'(x(s))y(s)ds\int_0^t q(s,y(s))ds \\ &\quad -\lambda\int_{t-\tau}^t y^4(s)ds + \mu Q(t)y^2\int_t^\infty |p(x(\theta))|d\theta - \mu|p(x(t))|\int_0^t Q(s)y^2ds. \qquad (6)\end{aligned}$$

Applying conditions (c_3) and the inequality $2ab \leq a^2 + b^2$, we have the following:

$$\begin{aligned} y(t)\int_{t-\tau}^{t} h'(x(s))y(s)ds &\leq |y(t)|\int_{t-\tau}^{t} |h'(x(s))||y(s)|ds \\ &\leq \frac{K_1}{2}\int_{t-\tau}^{t}[y^2(t)+y^2(s)]ds \\ &= \frac{K_1}{2}\tau y^2 + \frac{K_1}{2}\int_{t-\tau}^{t} y^2(s)ds. \qquad (7) \end{aligned}$$

$$\begin{aligned} y\int_{t-\tau}^{t} p'(x(s))y(s)ds\int_0^t q(s,y(s))ds &\leq |y(t)|\int_{t-\tau}^{t}|p'(x(s))||y(s)|ds\int_0^t |q(s,y(s))|ds \\ &\leq \frac{K_2}{2}\int_{t-\tau}^{t}[y^2(t)+y^2(s)]ds\int_0^t Q(\theta)|y(\theta)|d\theta \\ &\leq \frac{K_2}{4}\tau y^4\int_0^\infty Q(\theta)d\theta + \frac{K_2}{2}\tau\int_0^t Q(s)y^2(s)ds \\ &\quad + \frac{K_2}{4}\int_{t-\tau}^{t} y^4(s)ds\int_0^\infty Q(s)ds. \qquad (8) \end{aligned}$$

$$\begin{aligned} -yp(x(t))\int_0^t q(s,y(s))ds &\leq |y(t)||p(x(t))|\int_0^t |q(s,y(s))|ds \\ &\leq |y(t)||p(x(t))|\int_0^t Q(s)|y(s))|ds \\ &\leq \frac{1}{2}|p(x(t))|\int_0^t Q(s)[y^2(t)+y^2(s)]ds \\ &\leq \frac{|p(x(t))|}{2}y^2\int_0^\infty Q(s)ds + \frac{|p(x(t))|}{2}\int_0^t Q(s)y^2(s)ds. \qquad (9) \end{aligned}$$

Substituting inequalities (7), (8) and (9) in equation (6), we now have

$$\begin{aligned} V'(t) \leq{}& -a(t)f(t,x,y)y^2 - g(t,x,y)y + \lambda\tau y^4 + \frac{K_1}{2}\tau y^2 + \frac{K_1}{2}\int_{t-\tau}^{t} y^2(s)ds \\ &+\frac{|p(x(t))|}{2}y^2\int_0^\infty Q(s)ds + \frac{|p(x(t))|}{2}\int_0^t Q(s)y^2(s)ds + \frac{K_2}{4}\tau y^4\int_0^\infty Q(\theta)d\theta \\ &+\frac{K_2}{2}\tau\int_0^t Q(s)y^2(s)ds + \frac{K_2}{4}\int_{t-\tau}^{t} y^4(s)ds\int_0^\infty Q(s)ds - \lambda\int_{t-\tau}^{t} y^4(s)ds \\ &+\mu Q(t)y^2\int_t^\infty |p(x(\theta))|d\theta - \mu|p(x(t))|\int_0^t Q(s)y^2(s)ds. \end{aligned}$$

Using the conditions (c_1), (c_4) and (c_5), we obtain

$$V'(t) \leq -\Big[\alpha M - \frac{K_2}{4}\tau \int_0^{\infty} Q(s)ds - \lambda\tau\Big] y^4 - \Big[N - \frac{\Omega_2}{2}\int_0^{\infty} Q(s)ds - \mu\Phi \int_0^{\infty} P(s)ds - \frac{K_1}{2}\tau\Big] y^2$$
$$+\Big[\frac{K_2}{4}\int_0^{\infty} Q(s)ds - \lambda\Big]\int_{t-\tau}^{t} y^4(s)ds + \Big[\frac{\Omega_2}{2} + \frac{K_2}{2}\tau - \mu\Omega_1\Big]\int_0^{t} Q(s)y^2(s)ds$$
$$+\frac{K_1}{2}\int_{t-\tau}^{t} y^2(s)ds.$$

Hence, we choose $\lambda = \dfrac{K_2}{4}\displaystyle\int_0^{\infty} Q(s)ds$, $\mu = \dfrac{(\Omega_2 + K_2\tau)}{2\Omega_1}$ and by (c_6), c_7, it is clear that

$$V'(t) \leq -\Big[\alpha M - \frac{K_2}{2}\tau \int_0^{\infty} Q(s)ds\Big] y^4 - \Big[N - \frac{\Omega_2}{2}\int_0^{\infty} Q(s)ds$$
$$-\frac{(\Omega_2 + K_2\tau)}{2\Omega_1}\Phi \int_0^{\infty} P(s)ds - \frac{K_1}{2}\tau\Big] y^2$$
$$+\frac{K_1}{2}\int_{t-\tau}^{t} y^2(s)ds \leq 0.$$

Thus, the zero solution of system (5) is stable and the proof completed.

Remark. If $h(x(t-\tau)) = h(x)$, where $t > \tau$, $\tau > 0$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Then equation (3) reduces to the one considered for Theorem 3 in Zhao and Meng [28].

We now generalise equation (3) to the form with a variable delay $\tau(t)$. Such that its equivalent system now become

$$x' = y,$$
$$y' = -a(t)f(t,x,y)y - g(t,x,y) - h(x) + \int_{t-\tau(t)}^{t} h'(x(s))y(s)ds$$
$$- p(x)\int_0^{t} q(s,y(s))ds + \int_{t-\tau(t)}^{t} p'(x(s))y(s)ds \int_0^{t} q(s,y(s))ds. \tag{10}$$

We now give some new conditions to obtain additional result.

Corollary. Let conditions (c_1) to (c_5) of the above Theorem hold. Consider the system (10) with a variable delay $\tau(t)$ satisfying the following additional conditions:

- (c_8)**:** there are $\tau > 0$ and $0 < \beta < 1$, such that $0 \leq \tau(t) \leq \tau$ and $\tau'(t) \leq \beta$;
- (c_9)**:** $\dfrac{K_2\tau}{\alpha} \cdot \dfrac{(2-\beta)}{(1-\beta)}\displaystyle\int_0^{\infty} Q(s)ds \leq 4M.$

Then, the zero solution of system (10) with a variable delay $\tau(t)$ is stable.

Proof. Though the proof is similar to that of the above Theorem. But now, with the application of condition (c_8) we have

$$V'(t) \le -\Big[\alpha M - \frac{K_2}{4}\tau \int_0^\infty Q(s)ds - \lambda\tau\Big]y^4 - \Big[N - \frac{\Omega_2}{2}\int_0^\infty Q(s)ds - \mu\Phi\int_0^\infty P(s)ds - \frac{K_1}{2}\tau\Big]y^2$$
$$+\Big[\frac{K_2}{4}\int_0^\infty Q(s)ds - \lambda(1-\beta)\Big]\int_{t-\tau}^t y^4(s)ds + \Big[\frac{\Omega_2}{2} + \frac{K_2}{2}\tau - \mu\Omega_1\Big]\int_0^t Q(s)y^2(s)ds$$
$$+\frac{K_1}{2}\int_{t-\tau}^t y^2(s)ds.$$

Now, we choose $\lambda = \dfrac{K_2}{4(1-\beta)}\displaystyle\int_0^\infty Q(s)ds$, $\mu = \dfrac{(\Omega_2 + K_2\tau)}{2\Omega_1}$ and by (c_7), (c_9), it is obvious that

$$V'(t) \le -\Big[\alpha M - \frac{(2-\beta)}{4(1-\beta)}K_2\tau\int_0^\infty Q(s)ds\Big]y^4 - \Big[N - \frac{\Omega_2}{2}\int_0^\infty Q(s)ds$$
$$-\frac{(\Omega_2 + K_2\tau)}{2\Omega_1}\Phi\int_0^\infty P(s)ds - \frac{K_1}{2}\tau\Big]y^2$$
$$+\frac{K_1}{2}\int_{t-\tau}^t y^2(s)ds \le 0.$$

3.1. **Example.** To show the effectiveness of the results, we consider the following second order nonlinear delay integro-differential equation of the form:

$$(11)\quad x''(t) + 5(t + x^2(t) + x'^2(t))x'(t) + \frac{1}{4}x'(t) + 5x(t-1) = e^{-2(t-1)}\int_0^t e^{-s}x'(s)ds$$

which is a special case of equation (3). This equation (11) can be re-written as follows:
Let $x'(t) = y(t)$, then

$$(12)\quad y'(t) + 5(t + x^2(t) + y^2(t))y(t) + \frac{1}{4}y(t) + 5x(t-1) - e^{-2(t-1)}\int_0^t e^{-s}y(s)ds = 0.$$

In the example above,
$f(t,x,y) = (t + x^2 + y^2)$
$g(t,x,y) = \frac{1}{4}y$, $h(x) = 5x$
$p(x) = e^{-2t}$, $\int_0^t q(s,y(s))ds = \int_0^t e^{-s}y(s)ds = x(t)$ and $\tau = 1$.

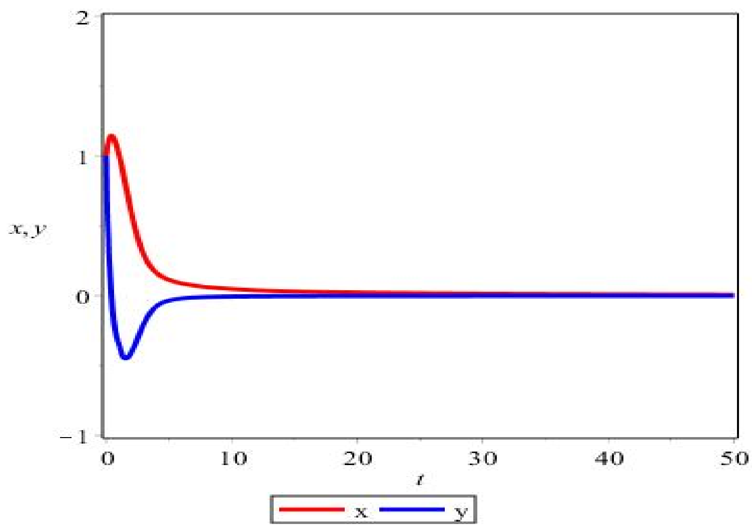

Figure 1: Stability of solutions for the nonlinear delay integro-differential equation (12)

Thus, by using Maple 2015 we have been able to establish the asymptotic stability of solutions for the second order nonlinear delay integro-differential equation (12) which is a special case of (3). This improve on the results achieve by Zhao and Meng [28].

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