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Some Generalizations of Opial-type Inequalities on Time Scales

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Abstract

Opial-type inequalities has grown into a substantial field with many applications in proving the existence and uniqueness of solutions of initial and boundary value problems for differential equations. We shall obtain Hua's inequality on time scales The methodology employed in this paper is the Hölder's inequality for convex functions.

1. Introduction

A Polish Mathematician called Zdzidlaw Opial established an inequality involving integrals of a function and its derivatives which is named after him as Opial inequality. Opial inequality has proved to be one of the most useful inequalities in analysis. The inequality has been receiving continual attention [14], [15], [16] and [17]. Its application in proving the existence and uniqueness of solutions of initial and boundary value problems for differential equations have been particularly striking. [10] established the following integral inequality:

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Theorem 1.1. *If $f(x)$ is absolutely continuous on $[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ on $(0,h)$, then*

$$\int_0^h \left|f(x)f'(x)\right| dx \leq \frac{h}{4}\int_0^h (f'(x))^2 dx. \tag{1}$$

In (1) , $\frac{h}{4}$ is the best possible constant.
Shortly after the publication of Opial inequality, [9] provided a modified version of Opial$'$ results. He observed that the absolute in (1) is not necessary if $f(x)$ is absolutely continuous and his result is stated as follows:

Theorem 1.2. *If $f(x)$ is absolutely continuous on $[0,h]$ with $f(0) = 0$, then*

$$\int_0^h f(x)f'(x)dx \leq \frac{h}{2}\int_0^h (f'(x))^2 dx. \tag{2}$$

A non-trivial generalization on Theorem 1.2 was established by Hua [8]

Theorem 1.3. *Let $x(t)$ be absolutely continuous on $[0,a]$ and $x(0) = 0$. If $l > 0$, then*

$$\int_0^a |x^l(t)x'(t)|dt \leq \frac{a^l}{l+1}\int_0^a |x'(t)|^{l+1} dt. \tag{3}$$

[12] established the following:

Theorem 1.4. *Let $f(t)$ and $g(t)$ be convex and increasing function on $[0,\infty]$ with $f(0) = 0$ and let $p(t) \geq 0$, $p'(t) > 0$, $t \in [\alpha, r]$ with $p(t) = 0$. If $x(t)$ is absolutely continuous on $[\alpha, r)$ and $x(\alpha) = 0$, then*

$$f\left(\int_\alpha^r p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \geq \int_\alpha^r p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\left[f'\left(p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\right]dt. \tag{4}$$

[6] got a result which is a special case of Shum-Opial$'$s inequality. This occurs when the main functions do not change in the interval $[a,b]$. In tir work, they took a class of functions that satisfied a condition and obtain generalization of the special case using adaptation of Jensen$'$s inequality for convex functions. They obtained the following result:

Theorem 1.5. *Let $f(t)$ be continuous and non-decreasing function on $[a,b]$, and $0 \leq a \leq b < \infty$ with $f(t) > 0$ for $t > 0$. Suppose that $p \geq l \geq 1, q > 0$, $0 <$*

$l+q \le p$ and $\delta > 0$. Then,

$$\begin{aligned}
&\int_a^b t^{\delta l-1} f^q(t)\left[\int_t^b f(s)ds\right]^l dt \\
(5)\qquad &\le [\delta^{-1}]^{(lp-(l+q)+p)/p}\left[\frac{p}{l+q}\right]\left[\int_a^b (f^p(s))s^{\frac{lp(1+\delta)}{l+q}-1}ds\right]^{\frac{(l+q)}{p}} \\
&+[\delta^{-1}]^{(lp-(l+q)+p)/p}\left[\frac{p}{l+q}\right]a^{\delta\frac{(l+q)}{p}}\left[\int_a^b f^p(s)s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)}ds\right]^{\frac{(l+q)}{p}}.
\end{aligned}$$

[1] generalized Opial-type inequalities for independent variables as follows:

Theorem 1.6. *Let $m \ge 2$ and let $f_i, D^1 f_i....D^m f_i$, $i = 1, ..., n$ be real valued continuous function on τ with $f_i(t)|_{t_i} = a_i = 0$, $i = 1, ..., n, j = 1, ...m$. or $f_i(t)|_{t_1} = a_1 = D_i(t)|_{t_2} = a_2 = ... = D_i(t)|_{t_m} = a_m = 0$ $i = 1, ..., n$*
Let F be a non-negative and differentiable function on $[0, \infty]$ with $F(0, ..., 0) = 0$ such that $D_i F$, $i = 1, \cdots, n$ are non-negative, continuous and non-decreasing on $[0, \infty]^n$. Then the integral inequality

(6)

$$\int_\tau \sum_{i=1}^n D_i F|f_1(t)|, ..., |f_n(t)||D^m f_i(t)|dt \le F\left[\int_\tau |D^m f_1(t)|dt, ..., \int_\tau |D^m f_n(t)|dt\right].$$

The time scale analysis discussed in [4] and [5] summarises the time scale calculus while [13] discussed several possible applications on time scales.
However, the study of dynamic inequalities of Opial types on time scales is initiated by [3] and only recently received a lot of attention and lots of papers have been written.
Throughout this work, we denote $f^\sigma := fo\sigma$, where the forward jump operator σ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. Also, if $t < sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense and if $t > inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by $\mu := \sigma(t) - t$. If $\mathbb{T}^k := \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left-scattered maximum m, otherwise $\mathbb{T}^k := \mathbb{T}$. We will assume that $\sup\mathbb{T} = \infty$ and define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b \cap \mathbb{T}]$. We will frequently use the results in the following theorem as discussed in [7].

Theorem 1.7. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(1) If f is differentiable at t, then f is continuous at t.

(2) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(3) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(4) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f(t).$$

In order to describe classes of functions that are integrable, the following theorem is introduced.

Theorem 1.8. *Assume $f : \mathbb{T} \to \mathbb{R}$.*

(1) If f is continuous, then f is rd-continuous,
(2) If f is rd-continuous, then f is regulated,
(3) The jump operator σ is rd-continuous.
(4) If f is regulated or rd-continuous, then so is f^{σ}.
(5) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then fog has that property too.

[11] obtained the following results:

Theorem 1.9. *Let $\mathbb{T}$ be a time scale with $t \in \mathbb{T}$. Let ς, ζ be real numbers, let $h, q \in C_{rd}([0,b]_{\mathbb{T}}, \mathbb{R})$ where h and q are positive rd-continuous functions on $[\alpha, \beta]_{\mathbb{T}}$ such that $\int_{[0,t]} r(t)\Delta(s) < \infty$. We define φ as convex function and if $x : [\alpha, \beta]_{\mathbb{T}} \longrightarrow \mathbb{R}$ is delta differentiable with $\alpha(0) = 0$, then we have*

(7)
$$\int_{[\alpha,\beta]} (\sqrt{q(s)})^{\varsigma+1} x^{\Delta}(s) x^{\varsigma}(s)\Delta(s)$$
$$\leq \frac{1}{1+\zeta}(\beta - \alpha)^{\varsigma-\zeta} \left(\int_{[\alpha,\beta]} \frac{\Delta(s)}{(\sqrt{h(s)})^{1+\zeta}} \right) \int_{[\alpha,\beta]} (\sqrt{h(s)q(s)})^{1+\zeta} x^{\Delta}(s)^{1+\zeta}\Delta(s).$$

[2] obtained the following results using modified Jensen's inequality on time scales:

Theorem 1.10. *Let $\mathbb{T}$ be a time scale with $t \in \mathbb{T}$. Let ς, ζ and σ be real numbers , let $h, q \in C_{rd}([0,b]_{\mathbb{T}}, \mathbb{R})$ and positive rd-continuous functions on $[\alpha, \beta]_{\mathbb{T}}$ such that*

$\int_{[0,t]} r(t)\Delta(s) < \infty$ *and* $q(t)$ *is non-increasing on* $[0,b]$ *. We define* φ *as convex function and if* $x : [\alpha,\beta] \longrightarrow \mathbb{R}$ *is delta differentiable with* $\alpha(0) = 0$*, then we have*

$$\begin{aligned} &\int_{[\alpha,\beta]} (\sqrt{q(t)})^{\varsigma+\sigma} x^{\Delta}(t)^{\sigma} x(t)^{\varsigma} \Delta(t) \\ &\leq \frac{\sigma}{\varsigma+\sigma} (\beta-\alpha)^{\varsigma} \left(\int_{[\alpha,\beta]} \frac{\Delta(s)}{(\sqrt{h(s)})^{\varsigma+\sigma}} \right) \int_{[\alpha,\beta]} (\sqrt{q(s)h(s)} x^{\Delta}(s))^{\varsigma+\sigma} \Delta(s) \end{aligned} \tag{8}$$

Theorem 1.11. *Let* $\mathbb{T}$ *be a time scale and* $\alpha,\beta \in \mathbb{T}$ *and let* $\varphi : [\alpha,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}^+$ *be a absolutely continuous convex function and* $C_{rd}([\alpha,\beta]_{\mathbb{T}}, \mathbb{R}^+)$ *be such that* $\xi(t)$ *is non-increasing on* $[\alpha,\beta]$ *and Suppose that* $x : [\alpha,\beta]_{\mathbb{T}} \longrightarrow \mathbb{R}^+$ *is delta differentiable with* $x(\alpha) = 0$*. Then,*

$$\begin{aligned} &\int_{[\alpha,\beta]} \xi(t) x^{\Delta}(t)^{\eta} x^{\varsigma}(t) \Delta(t) \\ &\leq \frac{\eta}{\eta+\varsigma} (\beta-\alpha)^{\varsigma\eta} \int_{[\alpha,\beta]} \left(\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^{\Delta}(t)^{\eta} \right)^{\frac{\eta+\varsigma}{\eta}} \Delta(t). \end{aligned} \tag{9}$$

The aim of this work is to generalize some inequalities of Opial-type by using Jensen′s and Hölder's inequalities for convex functions.

2. Adaptation of some inequalities

Adaptations of Jensen′s and Hölder's inequalities are considered in this section.

2.1. Adaptations of Jensen′s inequality. Let φ, ψ be continuous and convex and let $h(s,t)$ be non-negative, $s \geq 0, t \geq 0$ and λ be non-decreasing. Let $-\infty \leq \xi(t) \leq \eta(t) < \infty$, and suppose φ has a continuous inverse $(\varphi)^{-1}$(which is necessarily concave).
Then,

$$\varphi^{-1} \left[\frac{\int_{\xi(t)}^{\eta(t)} h(s,t) d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)} \right] \leq \left[\frac{\int_{\xi(t)}^{\eta(t)} (\varphi)^{-1} h(s,t) d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)} \right], \tag{10}$$

with the inequality reversed if φ is concave. The inequality (10) is known as Jensen's inequality for convex function. Setting $\varphi(u) = u^l$, $\xi(t) = 0$ and $\eta(t) = t$, then as a consequence of (10), we have for $l \geq 1$

$$(\phi(t))^l = \phi \left[\left| \frac{\int_0^t h(s,t) d\lambda(s)}{\int_0^t d\lambda(s)} \right| \right]^{\frac{1}{l}} \leq \left[\frac{\int_0^t h(s,t)^{\frac{1}{m}} d\lambda(s)}{\int_0^t d\lambda(s)} \right]. \tag{11}$$

2.2. **Adaptations of Hölder's inequality.** Hölder's inequality on time scale as expressed in [4] states that

$$\int_{\alpha}^{\beta} |f(t)g(t)|\,\Delta t \le \left[\int_{\alpha}^{\beta} |f(t)|^k \,\Delta t\right]^{\frac{1}{k}} \left[\int_{\alpha}^{\beta} |g(t)|^l \,\Delta t\right]^{\frac{1}{l}}, \tag{12}$$

where $\alpha, \beta \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{I}, \mathbb{R})$, $k > 1$ and $1/k + 1/l = 1$.

$$|c+d|^m \le 2^{m-1}(|c|^m + |d|^m), m \ge 1, \tag{13}$$

where c, d are positive real numbers.

3. Main Result

Theorem 3.1. *Let $x(t)$ be absolutely continuous on $[0, a]$ and $x(0) = 0$. Let $l > 0$. Then, the following inequality holds*

$$\int_0^a |x(t)|^l \left|x'(t)\right| dt \le \frac{a^l}{l+1} \int_0^a \left|x'(t)\right|^{l+1} dt. \tag{14}$$

In (14), equality holds if and only if,

$$x(t) = ct \tag{15}$$

Proof. Suppose $q = 1$ and $p = l + 1$ in (5). Then,

$$\begin{aligned} \left|\int_a^b t^{\delta l-1} f(t) \left[\int_t^b f(s)ds\right]^l dt\right| \le{}& [\delta^{-1}]^l a^\delta \left[\left|\int_a^b f^{l+1}(s) s^{(l-1)(1+\delta)} ds\right|\right] \\ &+ [\delta^{-1}]^l \left[\left|\int_a^b f^{l+1}(s) s^{l(1+\delta)-1} ds\right|\right]. \end{aligned} \tag{16}$$

We write (16) as

$$\begin{aligned} \left|\int_a^b t^{\delta l-1} f(t) \left[\int_t^b f(s)ds\right]^l dt\right| \le{}& [\delta^{-1}]^l a^\delta \left[\int_a^b \left|f^{l+1}(s)\right| s^{(l-1)(1+\delta)} ds\right] \\ &+ [\delta^{-1}]^l \left[\int_a^b \left|f^{l+1}(s)\right| s^{l(1+\delta)-1} ds\right]. \end{aligned} \tag{17}$$

Rearranging (17), we have

$$\begin{aligned} \left|\int_a^b t^{\delta l-1} f(t) \left[\int_t^b f(s)ds\right]^l\right| dt \le{}& [\delta^{-1}]^l \left[\int_a^b \left|f^{l+1}(t)\right| t^{l+\delta l-1} dt\right] \\ &+ [\delta^{-1}]^l a^\delta \left[\int_a^b \left|f^{l+1}(t)\right| t^{l-\delta+\delta l-1} dt\right]. \end{aligned} \tag{18}$$

Rearranging and factoring out $t^{\delta l-1}$, we have

(19)

$$0 \leq \int_a^b t^{\delta l-1}\left[[\delta^{-1}]^l\left|f^{l+1}(t)\right|t^l+[\delta^{-1}]^l a^\delta\left|f^{l+1}(t)\right|t^{l-\delta}-\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|\right]dt.$$

If $t \geq 0$ on $[a,b]$, (19) becomes

(20)

$$0 \leq \int_a^b \left[[\delta^{-1}]^l\left|f^{l+1}(t)\right|t^l+[\delta^{-1}]^l a^\delta\left|f^{l+1}(t)\right|t^{l-\delta}-\left|f(t)\left[\int_t^b f(s)ds\right]^l\right|\right]dt.$$

From (20)

$$\begin{aligned}\int_a^b \left|f(t)\left[\int_t^b f(s)ds\right]^l\right|dt &\leq \int_a^b [\delta^{-1}]^l\left|f^{l+1}(t)\right|t^l dt \\ &+ [\delta^{-1}]^l a^\delta \int_a^b \left|f^{l+1}(t)\right|t^{l-\delta}dt.\end{aligned} \tag{21}$$

Set $t^l = b^l$ for $t \in [a,b]$ and $l > 0$, (21) gives

$$\begin{aligned}\int_a^b \left|f(t)\left[\int_t^b f(s)ds\right]^l\right|dt &\leq [\delta^{-1}]^l b^l \int_a^b \left|f^{l+1}(t)\right|dt \\ &+ [\delta^{-1}]^l a^\delta \int_a^b \left|f^{l+1}(t)\right|t^{-\delta}dt.\end{aligned} \tag{22}$$

Since $|f(t)| = |-f(t)|$, (22) becomes

$$\begin{aligned}\int_a^b \left|-f(t)\left[\int_t^b f(s)ds\right]^l\right|dt &\leq [\delta^{-1}]^l b^l \int_a^b \left|-f^{l+1}(t)\right|dt \\ &+ [\delta^{-1}]^l a^\delta b^l \int_a^b \left|-f^{l+1}(t)\right|t^{-\delta}dt.\end{aligned} \tag{23}$$

Set $\delta = (1+l)^{\frac{1}{l}}$ (23), we obtain

$$\begin{aligned}\int_a^b \left|-f(t)\left[\int_t^b f(s)ds\right]^l\right|dt &\leq \frac{b^l}{l+1}\int_a^b \left|-f^{l+1}(t)\right|dt \\ &+ \frac{b^l}{l+1}a^{(l+1)^{\frac{1}{l}}}\int_a^b \left|-f^{l+1}(t)\right|t^{-\delta}dt.\end{aligned} \tag{24}$$

Set $x(t) = \int_t^b f(s)ds$ and $|x'(t)| = |-f(t)|$ into (24), we have

$$\begin{aligned}\int_a^b |x(t)|^l \left|x'(t)\right| dt \leq \frac{b^l}{l+1}\int_a^b \left|x'(t)\right|^{l+1} dt \\ + \frac{b^l}{l+1} a^{(l+1)^{\frac{1}{l}}} \int_a^b \left|x'(t)\right|^{l+1} dt.\end{aligned} \tag{25}$$

with $a \to 0$ in (25) and set $b = h$, $l = 1$ we have (2)

$$\int_0^h |x(t)| \left|x'(t)\right| dt \leq \frac{h}{2}\int_0^h \left|x'(t)\right|^2 dt. \tag{26}$$

The proof is complete. □

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