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Some Properties of a Certain Class of Analytic Functions Defined by a Convolution Operator

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Abstract

In this work, we study some properties of subclass $\mathcal{B}^{\alpha}_{n+1}(\beta)$ of the class of analytic functions defined by a convolution operator. In fact, this class generalizes the class of Yamaguchi functions. Thereafter, some geometric properties such as inclusion, Fekete-Szegö functional and upper bounds for some Hankel determinants are presented. Indeed, results from some of our corollaries and remarks show that when some involving parameters are varied, our results reduce to some existing ones.

1. Introduction

Let $\varDelta = \{z \in \mathbb{C} : |z| < 1\}$ be a unit disk and let $\mathcal{A}$ be the class of analytic functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad f(0) = f'(0) - 1 = 0, \quad z \in \varDelta. \tag{1}$$

Also let $\mathcal{S}$, a subset of $\mathcal{A}$, be the class of univalent functions analytic in $\varDelta$. Let

$$\phi(z) = z + \sum_{j=2}^{\infty} A_j z^j, \quad \psi(z) = z + \sum_{j=2}^{\infty} B_j z^j \in \mathcal{A},$$

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then the convolution (or Hadamard product) of functions $\phi(z)$ and $\psi(z)$ is define by

$$(\phi * \psi)(z) = z + \sum_{j=2}^{\infty} A_j B_j z^j, \quad z \in \Delta.$$

Pommerenke [13] defined the qth-Hankel determinants for $f \in \mathcal{S}$ as

$$\mathcal{H}_q(j) = \begin{vmatrix} 1 & a_{j+1} & \cdots & a_{j+q-1} \\ a_{j+1} & a_{j+2} & \cdots & a_{j+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j+q-1} & a_{j+q} & \cdots & a_{j+2(q-1)} \end{vmatrix}$$

where $j \geqslant 1$, $q \geqslant 1$ and $a_1 = 1$ for functions in $\mathcal{S}$. Now for $q = 2$ and $j = 1$,

$$|\mathcal{H}_2(1)| = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|, \tag{2}$$

for $q = 2$ and $j = 2$,

$$|\mathcal{H}_2(2)| = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \tag{3}$$

and for $q = 3$ and $j = 1$,

$$|\mathcal{H}_3(1)| = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

which implies that

$$|\mathcal{H}_3(1)| \leqslant |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{4}$$

Many properties of these determinants have been studied by many researchers for specific values of parameters j and q. In particular see [4, 10] for more details. Related to the coefficient estimates in (2) is the problem of estimating the upper bound of the functional

$$\mathcal{F}(\delta, f) := |a_3 - \delta a_2^2| \tag{5}$$

defined by Fekete and Szegö [9] where δ may be a real or complex value. The determination of sharp upper bounds for the non-linear functional $\mathcal{F}(\delta, f)$ for any subclass of $\mathcal{A}$ is what is usually termed "Fekete-Szegö problem". A remarkable relationship exists between the functionals (2) and (5) since $\mathcal{F}(1, f) = |\mathcal{H}_2(1)|$. See [1, 7, 10] for more details.

In [2, 3], Babalola defined a convolution operator $\mathscr{L}_n^\alpha : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\mathscr{L}_n^\alpha f(z) = (\tau_\alpha * \tau_{\alpha,n}^{(-1)} * f)(z) \tag{6}$$

where $\tau_{\alpha,n}(z) = \frac{z}{(1-z)^{\alpha-(n-1)}}$ and $\tau_{\alpha,n}^{(-1)}$ is such that

$$(\tau_{\alpha,n} * \tau_{\alpha,n}^{(-1)})(z) = \frac{z}{1-z} = z + \sum_{j=2}^{\infty} z^j$$

for fixed real number $\alpha \geqslant n+1$ and $n \in \mathbb{N} \cup \{0\}$. Simple calculation shows that (6) is equivalent to

$$\mathscr{L}_n^\alpha f(z) = z + \sum_{j=2}^{\infty} \left\{ \frac{(\alpha+j-1)!}{\alpha!} \frac{(\alpha-n)!}{(\alpha+j-n-1)!} \right\} a_j z^j, \quad z \in \Delta. \tag{7}$$

We note from [2, 3] that

$$\mathscr{L}_0^\alpha f(z) = \mathscr{L}_0^0 f(z) = f(z)$$

$$\mathscr{L}_1^1 f(z) = z f'(z) \tag{8}$$

$$\mathscr{L}_n^n f(z) = \mathcal{D}^n f(z) \tag{9}$$

$$\mathscr{L}_{n+1}^\alpha f(z) = z + \sum_{j=2}^{\infty} \left\{ \frac{(\alpha+j-1)!}{\alpha!} \frac{(\alpha-n-1)!}{(\alpha+j-n-2)!} \right\} a_j z^j \tag{10}$$

$$(\alpha-n)\mathscr{L}_{n+1}^\alpha f(z) = (\alpha-(n+1))\mathscr{L}_n^\alpha f(z) + z(\mathscr{L}_n^\alpha f(z))' \tag{11}$$

and

$$(\alpha-n)(\mathscr{L}_{n+1}^\alpha f(z))' = (\alpha-n)(\mathscr{L}_n^\alpha f(z))' + z(\mathscr{L}_n^\alpha f(z))''. \tag{12}$$

where (9) is the well-known Ruscheweyh operator introduced in [14]. Clearly, (8) $\Longrightarrow$ (9).
Now we define the class $\mathcal{B}_{n+1}^\alpha(\beta)$ as follows.
Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}_{n+1}^\alpha(\beta)$ if

$$\mathcal{R}e \frac{\mathscr{L}_{n+1}^\alpha f(z)}{z} > \beta, \quad z \in \Delta \tag{13}$$

for fixed number $\alpha \geqslant (n+1)$, $n \in \mathbb{N} \cup \{0\}$ and $0 \leqslant \beta < 1$.
It is interesting to note that (13) is the product combination of geometric expressions of functions in classes $\mathcal{B}_n^\alpha(\beta)$ and $\mathcal{S}_n^\alpha$ respectively studied in [2] and [3].
The following classes are equivalent to class $\mathcal{B}_{n+1}^\alpha(\beta)$.

(1) $\mathcal{B}_1^1(0) = \mathcal{T}$ studied in [12].
(2) $\mathcal{B}_0^\alpha(0) = \mathcal{Y}$ studied in [17].
(3) $\mathcal{B}_0^\alpha(\beta) = \mathcal{Y}(\beta)$ studied in [16].
(4) $\mathcal{B}_1^\alpha(\beta) = \mathcal{T}(\beta)$ in [16].

In this present work, some of the investigated geometric properties of functions in $\mathcal{B}^{\alpha}_{n+1}(\beta)$ are the inclusion, the Fekete-Szegö functional and the upper bounds of some Hankel determinants.

2. Lemmas

The following lemmas shall be used in proving the theorems that follows:
Firstly, let the class denoted by $\mathcal{P}$ consists of analytic functions of the form:

$$p(z) = 1 + \sum_{j=1}^{\infty} p_j z^j, \quad p(0) = 1, \ \mathcal{R}e\, p(z) > 0 \text{ and } z \in \Delta. \tag{14}$$

$p(z)$ is known as a function with positive real part in Δ.

Lemma 2.1. ([15]). Let $p \in \mathcal{P}$. Then $|p_j| \leqslant 2, \ j \in \mathbb{N}$.

Lemma 2.2. ([8]). Let $p \in \mathcal{P}$. Then

$$\left| p_2 - \nu \frac{p_1^2}{2} \right| \leqslant 2 \max\{1, |1-\nu|\}, \quad \nu \in \mathbb{C}.$$

Lemma 2.3. ([5]). Let $p \in \mathcal{P}$ and suppose

$$\mathcal{R}e \left(1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\beta - 1}{2\beta},$$

then

$$\mathcal{R}e(p(z)) > 2^{1-\frac{1}{\beta}}, \quad \frac{1}{2} \leqslant \beta < 1, \ z \in \Delta.$$

The constant $2^{1-\frac{1}{\beta}}$ is the best possible.

Lemma 2.4. ([6]). Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\psi(u, v)$ be a complex-valued function satisfying

(a) $\psi(u, v)$ is continuous in a domain Ω of $\mathbb{C}^2$,
(b) $(1, 0) \in \Omega$ and $\mathcal{R}e[\psi(1, 0)] > 0$,
(c) $\mathcal{R}e\, \psi(\xi + (1-\xi)u_2 i, v_1) \leqslant \xi$ when $(\xi + (1-\xi)u_2 i, v_1) \in \Omega$ and $2v_1 \leqslant -(1-\xi)(1+u_2^2)$ for real number $0 \leqslant \xi < 1$.

If $p \in \mathcal{P}$ such that $(p(z), zp'(z)) \in \Omega$ and $\mathcal{R}e[\psi(p(z), zp'(z))] > \xi$ for $z \in \Delta$. Then $\mathcal{R}e[p(z)] > \xi$ in $z \in \Delta$.

Lemma 2.5. ([11]). Let $p \in \mathcal{P}$. Then

$$p_2 = \frac{1}{2}p_1^2 + \frac{x}{2}(4 - p_1^2)$$

and

$$p_3 = \frac{1}{4}p_1^3 + \frac{1}{2}p_1(4-p_1^2)x - \frac{1}{4}p_1(4-p_1^2)x^2 + \frac{1}{2}(4-p_1^2)(1-|x|^2)z$$

for some x, z such that $|x| \leqslant 1$, $|z| \leqslant 1$.

Lemma 2.6. ([2]). Let $f \in \mathcal{B}^{\alpha}_{n+1}(\beta)$, then

$$a_j = (1-\beta)J_j p_{j-1} \quad and \quad |a_j| \leqslant 2(1-\beta)J_j$$

where

$$J_j = \frac{\alpha!(\alpha + j - (n+2))!}{(\alpha - (n+1))!(\alpha + j - 1)!}. \tag{15}$$

3. Main Results

Our results are as follows.

Theorem 3.1. $\mathcal{B}^{\alpha}_{n+1}(\beta) \subset \mathcal{B}^{\alpha}_{n}(\beta)$.

Proof. Let $f \in \mathcal{A}$ satisfy (13) so that for $p \in \mathcal{P}$, define the equation

$$\frac{z(\mathscr{L}^{\alpha}_{n} f(z))'}{\mathscr{L}^{\alpha}_{n} f(z)} = 1 + \frac{zp'(z)}{p(z)}. \tag{16}$$

Now using (12) in (16) gives

$$\frac{(\sigma - n)\mathscr{L}^{\alpha}_{n+1} f(z)}{\mathscr{L}^{\alpha}_{n} f(z)} - \frac{(\sigma - (n+1))\mathscr{L}^{\alpha}_{n} f(z)}{\mathscr{L}^{\alpha}_{n} f(z)} = 1 + \frac{zp'(z)}{p(z)} \tag{17}$$

$$\frac{(\sigma - n)\mathscr{L}^{\alpha}_{n+1} f(z)}{\mathscr{L}^{\alpha}_{n} f(z)} = (\sigma - n) + \frac{zp'(z)}{p(z)} \tag{18}$$

so that by divide through by $(\sigma - n)$ gives

$$\frac{\mathscr{L}^{\alpha}_{n+1} f(z)}{\mathscr{L}^{\alpha}_{n} f(z)} = 1 + \frac{zp'(z)}{(\sigma - n)p(z)}.$$

But (13) can be expressed as

$$\mathcal{R}e \left(p(z) + \frac{zp'(z)}{(\sigma - n)} \right) > \beta$$

so that

$$\mathcal{R}e \left(p(z) + \frac{zp'(z)}{(\sigma - n)} \right) - \beta > 0.$$

Now define the function

$$\psi(u, v) = u + \frac{v}{(\sigma - n)} - \beta$$

on a domain $\Omega = \mathbb{C} \times \mathbb{C}$ of $\mathbb{C}^2$.

Clearly $\psi(u, v)$ satisfies the condition (a) of Lemma 2.4. More so, $(1, 0) \in \Omega$ implies $\psi(1, 0) = 1 + 0 - \beta$ and $\mathcal{R}e\, \psi(1, 0) = 1 - \beta > 0$, $0 \leqslant \beta < 1$. Thus, with $\xi = 0$ in Lemma 2.4,

$$\psi(u_{2i}, v_1) = u_{2i} + \frac{v_1}{(\sigma - n)} - \beta$$

and $\mathcal{R}e\, \psi(u_{2i}, v_1) = \frac{v_1}{(\sigma - n)} - \beta < 0$ whenever $v_1 \leqslant \frac{-(1+u_2^2)}{2}$.

Therefore, ψ satisfies all the conditions of Lemma 2.4 so,

$$\mathcal{R}e\frac{\mathscr{L}_n^\alpha f(z)}{z} > 0 \implies f \in B_n^\sigma(\beta)$$

thus the proof is complete. □

Theorem 3.2. If $f \in \mathcal{A}$ satisfies the condition

$$\mathcal{R}e\frac{z(\mathscr{L}_{n+1}^\alpha f(z))'}{\mathscr{L}_{n+1}^\alpha f(z)} > \frac{3\beta-1}{2\beta},$$

then

$$\mathcal{R}e\frac{\mathscr{L}_{n+1}^\alpha f(z)}{z} > 2^{1-\frac{1}{\beta}}, \quad \frac{1}{2} \leqslant \beta < 1,\ z \in \Delta.$$

Proof. For $z \in \Delta$, define the function

$$p(z) = \frac{\mathscr{L}_{n+1}^\alpha f(z)}{z} \tag{19}$$

and by logarithmic differentiation,

$$\frac{p'(z)}{p(z)} = \frac{(\mathscr{L}_{n+1}^\alpha f(z))'}{\mathscr{L}_{n+1}^\alpha f(z)} - \frac{1}{z}. \tag{20}$$

Using (12) in (20) gives

$$\frac{p'(z)}{p(z)} = \frac{(\mathscr{L}_n^\alpha f(z))'}{\mathscr{L}_{n+1}^\alpha f(z)} + \frac{z(\mathscr{L}_n^\alpha f(z))''}{(\sigma-n)\mathscr{L}_{n+1}^\alpha f(z)} - \frac{1}{z} \tag{21}$$

so that

$$\mathcal{R}e\left(1+\frac{zp'(z)}{p(z)}\right) = \mathcal{R}e\left(\frac{z(\mathscr{L}_n^\alpha f(z))'}{\mathscr{L}_{n+1}^\alpha f(z)} + \frac{z^2(\mathscr{L}_n^\alpha f(z))''}{(\sigma-n)\mathscr{L}_{n+1}^\alpha f(z)}\right) > \frac{3\beta-1}{2\beta}$$

and using (12) implies

$$\mathcal{R}e\left(\frac{z(\mathscr{L}_{n+1}^\alpha f(z))'}{\mathscr{L}_{n+1}^\alpha f(z)}\right) > \frac{3\beta-1}{2\beta} \tag{4.10}$$

which by Lemma 2.3 implies $\mathcal{R}e\, p(z) > 2^{1-\frac{1}{\beta}}$, $\frac{1}{2} \leq \beta < 1$ as required. □

Corollary 3.3. If $f \in \mathcal{A}$ satisfies the condition of Theorem 3.2, then $f(z) \in \mathcal{B}_{n+1}^\alpha(2^{1-\frac{1}{\beta}})$.

Corollary 3.4. Let $n = 0$ in Theorem 3.2 and suppose

$$\mathcal{R}e\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{3\beta-1}{2\beta},$$

then

$$\mathcal{R}e f'(z) > 2^{1-\frac{1}{\beta}}.$$

Corollary 3.5. Let $\beta = \frac{1}{2}$ in Corollary 3.4 and suppose

$$\mathcal{R}e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2},$$

then

$$Ref'(z) > \frac{1}{2}.$$

Theorem 3.6. Let $f \in \mathcal{B}^{\alpha}_{n+1}(\beta)$. Then for $\delta \in \mathbb{C}$,

$$\left|a_3 - \delta a_2^2\right| \leq 2J_3(1-\beta)\max\left\{1, \left|1 - \frac{2\delta(1-\beta)J_2^2}{J_3}\right|\right\}$$

where $0 \leqslant \beta < 1$ and J_j is defined by (15).

Proof. Using Lemma 2.6 and for $\delta \in \mathbb{C}$,

$$\begin{aligned} \left|a_3 - \delta a_2^2\right| &= \left|(1-\beta)J_3 p_2 - \delta(1-\beta)^2 J_2^2 p_1^2\right| \\ &= (1-\beta)J_3\left|p_2 - \eta\frac{p_1^2}{2}\right| \end{aligned} \tag{22}$$

where

$$\eta = \frac{2\delta(1-\beta)J_2^2}{J_3}.$$

Using Lemma 2.4 implies

$$\left|p_2 - \eta\frac{p_1^2}{2}\right| \leq 2\max\left\{1, \left|1 - \frac{2\delta(1-\beta)J_2^2}{J_3}\right|\right\} \tag{23}$$

and putting (23) into (22) completes the proof. □

Corollary 3.7. Let $\delta = 1$. Then $|a_3 - a_2^2| \leq 2J_3(1-\beta)$.

Theorem 3.8. Let $f \in \mathcal{B}^{\alpha}_{n+1}(\beta)$. Then

$$|a_2a_4 - a_3^2| \leq \frac{9(1-\beta)^2 J_2 J_4}{2} + 4(1-\beta)^2 J_3^2$$

where $0 \leqslant \beta < 1$ and J_j is defined by (15).

Proof. Using Lemma 2.6 in (3) gives

$$\begin{aligned} a_2a_4 - a_3^2 &= (1-\beta)J_2p_1 \times (1-\beta)J_4p_3 - (1-\beta)^2 J_3^2 p_2^2 \\ &= (1-\beta)^2 J_2 J_4[p_1p_3 - \lambda p_2^2] \end{aligned} \tag{24}$$

where $\lambda = \frac{J_3^2}{J_2J_4}$. Now using Lemma 2.5 leads to

$$|a_2a_4 - a_3^2| = \frac{(1-\beta)^2J_2J_4}{4}\Big|p_1^4 + 2(4-p_1^2)p_1^2x - (4-p_1^2)p_1^2x^2$$
$$+ 2(4-p_1^2)(1-|x|^2)p_1z - \lambda p_1^4 - \lambda 2(4-p_1^2)p_1^2x - \lambda(4-p_1^2)^2x^2\Big|$$

Now for $|p_1| \leqslant 2$, letting $p_1 = p$, assume without restriction that $p \in [0,2]$ and applying triangle inequality with $\mu = |x|$ gives

$$|a_2a_4 - a_3^2| = \frac{(1-\beta)^2J_2J_4}{4}\Big\{p^4 + 2(4-p^2)p^2\mu + (4-p^2)p^2\mu^2$$
$$+ 2(4-p^2)(1-\mu^2)p + \lambda p^4 + \lambda 2(4-p^2)p^2\mu + \lambda(4-p^2)^2\mu^2\Big\}.$$

Factoring out μ gives

$$(25)\quad |a_2a_4 - a_3^2| \le \frac{(1-\beta)^2J_2J_4}{4}\Big\{(\lambda+1)p^4 + [2(4-p^2)(\lambda+1)p^2]\mu$$
$$+ (4-p^2)[p^2 + \lambda(4-p^2)]\mu^2 + 2(4-p^2)p - 2(4-p^2)p\mu^2\Big\} = F(\mu,p).$$

Now from (25) we have

$$\frac{\partial F(\mu,p)}{\partial\mu} = \frac{(1-\beta)^2J_2J_4}{4}\Big\{2(4-p^2)(\lambda+1)p^2$$
$$+ 2(4-p^2)[p^2 + \lambda(4-p^2)]\mu - 4(4-p^2)p\mu\Big\}$$

Observe that $\frac{\partial F(\mu,p)}{\partial\mu} > 0$ in the interval $\mu \in [0,1]$. This implies that $\frac{\partial F(\mu,p)}{\partial\mu}$ is an increasing function of μ on the closed interval $[0,1]$, thus from (25) the maximum point is at $\mu = 1$, hence

$$(26)\qquad F(1,p) \le \frac{(1-\beta)^2J_2J_4}{4}2\{-p^4 + 6p^2 + 8\lambda\} = G(p).$$

Now,

$$(27)\qquad G'(p) = \frac{(1-\beta)^2J_2J_4}{2}\{-4p^3 + 12p\}$$

so that at the critical points, $G'(p) = 0$ implies

$$\frac{(1-\beta)^2J_2J_4}{2}\{-4p^3 + 12p\} = 0.$$

Solving for p implies that $p_0 = 0$ or $p_1 = \sqrt{3}$ and from (27),

$$G''(p) = \frac{(1-\beta)^2 J_2 J_4}{2}\{-12p^2 + 12\}$$
$$G''(p_0) = \frac{(1-\beta)^2 J_2 J_4}{2}\{12\} > 0 \text{ (a minimum point)}$$
$$G''(p_1) = \frac{(1-\beta)^2 J_2 J_4}{2}\{-36 + 12\} < 0 \text{ (a maximum point)}.$$

From (26), $G(p)$ attains maximum at

$$G(p_1) = \frac{(1-\beta)^2 J_2 J_4}{2}\{-(\sqrt{3})^4 + 6(\sqrt{3})^2 + 8\lambda\}$$

hence using $\lambda = \frac{J_3^2}{J_2 J_4}$ and simplifying completes the proof. □

Theorem 3.9. Let $f \in \mathcal{B}_{n+1}^{\alpha}(\beta)$. Then

$$|a_2a_3 - a_4| \leqslant \frac{2(1-\beta)[2(1-\beta)J_2J_3 + 3J_4]}{3}\sqrt{\frac{2[2(1-\beta)J_2J_3 + 3J_4]}{3J_4}}$$

where $0 \leqslant \beta < 1$ and J_j is defined by (15).

Proof. Using Lemma 2.6 in (3) leads to

$$a_2a_3 - a_4 = (1-\beta)^2 J_2J_3p_1p_2 - (1-\beta)J_4p_3.$$

Now using Lemma 2.5 we have

$$a_2a_3 - a_4 = \frac{(1-\beta)^2 J_2J_3p_1[p_1^2 + (4-p_1^2)x]}{2} - \frac{AJ_4[p_1^3 + 2(4-p_1^2)p_1x - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)z]}{4}$$

where $A = (1-\beta)$ and it simplifies to

$$\begin{aligned}4(a_2a_3 - a_4) &= 2(1-\beta)^2J_2J_3p_1^3 - (1-\beta)J_4p_1^3\\ &+ 2(1-\beta)^2(4-p_1^2)J_2J_3p_1x - 2(1-\beta)(4-p_1^2)J_4p_1x\\ &+ (1-\beta)(4-p_1^2)J_4p_1x^2 - 2(1-\beta)(4-p_1^2)(1-|x|^2)J_4z.\end{aligned}$$

By Lemma 2.1, $|p_1| \leqslant 2$, then letting $p_1 = p$, assume without restriction that $p \in [0,2]$ and applying triangle inequality with $\eta = |x|$ we have

$$\begin{aligned} & 4|a_2a_3 - a_4| \\ & \quad \leqslant \{2(1-\beta)^2 J_2J_3p^3 + (1-\beta)J_4p^3 + 2(1-\beta)(4-p^2)J_4\} \\ & \qquad + \left\{2(1-\beta)^2(4-p^2)J_2J_3p + 2(1-\beta)(4-p^2)J_4p\right\}\eta \\ & \qquad + \left\{(1-\beta)(4-p^2)J_4p - 2(1-\beta)(4-p^2)J_4\right\}\eta^2 \\ & \quad = F(\eta, p). \end{aligned} \tag{28}$$

Now from (28) we have

$$\begin{aligned} \frac{\partial F(\eta,p)}{\partial \eta} = & \left\{2(1-\beta)^2(4-p^2)J_2J_3p + 2(1-\beta)(4-p^2)J_4p\right\} \\ & + \left\{2(1-\beta)(4-p^2)J_4p - 2(1-\beta)(4-p^2)J_4\right\}\eta \end{aligned} \tag{29}$$

Observe that $\frac{\partial F(\eta,p)}{\partial \eta} > 0$ in the interval $\eta \in [0,1]$. This implies that $\frac{\partial F(\eta,p)}{\partial \eta}$ is an increasing function of η on the closed interval $[0,1]$, thus from (28) the maximum point is at $\eta = 1$, hence

$$F(1,p) \leq -2(1-\beta)J_4p^3 + 4(1-\beta)[2(1-\beta)J_2J_3 + 3J_4]p = G(p). \tag{30}$$

Now,

$$G'(p) = -6(1-\beta)J_4p^2 + 4(1-\beta)[2(1-\beta)J_2J_3 + 3J_4] \tag{31}$$

Note that at the critical points, $G'(p) = 0$ which implies that

$$-6(1-\beta)J_4p^2 + 4(1-\beta)[2(1-\beta)J_2J_3 + 3J_4] = 0$$

so that $p_1 = \sqrt{\frac{2[2(1-\beta)J_2J_3+3J_4]}{3J_4}}$ and from (31),

$$G''(p) = -12(1-\beta)J_4p$$

$$G''(p_1) = -12(1-\beta)J_4\left(\sqrt{\frac{2[2(1-\beta)J_2J_3 + 3J_4]}{3J_4}}\right) < 0$$

Now $G(p)$ in (30) attains maximum at

$$G(p_1) \leq \left\{\frac{-4(1-\beta)[2(1-\beta)J_2J_3 + 3J_4]}{3} + 4(1-\beta)[2(1-\beta)J_2J_3 + 3J_4]\right\}\sqrt{\frac{2[2(1-\beta)J_2J_3 + 3J_4]}{3J_4}}$$

and simple simplification completes the proof. □

Theorem 3.10. Let $f \in \mathcal{B}^{\alpha}_{n+1}(\beta)$. Then

$$|\mathcal{H}_3(1)| \leqslant 9(1-\beta)^3 J_2 J_3 J_4 + 8(1-\beta)^3 J_3^3 + 4(1-\beta)^2 J_3 J_5 + \frac{4(1-\beta)^2 J_4[2(1-\beta)J_2J_3 + 3J_4]}{3}\sqrt{\frac{2[2(1-\beta)J_2J_3+3J_4]}{3J_4}}.$$

where $0 \leqslant \beta < 1$ and J_j is defined by (15).

Proof. Using Lemma 2.6, Theorems 3.8, 3.9 and Corollary 3.7 in (4) leads to our assertion. □

Conclusions: A class of generalized analytic functions defined by the well-known Babalola convolution operator which was earlier studied in [2] was further investigated in this paper. Some results obtained were its inclusion condition, the upper estimate of the Fekete-Szegö functional for complex parameter and some estimates for some Hankel determinants. Varying some parameters in the class made it to reduce to some known classes earlier studied by some authors. Finally, some relevant corollaries were presented and a few remarks discussed.

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