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On the Stability Analysis of a Sixth-Stage Fifth- Order Runge-Kutta Method for Solving Initial Value Problems in Ordinary Differential Equations

G. U. Agbeboh, B. C. Ononogbo* and M. E. Ehiemua

Abstract

This article investigates the stability of a sixth-stage, fifth-order Runge-kutta method which had already been derived in our previous results. The method is subjected to the test equation given as $y^{'} = \lambda y$, where a complex constant is λ, and the resulting expansion and algebraic manipulations were done with the help of Maple-18 and Matlab package, to ease the simplification of the computation. The region of stability in the method is determined to verify the nature of the stability of the method, using Matlab package.

1. Introduction

The Runge-Kutta methods for the solution of Initial value problems are one-step methods designed to approximate $y(x)$by Taylor series expansion, and have the advantage of not requiring explicit evaluation of the derivatives of $f(x, y)$, where x often represents time (t) [1]. The basic idea is to use a linear combination of values of $f(x, y)$ to approximate $y(x)$ and this linear combination is matched up as closely as possible with a Taylor series for $y(x)$ to obtain methods of the highest possible order. In order to carry out our work on stability, we will investigate the inherent properties of the Runge-Kutta method like consistency, divergence and

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Department of Mathematics, Ambrose Alli University, Ekpoma, Edo State, Nigeria
E-mail of the corresponding author: ononogbochibuike58@gmail.com
ORCID of the corresponding author: 0000-0003-3118-1011

stability of the method, we must acknowledge that in the course of implementing a method, the numerical solution should approximates the exact solution, and that the numerical solution tends to the exact solution as the step size tends to zero. But [7], observed that if the step length is too small, excessive computation time and round-off error results, and so we should also consider the opposite case, and ask whether there is any upper bound on step length. Often there is such a bound, and it is reached when the method becomes numerically unstable. This occurs when the numerical solution produced, no longer corresponds qualitatively with the exact solution [11].
The establishment of the consistency and convergence of a method, following [8], revealed that a Scheme is said to be consistent if the difference equation of the computational formula exactly approximates the differential equation it intends to solve. According [5], the traditional criterion for ensuring that a numerical method is stable is called Absolute Stability, and this analysis is carried out by subjecting the method to a linear test equation;$y' = \lambda y,\ \lambda \in \mathrm{C},\ Re\,(\lambda) < 0$. [6] further emphasized that, all Runge-Kutta methods including the implicit ones, when applied to the test equation, reduce to an equation of the form;$\frac{y_{n+1}}{y_n} = R(\mu)$. [10] acknowledged that the key issue for understanding the long term dynamics of Runge-Kutta methods near some fixed points, concerns the region where $R(\mu) \leq 1$; that is, the stability region of the numerical method. Furthermore, the polynomial for which $R(\mu) \leq 1$, is known as the Stability polynomial of the method, and this method is absolutely stable for a given $\mu = \lambda h$, if all roots of the polynomial function lie within the unit circle. The region containing all these points in the complex plane is said to be a region of absolute stability. However, Consistency and Stability properties of a method, play very significant roles in every numerical analysis [9]. According to [12], it is not possible to obtain accuracy in numerical solutions if a numerical method is not stable. For this reason, a numerical method must be stable to show its accuracy. Before we establish the stability region of this new Runge-Kutta method, we first define some very relevant terms for clearer understanding of this work. The following definition of stability is from [2].

Definition 1.1. The initial value problem in

$$y^{'} = f\,(x, y) \qquad a \leq x \leq b \;\; y\,(a) = y_0 \tag{1}$$

is said to be stiff in the interval $R = a \leq x \leq b$ if for $x \in \mathbb{R}$ the eigenvalues $\lambda_i\,(x)$ of the Jacobian $\frac{\partial x}{\partial y}$ satisfy

$$Re\lambda_i\,(x) < 0 \;\; i \;=\; 1, 2, ...m \tag{2}$$

$$\frac{\max|Re\lambda_i(x)|}{\min|Re\lambda_i(x)|} \gg 1 \tag{3}$$

The ratio

$$S = \frac{\max|Re\lambda_i(x)|}{\min|Re\lambda_i(x)|} \tag{4}$$

Is called the stiffness Ratio. If the partial derivative appearing in the Jacobian $\frac{\partial x}{\partial y}$are continuous and bounded in an approximate region, then the Lipschitz constant L may be defined as:

$$L = \left|\frac{\partial f}{\partial y}\right| \tag{5}$$

if $\max|Re\lambda| \gg 0$, it follows that $L \gg 0$ Thus, stiff system is occasionally referred to as system with large Lipschitz constants. Associated with stiff systems is the absolute stability requirement of numerical process which however use discretization process with step length h.

Definition 1.2. A one-step scheme is said to be absolutely stable at a point μ in the complex plane provided the stability function $y_i = (\mu)$ defined in (5) satisfies the following condition

$$|\lambda(\mu) < 1| \tag{6}$$

and the corresponding region of absolute stability is

$$R = \{\mu\,|\lambda y(\mu)| < 1\} \tag{7}$$

Definition 1.3. The numerical integration scheme is said to be A-stable provided that the region of absolute stability includes the entire complex half-plane with negative real part.

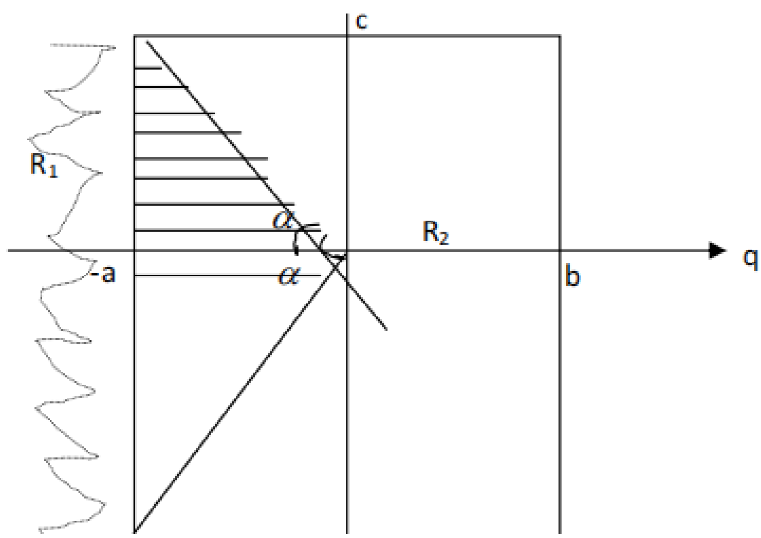
According to [2], A-stability concept was introduced by [14] as a very desirable property for any numerical integration algorithm, particularly if the Initial Value Problem are stiff and highly oscillatory. But A-stability requirement is rather too stringent, weaker and a less desirable stability criterion which accommodates higher orders has since been proposed. These include A (α)-stability introduce by [7] stiff- Stability [15], A(0)-stability and A(0)- stability, [16].

Definition 1.4. A numerical process is said to be A(α)-stable for $x \in \left(0, \frac{\pi}{2}\right)$if it's solution $\{y_n \to 0\}$ $as\ n \to \infty$

when this process is applied with fixed position into the test problem in (1) where in this case $\lambda \in (s(\alpha))$ *and* $s(\alpha) = \{z \in c := 0, |\arg(-z)| < \alpha\}$or if its region of absolute stability is continuous in the infinite wedge $SW_\alpha = \{q = \lambda h, -\alpha < \pi - \arg q < \alpha\}$ see [13].
Also a numerical process is said to be A(0)-stable if it is A(α)-stable for all (some) $\alpha \in \left(0, \frac{\pi}{2}\right)$such that $0 \le \alpha \le \frac{\pi}{2}$

Definition 1.5. A numerical method is said to be stiffly stable if (i) Its region of absolute stability contains R_1 and R_2 as shown in the figure below: (ii) It is accurate for all $q \in R$ when applied to the scalar test equation $\frac{y_{n+1}}{y_n} = R(z)$ where $R_1 = \{\lambda h \,|R_c(\lambda h) < -a|\}$, $R_2 = \{\lambda h\,|-a \le R_c(\lambda h) \le b_j, \cdots, c \le |\, \mu(\lambda h) \le c\}$ and a, b and c are positive constants.

The reason for this definition is to represent eigenvalue with rapidly decaying terms in the transient solution by corresponding λy *in* R_1



Definition 1.6. A one-step method is said to be L-stable if it is A-stable and in addition when applied to the stable test equation $y' = \lambda y$, $\lambda \in \mathrm{C}$, $Re(\lambda) < 0$ it yields $y_{n+1} = R(\lambda h)\, y_n$, $|Re(\lambda h)| \to 0$ *as* $Re\lambda h \to \infty$ (An L-stable one-step numerical method is necessarily A-stable and A(0)-stable).

We may now state the existence and uniqueness theorem without proof.

Definition 1.7. A Runge–Kutta method applied to the non-linear system $y^{'} = f(y)$ which verifies $(f(y) - f(\mu), y - \mu) < 0$ is called *B-stable*, if this condition implies $\|y_{n+1} - \mu_{n+1}\| \le \|y_n - \mu_n\|$ for two numerical solutions [17].

Theorem 1.8. *Let f(x,y) be defined and continuous on R given by*$R^{m+1} = [a,b]X\left\{y\left|y_\infty \leq \tau < \infty\right|\right\}$*and in addition, satisfy the inequalities* $f\left(x,y\right) - f\left(x,z\right)_\infty \leq Ly - Z_\infty$*where* ∞*is the maximum norm defined as* $y_\infty = \max_{1\leq r\leq m} |y_r|$ *Then the initial value problem* (1) *has a unique solution in R.*

This theorem will be our guide in the proof of the stability region for the new method. With the above definition and theorem, we will now analyze the inherent properties of the new method such as the consistency and stability the scheme.

2. Consistency Analysis of the Method

In order to establish the consistency and convergence of the method, the following steps will be taken.

2.1. **Let** $h_p = \frac{b-a}{2^p} \quad \forall\ p = 0, 1, 2, ...$ **where** $y_p(x)$ **is the piecewise continuous linear function whose graph is the polygon with the vertexes at the approximate solution point** (x_n, y_n)**.** To describe the $y_p(x)$ analytically, specify $x \in (a,b]$ by $x_{[p]}$ the nearest lattice point to the left of x which must not coincide with x in the p^{th} subdivision so that

$$x_{[p]} = a + nh_p \tag{8}$$

with n as the unique integer satisfying

$$a + nh_p < x \leq a + a + (n+1)\,h_p$$

Then we define the function $y_p(x) = \eta$ and

$$y_p(x) = y_p\left(x_{[p]}\right) + \left(x - x_{[p]}\right) f\left(x_{[p]}, y_p\left(x_{[p]}\right)\right) \tag{9}$$

These functions are trivially continuous in each of the intervals $\left(x_{[p]}, x_{[p]} + h_p\right)$and by construction, also at the points $x_{[p]}$ themselves.

Theorem 2.1. *The sequence of vector-valued functions* $y_{p(x)}$ *convergences for* $p \to \infty$ *uniformly for*$x \in [a,b]$*, to a continuous function* $y\left(x\right)$*.*

This will be prove later in this paper.

(1) The existence of solution of Initial Value Problem. In discussing the existence of solution of initial value problems, we assume that the vector-valued function $f(x,y)$ of the scalar variable x and the vector

$y = \left(y^{'}, y^2, ...y^s\right)$ satisfies the following two hypotheses:

(1) that f(x, y) is defined and continuous in the region $a \leq x \leq b$,
(2) there exist a constant L, the Lipschitz constant such that for any arbitrary $x \in [a,b]$ and any two vectors $y\ and\ y^*$ we have

$$\|f(x,y) - f(x,y^*)\| \leq L\,\|y - y^*\| \tag{10}$$

The sequence of vector-value functions $y_p(x)$converges for $p \to \infty$ uniformly for $x \in [a, b]$ to a continuous function $y(x)$.
Note: A **norm** is a function from a real or complex vector space to the non-negative real numbers that behaves in certain ways like the distance from the origin: it commutes with scaling, obeys a form of the triangle inequality, and is zero only at the origin. In particular, the Euclidean distance of a vector from the origin is a norm, called the Euclidean norm, or Norm which may also be defined as the square root of the inner product of a vector with itself.

Proof. Using condition (ii) with $y^* = 0$ we find

$$\|f(x\ ,\ y)\| \leq \|y\| + c \tag{11}$$

Where

$$c = \max_{x\in[a,b]} \|f(x\ ,\ 0)\| \tag{12}$$

From equation (9) we get

$$y_p(x_{[p]} + h_p) = y_p\left(x_{[p]}\right) + h_p f\left(x_{[p]}, y_p\left(x_{[p]}\right)\right) \quad \forall\, x \in (a, b - h_p] \tag{13}$$

Thus, from (11) we have

$$\left\|y_p\left(x_{[p]} + h_p\right)\right\| \leq (1 + h_p L)\left\|y_p\left(x_{[p]}\right)\right\| + h_p c \tag{14}$$

Now if the numbers $\lambda_n$satisfy $|\lambda_{n+1}| < A|\lambda_n| + B$
Where A and B are certain non-negative independent of n. it is then desirable to have an estimate for $|\lambda_n|$in terms of $|\lambda_n|$instead of $|\lambda_{n-1}|$
Hence for $n = 0, 1, 2, c \ldots, \mathbb{N} - 1$, we have

$$|\lambda_n| \leq A^n |\lambda_0| + \begin{cases} \frac{A^n-1}{A-1}B, & A \neq 1 \ \ \forall\, n = \ 1, 2, \cdots, \mathbb{N} \\ & A = 1 \end{cases}$$

Therefore if

$$\lambda_n = \|y\left(a + nh_p\right)\|, A = (1 + h_p L), B = h_p c$$

It follows that

$$\left\|y_p\left(x_{[p]}\right)\right\| \leq e^{\left(x_{[p]}-a\right)L\|\eta\| + E_L\left(x_{[p]}-a\right)c} \tag{15}$$

Where $E_L$is the Lipschitz function since the function $y_p\left(x\right)$are linear between any two adjacent points $x_p$we have

$$\left\|y_p\left(x_{[p]}\right)\right\| \leq y, \ \ \forall\, x \in [a, b] \tag{16}$$

Where

$$Y = e^{(b-a)L}\|\eta\| + E_L\left(b - a\right)c \tag{17}$$

Let R be the compact region in (x, y) space defined by $x \in [a, b], \|y\| \leq y$. Then from (16) we have that the functions $y_p\left(x\right)$ stay within R for $x \in [a, b]$ and p =

0,1,2... in view of the compactness of R, a function which is continuous in R has a finite Maximum in R. so that

$$m = \max_{(x,y)\in R} \|f(x,y)\| \tag{18}$$

Therefore, given any $\delta \geq 0$ we set

$$w(\delta) = \max \|f(x,y) - f(x^*,y)\| \tag{19}$$

Where the maximum is taken with respect to all points $(x,y)\, and (x^*,y)\, in\, R$such that $|x - x^*| \leq \delta$. so that

$$\underset{\delta\to 0}{Lim} w(\delta) = 0 \tag{20}$$

Applying Cauchy criterion, we put

$d(x) = y_p(x) - y_q(x)$ where p and q are any two nonnegative integers, $p < q$. □

Following same procedure we can show that our method

$$y_{n+1} - y_n = \frac{h}{144}\left(14k_1 + 48k_2 + 162k_3 + 33k_4 - 125k_5 + 12k_6\right) \tag{21}$$

whose slopes are;

$$\left.\begin{array}{l} k_1 = f(y_n) \\ k_2 = f\left(y_n + \frac{1}{3}hk_1\right) \\ k_3 = f\left(y_n + \frac{2}{3}hk_2\right) \\ k_4 = f\left(y_n + h\left(-\frac{167765027}{45900120}k_1 + \frac{43549}{7217}k_2 - \frac{30361}{14840}k_3\right)\right) \\ k_5 = f\left(y_n + h\left(-\frac{51638854921283}{283667160186l5}k_1 + \frac{35525}{9169}k_2 - \frac{27646}{19955}k_3 - \frac{10643}{155037}k_4\right)\right) \\ k_6 = f\left(y_n + h\left(-\frac{9039268043}{1401332565}k_1 + \frac{736810}{53619}k_2 - \frac{28702}{5227}k_3 - \frac{7}{5}k_4 + \frac{3}{5}k_5\right)\right) \end{array}\right\} \tag{22}$$

is consistent with the initial value problem:

$$y' = f(x,y)\,,\;\; y(a) = \delta,\;\; a \leq x \leq b \tag{23}$$

whose solution function $x \in [a, b \to R]$where a and b are finite.

Proof. Let

$$y_{n+1} - y_n = \frac{h}{144}\left(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4 + b_5k_5 + b_6k_6\right) + T_n\left(h^6\right) \tag{24}$$

where the values of the b_i^s for $i = 1, 2, 3, \cdots, 6$ are as in (21) above, with

$$(25)\qquad \begin{cases} k_1 = f(y_n) \\ k_2 = f(y_n + a_1 h k_1) \\ k_3 = f(y_n + a_3 h k_2) \\ k_4 = f(y_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3)) \\ k_5 = f(y_n + h(a_7 k_1 + a_8 k_2 + a_9 k_3 + a_{10} k_4)) \\ k_6 = f(y_n + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4 + a_{15} k_5)) \end{cases}$$

Substituting (25) into (24), we have;
(26)

$$y_{n+1} - y_n = \frac{h}{144}\begin{pmatrix} \begin{pmatrix} b_1 f(y_n) + b_2 f(y_n + a_1 h k_1) + b_3 f(y_n + a_3 h k_2) \\ +b_4 f(y_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3) \end{pmatrix} \\ b_5 f(y_n + h(a_7 k_1 + a_8 k_2 + a_9 k_3 + a_{10} k_4) \\ +b_6 f(y_n + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4 + a_{15} k_5) \end{pmatrix} + T_n\left(h^6\right)$$

Rearranging (26), we have:
(27)

$$T_n\left(h^6\right) = y_{n+1} - y_n - \frac{h}{144}\begin{pmatrix} \begin{pmatrix} b_1 f(y_n) + b_2 f(y_n + a_1 h k_1) + b_3 f(y_n + a_3 h k_2) \\ +b_4 f(y_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3) \end{pmatrix} \\ b_5 f(y_n + h(a_7 k_1 + a_8 k_2 + a_9 k_3 + a_{10} k_4) \\ +b_6 f(y_n + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4 + a_{15} k_5) \end{pmatrix}$$

Dividing all through by h
(28)

$$T_n\left(h^5\right) = \frac{(y_{n+1} - y_n)}{h} - \frac{1}{144}\begin{pmatrix} \begin{pmatrix} b_1 f(y_n) + b_2 f(y_n + a_1 h k_1) + b_3 f(y_n + a_3 h k_2) \\ +b_4 f(y_n + h(a_4 k_1 + a_5 k_2 + a_6 k_3) \end{pmatrix} \\ b_5 f(y_n + h(a_7 k_1 + a_8 k_2 + a_9 k_3 + a_{10} k_4) \\ +b_6 f(y_n + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4 + a_{15} k_5) \end{pmatrix}$$

Now take limits of both sides of (28) as h→0, we obtain: $\frac{y_{n+1} - y_n}{h} \to y^{'}$, Hence

$$(29)\quad 0 = y' - \frac{1}{144}\left(b_1 f(y_n) + b_2 f(y_n) + b_3 f(y_n) + b_4 f(y_n) + b_5 f(y_n) + b_6 f(y_n)\right)$$

$$(30)\qquad 0 = y' - \frac{1}{144}\left(b_1 + b_2 + b_3 + b_4 + b_5 + b_6\right) f(y_n)$$

which implies

$$0 = y^{'} - \frac{1}{144}\left(14 + 48 + 162 + 33 - 125 + 12\right) f\left(y_n\right)$$

Therefore

$$(31)\qquad 0 = y' - f(y_n)$$

$$(32)\qquad y' = f(y_n)$$

Hence the method is consistent. Consequently, as stated earlier by [5], if a method is consistent, it invariably converges. □

3. Stability Analysis of the Method

Stability plays a significant role in evaluating numerical approaches. Here, we study the stability of the proposed fifth-order Runge-Kutta Method and we will show that it has sufficiently-wide stability region. Following an approach similar to that of [2], the analysis presented below is based on examining what happens when a method is applied to a simple scalar linear first-order ODE of the form.

$$y^{'} = \lambda h \qquad \lambda \in \mathrm{C}; Re\left(\lambda\right) < \ 0 \tag{33}$$

where λ is a complex plane.
According to [3], the traditional criterion for ensuring that a numerical method is stable is called Absolute Stability and this analysis is carried out by subjecting the method to a linear test equation as can been seen in equation (33). Furthermore, all Runge – Kutta methods including implicit ones, when applied to the test equation, reduce to an equation of the form:

$$y_{n+1} = R\left(\lambda h\right) \tag{34}$$

where $R\left(\lambda h\right)$ is called the stability polynomial function. And so when we write $\mu = \lambda h$
it produces a linear system for the computation of $k_1, k_2, k_3, k_4, k_5,\ and\ k_6$ which will be solved for, and inserted into our method, leading to the equation

$$\frac{y_{n+1}}{y_n} = R\left(\mu\right) \tag{35}$$

[4], says, the key issue for understanding the long term dynamics of Runge-kutta methods near some fixed points, concern the region where $R\left(\mu\right) \leq 1$; that is, the stability region of the numerical method. The polynomial, for which $R\left(\mu\right) \leq 1$ is known as the stability polynomial of the method, and this method is absolutely stable for a given $\mu = \lambda h$ if all the roots of the polynomial function lie within the unit circle. It is also possible according to [4], that applying a method to the test equation (33) yields.

$$Y_i = y_n + \mu \sum_{j=1}^{s} a_{ij} Y_j \tag{36}$$

$$y_{n+1} = y_n + \mu \sum_{i=1}^{s} b_i Y_i \tag{37}$$

Now defining $Y, e \in R^s$ by $Y = [Y_1, Y_2, ..., Y_s]^T$and $e = [1, 1, ..., 1]^T$;we write by

(38) $$Y = y_n e + \mu AY \qquad and \quad y_{n+1} = y_n + \mu b^T Y$$

Solve the first of these for Y and substituting in the second gives:

(39) $$y_{n+1} = y_n \left[1 + \mu b^T (I - \mu A)^{-1} e\right]$$

Where I is the **s x s** unit matrix, the stability function is therefore given by

(40) $$R(\mu) = \mathrm{I} + \mu b^T (I - \mu A)^{-1} e$$

An operation according to [3] gives an alternative form of $R(\mu)$, such that the solution for y_{n+1} by Cramers's rule is $y_{n+1} = \frac{N}{D}$ where $y_{n+1} = y_n$+det $\left[I - \mu A + \mu e b^T\right]$
$D = \det [I - \mu A]$
Hence

(41) $$y_{n+1} = R(\mu) y_n = \frac{N}{D}$$

Then we have,

(42) $$\frac{y_{n+1}}{y_n} = R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} e = \frac{\det \left[I - \mu A + \mu e b^T\right]}{\det [I - \mu A]}$$

where e stands for the vector of ones. The function R is called the *stability function.* It follows from the formula that R is the quotient of two polynomials of degree S if the method has S stages. Explicit methods have a strictly lower triangular matrix A, which implies that $\det(I - \mu A) = 1$and that the stability function is a polynomial. The numerical solution to the linear test equation decays to zero if $|R(\mu)| < 1 \; with \; \mu = h\mu$. The stability function of an explicit Runge–Kutta method is a polynomial, so explicit Runge-Kutta methods can never be A-stable [17].
If the method has order p, then the stability function satisfies $R(\mu) = e^\mu + O\left(\mu^{p+1}\right) \; as \; \mu \to 0$.Thus, it is of interest to study quotients of polynomials of given degrees that approximate the exponential function the best. These are known as Padé approximant. A Padé approximant with numerator of degree m and denominator of degree n is A- stable if and only if $m \leq n \leq m + 2$ [17]. The Gauss–Legendre method with s stages has order $2s$, so its stability function is the Padé approximant with $m = n = s$. It follows that the method is A-stable. This shows that A-stable Runge–Kutta can have arbitrarily high order. In contrast, the order of A-stable linear multistep methods cannot exceed two.

3.1. **Stability of the Method.** The *A-stability* concept for the solution of differential equations is related to the linear autonomous equation $y^{'} = \lambda y$. Dahlquist proposed the investigation of stability of numerical schemes when applied to nonlinear systems that satisfy a monotonicity condition. The corresponding concepts were defined as *G-stability* for multistep methods (and the related one-leg methods) and *B-stability* for Runge-Kutta methods [18].

A Runge–Kutta method applied to the non-linear system $y^{'} = f\left(y\right)$which verifies $\left(f\left(y\right) - f\left(\mu\right), y - \mu\right) < 0$is called *B-stable*, if this condition implies $\|y_{n+1} - \mu_{n+1}\| \leq \|y_n - \mu_n\|$ (**??**).

for two numerical solutions.

Let , **B**, **M** and **Q** be three matrices defined by

$$\mathrm{B} = diag\left(b_1, b_2, ..., b_s\right), \qquad \mathrm{M} \; = \; \mathrm{BA} + \mathrm{A}^T B - bb^T \qquad Q = \; BA^{-1} + A^{-T}B - A^{-T}bb^T A^{-1}$$

A Runge–Kutta method is said to be *algebraically stable,* if the matrices **B and M**

are both non-negative definite [17]. A sufficient condition for *B-stability* is **B** and **Q** are non-negative definite. Following the same approach we presented the stability region of our new method in equation (21) by applying the linear test equation analytically.

Proof:

$$k_1 = y^{'} = \lambda y_n$$

$$k_2 = \lambda y_n(1 + \frac{1}{3}h\lambda)$$

$$k_3 = \lambda y_n(1 + \frac{2}{3}h\lambda + \frac{2}{9}h^2\lambda^2)$$

$$k_4 = \lambda y_n(1 + \frac{1}{3}h\lambda + \frac{14859749}{22950060}h^2\lambda^2 - \frac{30361}{66780}h^3\lambda^3)$$

$$k_5 = \lambda y_n(1 + \tfrac{3}{5}h\lambda + \tfrac{29359148574014}{85100148055845}h^2\lambda^2 - \tfrac{1667691084477203}{4733470277603340}h^3\lambda^3 + \tfrac{323132123}{10353370860}h^4\lambda^4)$$

$$k_6 = \lambda y_n(1 + h\lambda + \tfrac{17091887746}{21019988475}h^2\lambda^2 - \tfrac{17607999141369252004177}{91721569315685354253000}h^3\lambda^3 + \tfrac{1509175664474488 3}{35501027082025050}h^4\lambda^4 + \tfrac{323132123}{17255618100}h^5\lambda^5)$$

Substituting all k_1, k_2 , k_3 , k_4 , k_5 , k_6 into equation (21), simplifying the expression, letting

$z = h\lambda$ Also dividing both side by $y_n$we have;

$$\frac{y_{n+1}}{y_n} = \left(\begin{array}{l} 1 + z + \frac{1}{2}z^2 + \frac{2.754\times10^{29}}{1.652\times10^{29}}z^3 + \frac{6.420\times10^{22}}{1.540\times10^{24}}z^4 \\ + \frac{3.651\times10^{16}}{4.381\times10^{18}}z^5 \end{array} \right)$$

To get the roots of the stability polynomial, we equate the RHS to Zero, and using a symbolic manipulation tool (MATLAB) to reduce complexity, we have the following;

$$z_1 = \; 0.2398 \; + \; 3.1283\mathrm{i}$$
$$z_2 = 0.2398 \; - \; 3.1283\mathrm{i}$$
$$z_3 = -2.1806$$
$$z_4 = \; -1.6495 \; + \; 1.6939\mathrm{i}$$
$$z_5 = \; -1.6495 \; - \; 1.6939\mathrm{i}$$

That is $Z = a \pm ib$ *where* $a \in \mathbb{R}$ *and* $ib \in \mathbb{C}$.
The figure below shows the stability region of new sixth-stage fifth-order Runge-Kutta method

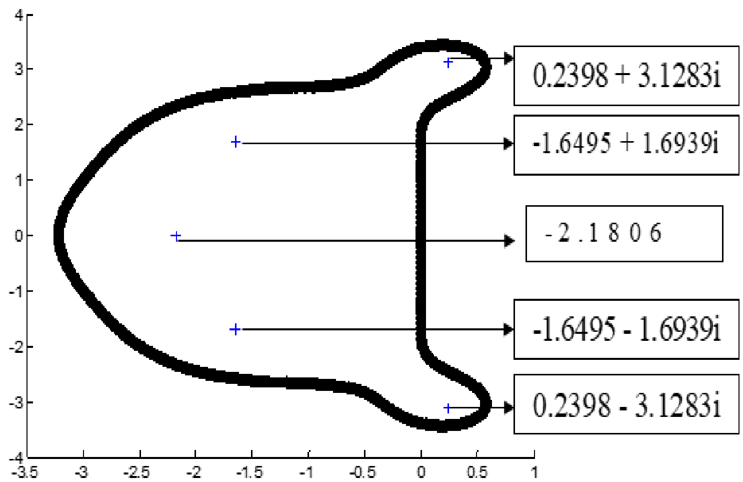


Figure 1: Region of Absolute Stability of the Method

This implies that the method is absolutely stable since all the polynomial roots lies within the unit cycle [5].

Discussion. In this paper, the inherent properties of the proposed Sixth-Stage Fifth-Order Runge-Kutta method were properly investigated. The polynomial function of the method was derived through the help of Maple-18 software. Furthermore, the roots of the polynomial function as well as the curve were obtained with MATLAB to determine the region of Absolute stability of the method. Moreover, the stability region is on the left half of the complex plane, and the roots of the stability function were found to be within the region of stability. Furthermore, we established the condition of a method to be B-stable, we can therefore, say that the new method is B-stable.

Conclusions: In conclusion, this Sixth-Stage Fifth-Order Runge-Kutta method can be used to proffer solutions to initial value problems as well as real life problems as seen in [1] which can be modeled to first order differential equations.

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APPENDIX I
MATLAB CODE FOR ROOTS OF THE POLYNOMIAL FUNCTION OF THE NEW METHOD

```
m=[36515342249279243/4381841056981377600 642050984799714823335337/15409223645035139514504000
27540905648531089258552715461/165245432524627829125637995200 1/2 1 1]
m =0.0083 0.0417 0.1667 0.5000 1.0000 1.0000
> > roots_m=roots(m)
roots_m =
0.2398 + 3.1283i
0.2398 - 3.1283i
-2.1806
-1.6495 + 1.6939i
-1.6495 - 1.6939i
```

APPENDIX II
MATLAB CODE FOR THE ABSOLUTE STABILITY REGION OF THE NEW METHOD

```
Q=0:0.001:2*pi;
a=zeros(5,length(Q));
for k=1:length(Q)
c=[1/120 1/24 1/6 1/2 1 1-exp(i*Q(k))];
a(:,k)=roots(c);
end
hold on
plot(a(1,:),'ko')
plot(a(2,:),'ko')
plot(a(3,:),'ko')
plot(a(4,:),'ko')
plot(a(5,:),'ko')
P= [36515342249279243/4381841056981377600 642050984799714823335337/15409223645035139514504000
27540905648531089258552715461/165245432524627829125637995200 1/2 1 1 ];
M=roots(P);
plot(M,'+')
```