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New Approximate Method for Systems of Non-linear Volterra Integro-Differential Equations

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Abstract

In this paper, we present a new iterative scheme for approximating nonlinear systems of Volterra-integro-differential equations. The algorithm relies on the principles of the Variational Iteration Method (VIM). The modified method provides successive approximations by variational theory, decomposition techniques, and correction functional on Lagrange's multiplier. In comparing with numerous results in literatures, which frequently rely on discretization or linearization techniques, this result is of tremendous importance and fewer iterations are enough to obtain a higher degree of accuracy in the solution. We considered three numerical examples to demonstrate the effectiveness and efficiency of the method.

1. Introduction

Mathematical models are usually formulated to aid the understanding of dynamical systems and these models often result in integro-differential equations that

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involve both integral and derivatives of an unknown function. One of the simplest nonlinear integro-differential models studied in literature is the Volterra model [26] which may be used to describe the evolution of a population in a closed environment [6]. Moreover, nonlinear Volterra equations naturally appear in memory-dependent problems like epidemiology [13]. Some other applications of nonlinear integral equations can be found in [14]. Methods for solving nonlinear Volterra equations have enjoyed lots of attention from researchers. These include the convolution integral approach [18], application of Legendre polynomials [23], collocation [4, 12, 14], and method of Euler polynomials [17]. Bernstein operation matrix was used by [17] to solve systems of a higher-order Volterra-Fredholm integro-differential equations [17]. The variational iteration approach is an effective tool for solving a variety of linear and nonlinear functional differential equations. Ji-Huan He [10, 13, 16] pioneered this concept, which has since been adopted by many scientists to solve various kinds of differential equations [1, 2, 5, 8, 20, 22, 23, 24, 25]. The fundamental feature of this approach is that the solution of a mathematical problem under the assumption of linearization is employed as an initial approximation or trial function, and it is based on the general Lagrange's multiplier method.

In this paper, modifying the traditional variational method and Wazwaz techniques [28], we propose and implement a modified variational iterative method MVIM to find approximate solution of the following system of the nonlinear VID Equations:

$$\text{(1)} \qquad \begin{aligned} \mathrm{u}^{(n)}(\mathrm{x}) &= \mathrm{g}_1(\mathrm{x}) + \lambda_1 \int_{\mathrm{a}}^{\mathrm{x}} (\mathrm{K}_1(\mathrm{x},\mathrm{s})\,\mathrm{u}^{k}(\mathrm{s}) + \widetilde{\mathrm{K}}_1(\mathrm{x},\mathrm{s})\,\mathrm{v}^{k}(\mathrm{s}))\mathrm{ds}, \\ \mathrm{v}^{(n)}(\mathrm{x}) &= \mathrm{g}_2(\mathrm{x}) + \lambda_2 \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{K}_2(\mathrm{x},\mathrm{s})\mathrm{u}^{m}(\mathrm{s}) + \widetilde{\mathrm{K}}_2(\mathrm{x},\mathrm{s})\,\mathrm{v}^{m}(\mathrm{s})\,\mathrm{ds}, \end{aligned}$$

for all

$$0 \le a \le x, s \le b,$$

with the initial conditions:

$$\text{(2)} \qquad \mathrm{u}^{(j)}(0) = c_j,\ \mathrm{v}^{(j)}(0) = d_j,\ 0 \le j \le (n-1)$$

In (1) , $u^{(n)}(\mathrm{x})$, $\mathrm{v}^{(\mathrm{n})}(\mathrm{x})$ indicates the $\mathrm{n-th}$ derivatives. $\mathrm{K}_1(\mathrm{x},\mathrm{s})$, $\mathrm{K}_2(\mathrm{x},\mathrm{s})$, $\widetilde{\mathrm{K}}_1(\mathrm{x},\mathrm{s})$, $\widetilde{\mathrm{K}}_2(\mathrm{x},\mathrm{s})$ are non-linear functions while λ_1 and λ_2 are all real finite constants. The integers k and m satisfy $k, m < n$.

The remaining part of this paper is organized as follows: Section 2 starts with a brief overview of the conventional variation iteration method (VIM) and ends with the presentation of our proposed modified variational iteration scheme (MVIM). The convergence analysis of the MVIM is proved in Section 3. The numerical implementation of MVIM and discussions of the results are in Section 4.

List of Abbreviations
VIM: Variational Iteration Method
MVIM: Modified Variational Iteration Method
SNVIDE: Systems of nonlinear Volterra-Integro-Differential Equations
VIDE: Volterra-Integro-Differential Equations

2. Conventional Variational Iteration Method

The variational iteration method has found wide applications for the solution of nonlinear differential equations but it is based upon some restricted variations and correctional functions for the calculation of the Lagrange multiplier. [13] used it for solving some integro-differential equations by choosing the initial approximate solution in the form of an exact solution with unknown constants. It is worth mentioning that the origin of the variational iteration method can be traced back to [11], however, the real potential of the method was explored by [21] who used the variational iteration method to solve integro-differential equations and he investigated in his research that Lagrange multiplier can be regarded as a Green function.
Let us consider the general Volterra integro-differential equation defined by

$$u^{(n)}(x) = g_i(x) + \sum_{i=0}^{m} \lambda_i \int_a^x (K_i(x,t)G(u(t))dt, \tag{3}$$

with the initial conditions $u^{(j)}(0) = c_j,\ 0 \leq j \leq (n-1)$, where $u^{(n)}(x)$ is the $n-th$ derivatives of $u(x)$ and $G(u(t))$ is non-linear. It is of important to point out that $u(x),\ g(x)$ are assumed to be real and, λ_i $(i{=}\ 0,\ 1,\ ...j-1)$ are all real finite constants.
Given a differential equation:

$$Lu + \aleph(u) = g(x), \tag{4}$$

where the inhomogeneous term is $g(x)$, L, $\aleph$ are linear and nonlinear operators, respectively. A correction functional for equation (3) is presented by the variational iteration method in the form:

$$\mathrm{u}_{n+1}(\mathrm{x}) = \mathrm{u}_n(\mathrm{x}) + \int_{\mathrm{a}}^{\mathrm{x}} \lambda(\tau)(Lu_n(\tau) + \aleph\overline{u}_n(\tau) - \mathrm{g}(\tau))\mathrm{d}\tau. \tag{5}$$

The parameter λ is a general multiplier of Lagrange, which can be optimally defined by variational theory, i.e. integration by parts and the use of a minimal variation. λ can be a constant or a function and δ is a restricted value, meaning $\overline{u}_n$is considered as restricted variation($\delta\overline{u}_n = 0$), where δ is the variational derivative. $\delta \mathrm{u}_{n+1} = 0$, a requirement for an extremum condition for u_{n+1} , produces the stationary conditions:

$$1 + \lambda|_{\tau=x} = 0, \quad \lambda'\Big|_{\tau=x} = 0, \text{ so }, \lambda = -1. \tag{6}$$

In fact, it is proved in [28] that

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1}, \tag{7}$$

is the general Lagrange formulas for n-th order differential equation. Upon using selective function $u_n(x)$, successive approximations $u_{n+1}, \quad n \geq 0$, of the solution $u(x)$ can be quickly obtained and the solution $u(x) = \lim_{n\to\infty} u_n(x)$. The convergence of variational iteration method relies primarily on Banach fixed point theorem [25, 21].

2.1. **Modified Variational Iteration Scheme.** The theory and implementation of the method of decomposition with initial conditions on Integro-differential problems [3, 29] is used to propose a modified variational iteration method (MVIM) for solving the general nth-order nonlinear Integro-differential equations. The simplification of the non-linear term will help to reduce a lot of computational work involved. The non-linear term is expressed in a special way that offers a better approximation than Adomian, Bell and orthogonal polynomials. We use this MVIM to solve the system of nonlinear Volterra Integro-differential equations (SVIDEs) in (1) and (2).
Considering a special case of general n-th-order nonlinear integro-differential equation,

$$\mathrm{u}^{(\mathrm{n})}(x) = \mathrm{g}(x) + \lambda \int_{\mathrm{a}}^{\mathrm{h(x)}} \mathrm{K}(x,\mathrm{t})\, \mathrm{u}^p(\mathrm{t})\mathrm{dt} \tag{8}$$

subject to the initial conditions:

$$\mathrm{u}^{(r)}(0) = c_r. \tag{9}$$

where $c_r,\ r = 0,\ 1, \ldots (n-1)$are real constant and p an integers with $p \leq n$.
Equation (8) can take the functional form:

$$\mathcal{L}u = f + \aleph(\mathrm{u}), \tag{10}$$

where $\aleph$ is the non-linear operator and f is a known analytical function. The nonlinear operator $\aleph(\mathrm{u})$ is decomposed as follows:

$$\aleph\left(\int_{i=0}^{\infty} u_i\right) = \aleph(u_0) + \sum_{k=1}^{\infty}\left\{\aleph\left(\sum_{k=0}^{i} u_k\right) - \aleph\left(\sum_{k=0}^{i-1} u_k\right)\right\}, \tag{11}$$

where $u_k(x)$ are polynomials of x. Suppose the solution u is of the form:

$$u = \sum_{k=0}^{\infty} u_k. \tag{12}$$

Then, in view of equations (8) and (9), for sequential integro-differential equation, we get

$$\aleph\left(\int_{i=0}^{\infty} u_i\right) = f + \aleph(u_0) + \sum_{i=1}^{\infty}\left\{\aleph\left(\sum_{k=0}^{i} u_k\right) - \aleph\left(\sum_{k=0}^{i-1} u_k\right)\right\}. \tag{13}$$

Thus, we define the recurrence relation as:

(14)

$$\left.\begin{array}{c} \mathrm{u}_0 = f, \\ \mathrm{u}_1 = \aleph\left(\mathrm{u}_0\right), \\ \mathrm{u}_2 = \aleph\left(\mathrm{u}_0 + \mathrm{u}_1\right) - \aleph\left(\mathrm{u}_0\right), \\ \mathrm{u}_3 = \aleph\left(\mathrm{u}_0 + \mathrm{u}_1 + \mathrm{u}_2\right) - \aleph\left(\mathrm{u}_0 + \mathrm{u}_1\right), \\ \vdots \\ \mathrm{u}_{n+1} = \aleph\left(\mathrm{u}_0 + \mathrm{u}_1 + \cdots + \mathrm{u}_n\right) - \aleph\left(\mathrm{u}_0 + \mathrm{u}_1 + \cdots + \mathrm{u}_{n-1}\right)\ ,\ n = 1,\ 2, \cdots \end{array}\right\}.$$

In (5), rewrite the non-linear term as $\aleph\tilde{u}_n(\tau) = \aleph\mathrm{u}_n(\tau)$. Summing the Left Hand Side of (12), from $n = 0,\ 1,\ 2, \cdots$ and from (13), we get the n-th term approximate solution

$$\mathrm{u}_0 + \mathrm{u}_1 + \ldots \mathrm{u}_{n+1} = \aleph\left(\mathrm{u}_0 + \mathrm{u}_1 + \cdots + \mathrm{u}_n\right),$$

$$u = \sum_{n=0}^{n-1} u_n(x).$$

We define a linear operator $\mathcal{L}$ as $\frac{d}{dx}$ and its inverse as:

$$\mathcal{L}^{-1} = \int_0^{\mathrm{x}} (.)\, \mathrm{dx}. \tag{15}$$

Taking the inverse of (10), we get

$$u(x) = \left(u(0) + \mathcal{L}^{-1} f\right) + \mathcal{L}^{-1}\left(\aleph(u)\right). \tag{16}$$

The recurrence relation below is used for the determination of the components:

(17)

$$\begin{aligned} \mathrm{u}_0(x) &= u(0) + \mathcal{L}^{-1}(f) \\ \mathrm{u}_1(x) &= \aleph(\mathrm{u}_0) = \mathcal{L}^{-1}(\aleph(\mathrm{u}_0)) \\ \mathrm{u}_2(x) &= \aleph(\mathrm{u}_0+\mathrm{u}_1) - \aleph(\mathrm{u}_0) = \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1) - \mathrm{u}_1 \\ \mathrm{u}_3(x) &= \aleph(\mathrm{u}_0+\mathrm{u}_1+\mathrm{u}_2) - \aleph(\mathrm{u}_0+\mathrm{u}_1) = \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1+\mathrm{u}_2) - \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1) \\ \mathrm{u}_4(x) &= \aleph(\mathrm{u}_0+\mathrm{u}_1+\mathrm{u}_2+\mathrm{u}_3) - \aleph(\mathrm{u}_0+\mathrm{u}_1+u_2) = \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1+\mathrm{u}_2+\mathrm{u}_3) - \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1+u_2) \\ &\vdots \\ &\vdots \end{aligned}$$

Progressing in this way, the $(n+1)\,th$ approximation of the exact solutions for the unknown functions $u(x)$ can be obtained as follows:

$$\begin{aligned} &\mathrm{u}_{n+1}(x) = \aleph(\mathrm{u}_0+\mathrm{u}_1+\cdots+\mathrm{u}_n) - \aleph(\mathrm{u}_0+\mathrm{u}_1+\cdots+\mathrm{u}_{n-1}) \\ &= \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1+\cdots+\mathrm{u}_n) - \mathcal{L}^{-1}(\aleph(\mathrm{u}_0+\mathrm{u}_1+\cdots+\mathrm{u}_{n-1}). \end{aligned} \tag{18}$$

Based on the above, we construct the solution

$$u(x) = \mathcal{L}^{-1}\sum_{n=0}^{n-1} u_n(x), \qquad n \geq 0. \tag{19}$$

In view of (5), (18) and (19) we then formulate the modified algorithms as:

$$\mathrm{u}_{n+1}(x) = \mathrm{u}_n(\mathrm{x}) + \int_{\mathrm{a}}^{x} \lambda(\tau)\left[Lu_n(\tau) - \mathrm{g}(\tau) + \mathcal{L}^{-1}\sum_{n=0}^{n-1} u_n(\tau)\mathrm{dr}\right]\mathrm{d}\tau. \tag{20}$$

Applying (7), we get

(21)

$$\begin{aligned} &u_{n+1}(x) = u_n(x) \\ &+ \int_a^x (-1)^n \tfrac{1}{(n-1)!}(\tau-x)^{n-1}[Lu_n(\tau) - g(\tau) + L^{-1}\textstyle\int_a^\tau k(\tau,r)\sum_{n=0}^{n-1} u_n(\tau))dr]d\tau. \end{aligned}$$

(21) is our modified variational iteration scheme for finding approximate solution to (8).

2.2. **Convergence Analysis of the Modified Variational Iteration Method.** In this section, we present the convergence of the solution obtained by the algorithm in (18) to the exact solution. This proof requires that the nonlinearity $\aleph\tilde{u}_k$ be analytic in x and u. For simplicity and clarity's sake, we consider the first order nonlinear (IDE) of the second kind:

$$\mathrm{u}'(x) = \mathrm{f}(x) + \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{K}(x,\mathrm{t})\mathrm{F}(\mathrm{u}(\mathrm{t}))\mathrm{dt}, \tag{22}$$

subject to the initial conditions

$$\text{u}(x) = h(x) = \text{u}_\text{n},\; n = 0,\; 1,\; 2\ldots,\quad x \in (\alpha,\; 1),\quad 0 \leq \alpha < 1. \tag{23}$$

The unknown function $\text{u}(\text{x})$ is obtained by the zeroth approximation and assumed that $\text{u}_\text{n}$is bounded for all $x \in [0, T]$ and $|x - t| \leq \alpha$ for all $0 \leq x,\; t \in T$. Consider the operator form in (4) and assume that the nonlinear operator $\aleph$ is Lipchitz continuous with

$$|\aleph(u_k) - \aleph(u_n)| \leq L\,|u_k - u_n| \leq \alpha T. \tag{24}$$

By using (5) and applying (15) to obtain:

$$\text{u}_{n+1}(\text{x}) = \text{u}_n(\text{x}) + \mathcal{L}^{-1}\left[\lambda_i(\xi)\left(Lu_n(\xi) + \aleph\tilde{u}_n(\xi) - \text{g}(\xi)\right)\right]. \tag{25}$$

Applying (6), we have

$$\text{u}_{n+1}(\text{x}) = \text{u}_n(\text{x}) - \mathcal{L}^{-1}\left[Lu_n(\xi) + \aleph\tilde{u}_n(\xi) - \text{g}(\xi)\right]. \tag{26}$$

We start with the initial approximation $\text{u}(x) = u_0(x)$ then $u(x) = \lim_{n\to\infty} u_n$.

Theorem 2.1. *Let $\{u_n\}^i$ be approximate solution of the sequence equation (26) obtained by MVIM. Suppose in addition, the nonlinear operator $\aleph u_n(\xi)$ has Lipchitzian derivatives in x with Lipchitzian condition with the constant L, $\|\aleph(u_k) - \aleph(u_n)\| \leq L\,\|u_k - u_n\|$ for all $k,\; n \in \mathbb{N}$. Then for any approximation $u_0(x)$ and a constant $0 \leq \alpha < 1$, the sequence $\{u_n\}^i$ converges to the exact solution say u of equation (10).*

Proof. Let u_k and $u_n$be two solution of the sequence equation (26). In view of equation (24), we have

$$\begin{aligned} |u_k - u_n| &= \left|\int_0^{\text{x}} (x - \text{t})\,\aleph(u_k(x)) - \aleph(u_n(t))\text{dt}\right|, \\ &\leq \int_0^x |x - t|\,|\aleph(u_k(x)) - \aleph(u_n(t))|\,dt. \\ &\leq \alpha T L\,|u_k - u_n|, \\ &= \alpha\,|u_k - u_n|. \end{aligned}$$

It follows that $(1 - \alpha)\,|u_k - u_n| \leq 0$. Since $0 \leq \alpha < 1$ and $|u_k - u_n| \geq 0$, we get $|u_k - u_n| = 0$. Thus, $u_k = u_n$, which shows uniqueness of solution.
To show convergence, we let let $B = (C\,|m|,\; \|.\|)$ denotes the Banach space of all continuous function k with the norm $\|u(x)\| = \max|u(x)|$ for all x in m. Let u_k and u_n be arbitrary partial sums with $k > n$, we need to show that u_n is Cauchy in B

$$\|u_k - u_n\| = \left\|u_{k-1} - \mathcal{L}^{-1}\left[Lu_{k-1} + \aleph\tilde{u}_{k-1} - \text{g}\right] - u_{n-1} - \mathcal{L}^{-1}\left[Lu_{n-1} + \aleph\tilde{u}_{n-1} - \text{g}\right]\right\|$$

$$= \left\| u_{k-1} - u_{n-1} - \mathcal{L}^{-1}\left[Lu_{k-1} - Lu_{n-1}\right] - \mathcal{L}^{-1}[\aleph\tilde{u}_{k-1} - \aleph\tilde{u}_{n-1}]\right\|$$

for all k and by Lipchitzian of L

$$\begin{aligned}\|u_k - u_n\| &\leq \left\| L\mathcal{L}^{-1}[\aleph\tilde{u}_{k-1} - \aleph\tilde{u}_{n-1}]\right\| \\ &\leq \mathcal{L}^{-1}\left\| L[\aleph\tilde{u}_{k-1} - \aleph\tilde{u}_{n-1}]\right\| \\ &\leq L\mathcal{L}^{-1}\left\|[\aleph\tilde{u}_{k-1} - \aleph\tilde{u}_{n-1}]\right\| \\ &\leq \alpha\left\|[\tilde{u}_{k-1} - \tilde{u}_{n-1}]\right\|\end{aligned}$$

Consequently, for $\tilde{u}_1 - \tilde{u}_0 \leq \tilde{u}_2 - \tilde{u}_1 \leq \cdots \leq \tilde{u}_{k-1} - \tilde{u}_{n-1}$.
Let $k = n + 1$ then

$$\|u_k - u_n\| \leq \alpha\|u_k - u_n\| \leq \alpha^2\|u_{n-1} - u_{n-2}\| \leq \alpha^2\|u_{n-2} - u_{n-3}\| \leq \ldots \leq \alpha^n\|[u_1 - u_0]\|$$

By using the triangular inequality,

$$\begin{aligned}\|u_k - u_n\| &\leq \|[\tilde{u}_k - \tilde{u}_{n-1}]\| + \|[\tilde{u}_{n-1} - \tilde{u}_{n-2}]\| + \|[\tilde{u}_{n-2} - \tilde{u}_{n-3}]\| + \cdots + \|[\tilde{u}_{n+1} - \tilde{u}_n]\| \\ &\leq (\alpha^n + \alpha^{n-1} + \alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{k-n-1})\|u_1 - u_0\| \\ &\leq \alpha^n\left(\frac{1-\alpha^{n-k}}{1-\alpha}\right)\|u_0\|\end{aligned}$$

with $0 < \alpha < 1$ *and* $(1 - \alpha^{n-k}) < 1$, then

$$\|u_k - u_n\| \leq \left(\frac{\alpha^n}{1-\alpha}\right) max\,\|u_0\|$$

$u_0(x) < \infty$ as $\lim_{n\to\infty} \mathrm{u}_n$, $\|u_k - u_n\| \to 0$. Hence $\{u_k\}$ is a Cauchy sequence in the Banach space. Therefore, the series converges.
Integro-differential equation (IDE) can be implemented in a reliable and efficient way to handle nonlinear equations; and using equation (17) for approximation purpose, the approximate solution $u(x) = \sum_{i=0}^{\infty} u_i(x)$of $n - th$ order is truncated to the series

$$u(x) = \sum_{i=0}^{n} u_i(x). \tag{27}$$

Using the initial approximations

$$u_0 = \sum_{k=0}^{k-1} u^k(0)\frac{x^k}{k!} \tag{28}$$

and substituting (28) into equation (4), yield

$$\mathrm{u}_{n+1} = \mathrm{u}_0 + \mathcal{L}^{-1}[\lambda_i(\xi)\left(Lu(\xi) + \aleph(\xi) - \mathrm{g}(\xi)\right].\tag{29}$$

Now using equation (5), we have

(30)

$$\begin{aligned}
&\mathrm{u}_{n+1}{=}\mathrm{u}_0 + \lambda_i(\xi)\mathcal{L}^{-1}Lu\,(\xi) + \lambda_i(\xi)\mathcal{L}^{-1}\aleph\tilde{u}\,(\xi) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&\mathrm{u}_1{=}\mathrm{u}_0 + \lambda_i(\xi)\mathcal{L}^{-1}(L\mathrm{u}_0\,(\xi) + \lambda_i(\xi)\mathcal{L}^{-1}\aleph f\,(\xi) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&\mathrm{u}_2{=}\mathrm{u}_1 + \lambda_i(\xi)\mathcal{L}^{-1}(L\mathrm{u}_1\,(\xi) + \lambda_i(\xi)\mathcal{L}^{-1}\aleph\mathrm{u}_0 - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&\mathrm{u}_3{=}\mathrm{u}_2 + \lambda_i(\xi)\mathcal{L}^{-1}(L\mathrm{u}_2\,(\xi) + \lambda_i\,(\xi)\,\mathcal{L}^{-1}\aleph\,(\mathrm{u}_0{+}\mathrm{u}_1) - \aleph\,(\mathrm{u}_0)\,) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&{=}\mathrm{u}_2 + (\aleph\,(\mathrm{u}_0)\,) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)) + \lambda_i\,(\xi)\,\mathcal{L}^{-1}\aleph\,(\mathrm{u}_0{+}\mathrm{u}_1) + \lambda_i(\xi)\mathcal{L}^{-1}L\mathrm{u}_2\,(\xi)\\
&\mathrm{u}_4{=}\mathrm{u}_3 + \lambda_i(\xi)\mathcal{L}^{-1}(L\mathrm{u}_3\,(\xi) + \lambda_i\,(\xi)\,\mathcal{L}^{-1}\aleph\,(\mathrm{u}_0{+}\mathrm{u}_1 + \mathrm{u}_2) - \aleph\,(\mathrm{u}_0 + \mathrm{u}_1)\,) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&{=}\mathrm{u}_3 + \lambda_i(\xi)\mathcal{L}^{-1}(L\mathrm{u}_3\,(\xi) + \lambda_i\,(\xi)\,\mathcal{L}^{-1}\aleph\,(\mathrm{u}_0{+}\mathrm{u}_1 + \mathrm{u}_2) - \aleph\,(\mathrm{u}_0 + \mathrm{u}_1)\,) - \lambda_i(\xi)\mathcal{L}^{-1}\mathrm{g}\,(\xi)\\
&\vdots\\
&u_{n+1} = u_n + \lambda_i\,(\xi)\,L^{-1}\left[Lu_n\,(\xi) + L^{-1}\lambda_i\,(\xi)\,\aleph\textstyle\sum_{n=0}^{n-2}u_n - L^{-1}\lambda_i\,(\xi)\,\aleph\sum_{n=0}^{n-3}u_n - \lambda_i\,(\xi)L^{-1}g\,(\xi)\right]
\end{aligned}$$

To solve these equations, the relations in equation (4) and equation (25) are used. The approximate solution is determined by

$$\begin{aligned}
&\mathrm{u}\,(\mathrm{x}) = \mathrm{u}\,(0) + \mathrm{L}^{-1}\,(\mathrm{f}) + L^{-1}\,(\mathrm{f})\,u_1 + \cdots + L^{-1}(\mathrm{f})u_n\\
&= \textstyle\sum_{k=0}^{k-1}u^k\,(0)\,\frac{x^k}{k!} + \lambda_i\,(\xi)\,L^{-1}u_0\,(\xi) + \quad L^{-1}\,(f)\\
&+L^{-1}\lambda_i(\xi)\textstyle\sum_{n=0}^{\infty}u_0L^{-1}L^{-1}\,(f) - L^{-1}\lambda_i\,(\xi)\sum_{n=0}^{n-2}\aleph\,(u_n) - \lambda_i\,(\xi)\sum_{n=0}^{n-3}\aleph\,(u_n) - \lambda_i\,(\xi)L^{-1}g\,(\xi)\,.
\end{aligned}$$

Using equation (11) and considering the left hand side, we have

$$u_k = \sum_{n=0}^{n-1} u_n. \tag{31}$$

$\int_{n\to\infty} lim u_k \to u$, it implies that u_k is an appropriate approximation of u, the term becomes zero where $\frac{1}{\mathrm{k!}}$ is the coefficient of the calculation derived from the integration. k is the number of terms and n is the operator derivative and so it has a rapid convergence. □

3. Numerical Results and Discussions

In this section, we demonstrate the usefulness of our modified technique with numerical examples. The error estimates are tabulated. Graphical solutions of this method are compared with its exact solution.

Example 3.1. Consider the system of nonlinear third-order Volterra Integro-differential equation (SNVIDEs)

$$u'''\,(x) = -2\mathrm{x} - 2\mathrm{x}^3 - \frac{2}{5}\mathrm{x}^5 + \int_0^x \left(\mathrm{u}^2\,(\mathrm{s}) + \mathrm{v}^2\,(\mathrm{s})\right)\mathrm{ds},\ u\,(0) = 1,\ \ \mathrm{u}'\,(0) = 1, \mathrm{u}''\,(0) = 2$$

$$v'''\,(x) = -\frac{2}{3}\mathrm{x}^3 - \frac{1}{5}\mathrm{x}^5 + \int_0^x \left((x - s)\,\mathrm{u}^2\,(\mathrm{s}) - \mathrm{v}^2\,(\mathrm{s})\right)\mathrm{ds},\ v\,(0) = 1,\ \mathrm{v}'\,(0) = -1, \mathrm{v}''\,(0) = 2$$

Using the (MVIM) method and forcing the correction functional to be stationary with respect to u_n. Suppose $\tilde{u}_n$ is a restriction variation, $\delta\tilde{u}_n = 0$, gives

(32)
$$\left.\begin{array}{r}u_{n+1}(x) = u_n(x) + \int_a^x \lambda(\tau)\left[Lu_n(\tau) + 2\tau + 2\tau^3 + \frac{2}{5}\tau^5 - \int_a^x \mathcal{L}^{-1}\sum_{k=0}^{i}(\widetilde{u}_k^2(r) + \widetilde{v}_k^2(r))dr\right]d\tau \\ v_{n+1}(x) = v_n(x) + \int_a^x \lambda(\tau)\left[Lu_n(\tau) + \frac{2}{3}\tau^3 + \frac{1}{5}\tau^5 - \int_a^x \mathcal{L}^{-1}\sum_{k=0}^{i}(\tau - r) * (\widetilde{u}_k^2(r) - \widetilde{v}_k^2(r)))dr\right]d\tau.\end{array}\right\}$$

Lagrange's multiplier $\lambda(\tau)$ can be identified optimally through integration by parts in combination to a restricted variation,

$$\delta u_{n+1} = \delta u_n + \delta\lambda(\tau)\, u''_n(\tau) - \lambda'(\tau)\,\delta u'_n + \delta\lambda''(\tau)\, u_n - \int_a^x \lambda'''\delta(u_n)\, d\tau,$$

$$\delta u_{n+1} = \delta u_n(\tau)\left(1 + \lambda''\Big|_{\tau=x}\right) + \delta\lambda(\tau)\, u''_n(\tau) - \lambda'(\tau)\,\delta u'_n - \int_a^x \lambda'''\delta(u_n)\, d\tau.$$

In order to obtain optimal $\lambda(\tau)$, we calculate variation with respect to u_n. Thus, we get the following stationary conditions: In view of the formulas in Equation (7), with $n = 3$ we get

$$\lambda(\tau) = -\frac{1}{2!}(\tau - x)^2.$$

Therefore,

(33)
$$u_{n+1}(x) = u_n(x) - \int_a^x \frac{1}{2!}(\tau - x)^2 [Lu_n(\tau) + 2\tau + 2\tau^3 + \frac{2}{5}\tau^5 - \int_a^x \mathcal{L}^{-1}\sum_{k=0}^{i}(\widetilde{u}_k^2(r) + \widetilde{v}_k^2(r))dr]d\tau,$$

and

(34)
$$v_{n+1}(x) = v_n(x) - \int_a^x \frac{1}{2!}(\tau - x)^2 [Lu_n(\tau) + \frac{2}{3}\tau^3 + \frac{1}{5}\tau^5 - \int_a^x \mathcal{L}^{-1}\sum_{k=0}^{i}(\tau - r) * (\widetilde{u}_k^2(r) - \widetilde{v}_k^2(r)))dr]d\tau.$$

Applying the initial condition, we have the zeroth approximation,

(35)
$$u_0 = 1 + x + x^2$$

(36)
$$v_0 = 1 - x + x^2$$

Additionally, other approximations are obtained as:

$$u_1 = 1 + x + x^2 - \frac{1}{12}x^4 - \frac{1}{60}x^6 + \frac{1}{72}x^7 - \frac{1}{840}x^8 + \frac{1}{360}x^9 + \frac{1}{5040}x^{11}$$

$$v_1 = 1 - x + x^2 - \frac{1}{180}x^6 - \frac{1}{144}x^7 + \frac{1}{280}x^8 - \frac{1}{1080}x^9 + \frac{1}{1680}x^{10} - \frac{1}{20160}x^{11} + \frac{1}{90720}x^{12}$$

$$u_2 = 1 + x + x^2 - \frac{1}{12}x^4 - \frac{1}{60}x^6 + \frac{1}{72}x^7 - \frac{1}{840}x^8 + \frac{1}{360}x^9 + \frac{11}{60480}x^{11} - \frac{1}{108864}x^{12} - \frac{19}{2721600}x^{13} - \frac{1}{5702400}x^{14} + \frac{11}{7983360}x^{15} + \frac{61}{778377600}x^{16} + \frac{61}{544864320}x^{17} + \frac{1}{396264960}x^{18} + \frac{67}{7925299200}x^{19} - \frac{13}{7772889600}x^{20} + \frac{191}{319798886400}x^{21} - \frac{77}{217006387200}x^{22} + \frac{1859}{13291641216000}x^{23} - \frac{1}{31274449920}x^{24} + \frac{2881}{160542176256000}x^{25} - \frac{17}{10411099776000}x^{26} + \frac{31}{29151079372800}x^{27} - \frac{109}{2498663946240000}x^{28} + \frac{67}{2624859095040000}x^{29} - \frac{1}{2311017246720000}x^{30} + \frac{1}{24265681090560000}x^{31}$$

$$v_2 = 1 - x + x^2 - \frac{1}{180}x^6 - \frac{1}{1680}x^8 + \frac{1}{1080}x^9 + \frac{1}{10080}x^{11} - \frac{1}{544320}x^{12} - \frac{1}{1088640}x^{13} - \frac{17}{29937600}x^{14} - \frac{1}{205286400}x^{15} - \frac{1}{84913920}x^{16} + \frac{37}{990662400}x^{17} - \frac{1}{5108103300}x^{18} + \frac{41}{19020718080}x^{19} - \frac{71}{673650432000}x^{20} - \frac{11}{139912012800}x^{21} + \frac{733}{30380894208000}x^{22} - \frac{13}{1215235768320}x^{23} + \frac{227}{55824893107200}x^{24} + \frac{1}{17200947456000}x^{25} + \frac{1151}{3692470053888000}x^{26} + \frac{13}{349812952473600}x^{27} + \frac{1}{66252453120000}x^{28} + \frac{109}{6495262602240000}x^{29} + \frac{29}{70871195566080000}x^{30} + \frac{1}{64708482908160000}x^{31} - \frac{1}{703704751626240000}x^{32}$$

$$u_3 = 1 + x + x^2 - \frac{1}{12}x^4 - \frac{1}{60}x^6 + \frac{1}{72}x^7 - \frac{1}{840}x^8 + \frac{1}{360}x^9 + \frac{11}{60480}x^{11} - \frac{1}{108864}x^{12} - \frac{19}{2721600}x^{13} + \frac{1}{8553600}x^{14} - \frac{67}{359251200}x^{15} + \frac{257}{778377600}x^{16} + \frac{1}{49533120}x^{17} + \frac{19}{660441600}x^{18} + \frac{59}{23775897600}x^{19} - \frac{1073}{932746752000}x^{20} + \frac{1741}{3769058304000}x^{21} - \frac{473}{1458282921984}x^{22} + \frac{5189507}{45616912653312000}x^{23} - \frac{849841}{56350303865856000}x^{24} + \frac{19567}{1444879586304000}x^{25} - \frac{668177}{333488348024832000}x^{26} + \frac{6733}{9968003369533440}x^{27}$$

$$- \frac{1574437}{6338349968670720000}x^{28} + \frac{4720411}{81836741300041728000000}x^{29} - \frac{46229}{1317821921095680000}x^{30}$$
$$+ \frac{189308519}{706108752651881779200000}x^{31} - \frac{70525969}{4368459483072975273984000}x^{32}$$
$$+ \frac{365625149}{1293886093458255897600000}x^{33} - \frac{104185531511}{1197966004396550527057920000}x^{34}$$
$$+ \frac{10855335637}{421852774657448765030400000}x^{35} - \frac{209713385701}{42843513260684516386406400000}x^{36}$$
$$+ \frac{1394809936517}{847522589593177342334730240000}x^{37} - \frac{1655797471967}{26314209434949457806360576000000}x^{38}$$
$$+ \frac{108487796799799}{600950757970658242652759654400000}x^{39} - \frac{2198158324813543}{525558753788884754029049806848000000}x^{40}$$
$$+ \frac{43850476512711283}{2349556781644425959188693254144000000}x^{41} - \frac{2361523494691}{1367856612690440549460502118400000}x^{42}$$
$$+ \frac{53401549744787}{50154742465316153480218411008000000}x^{43} - \frac{4542045742477}{60629320130062770300411445248000000}x^{44}$$
$$+ \frac{482764505047439}{15546634088087674573873924276224000000}x^{45} - \frac{2946424644353}{487635467086569633517398287974400000}x^{46}$$
$$+ \frac{24697127994830531}{13222262805052335612824254578425856000000}x^{47}$$
$$- \frac{1341768710991271}{2404394737244327157188419796336640000000}x^{48}$$
$$+ \frac{11141855689973}{355032421663051135226903474995200000000}x^{49}$$
$$- \frac{9966409031913679}{168383517819740180431738521916538880000000}x^{50}$$
$$+ \frac{8870613685101523}{265914750219226051175316938507157504000000}x^{51}$$
$$- \frac{1454164227739891697}{197618978537921493698456371467235885056000000}x^{52}$$
$$+ \frac{21909853367832697}{7811026819680691450531872390009323520000000}x^{53}$$
$$- \frac{135720918754072661}{174813273105353772755919431281724620800000000}x^{54}$$
$$+ \frac{1017451411190197}{4366493152335088197405760937263104000000000}x^{55}$$
$$- \frac{1738206671876629}{29210798299826294015504016916611072000000000}x^{56}$$
$$+ \frac{252745753279066139}{145750199196813276619758842807122604851200000000}x^{57}$$

$$
\begin{aligned}
&- \frac{37819999725620527}{1178861905268342678542167110939962245120000000000}x^{58} \\
&+ \frac{485758689463107097}{500562901313942429657904804029891661004800000000000}x^{59} \\
&- \frac{910653191998717}{76283170737248120630014520207748366336000000000}x^{60} \\
&+ \frac{200499456630029}{54622517318029518475812866321597595648000000000}x^{61} \\
&- \frac{3217498685329}{10986574506012755420703269703321334579200000000}x^{62} \\
&+ \frac{24011417531753}{282511915868899425103798363799691460608000000000}x^{63} \\
&- \frac{4577416757}{1077851384933099723634593626155122688000000000}x^{64} \\
&+ \frac{1857378504857}{1883001059865391556599863465909619261440000000000}x^{65} \\
&+ \frac{176290823}{6313382942892691766132155151139471360000000000}x^{66} \\
&- \frac{1743477607}{64649041335221163685193268747668186726400000000000}x^{67} \\
&+ \frac{3733}{13098299417084305931908392268333056000000000000}x^{68} \\
&+ \frac{61}{8177662109523562020791021727645696000000000000}x^{69} \\
&- \frac{1}{2512956813130421714262228367048704000000000000}x^{70} \\
&+ \frac{1}{58072861353435839303028683669766144000000000000}x^{71}
\end{aligned}
$$

$$
\begin{aligned}
v_3 = {}& 1 - x + x^2 - \frac{1}{180}x^6 - \frac{1}{1680}x^8 + \frac{1}{1080}x^9 + \frac{1}{10080}x^{11} - \frac{1}{544320}x^{12} - \frac{1}{1088640}x^{13} \\
& - \frac{17}{29937600}x^{14} - \frac{1}{34214400}x^{15} + \frac{19}{1556755200}x^{16} + \frac{211}{10897286400}x^{17} + \frac{17}{4086482400}x^{18} \\
& + \frac{47}{95103590400}x^{19} + \frac{193}{866121984000}x^{20} - \frac{103}{799497216000}x^{21} + \frac{3917}{167094918144000}x^{22} \\
& - \frac{114269}{8020556070912000}x^{23} + \frac{5293493}{957955165719552000}x^{24} - \frac{573127}{1239706685048832000}x^{25} \\
& + \frac{2493817}{475220895935385600 0}x^{26} - \frac{1612661}{240111610577879040000}x^{27} + \frac{88833401}{510237172477992960000 0}x^{28}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{527861}{505377770835345408000}x^{29}-\frac{126533999}{441918403020225331200000}x^{30}-\frac{36197443}{537987621068100403200000}x^{31}\\
&+\frac{3748673}{341285897115076193280000}x^{32}+\frac{1542830881}{393161353476567774658560000}x^{33}\\
&+\frac{450445361}{580910239200913509888000000}x^{34}-\frac{67031469181}{383349121406896168658534400000}x^{35}\\
&+\frac{1054975833689}{165661584607980130027438080000000}x^{36}-\frac{203406376331}{145667945086327355713781760000000}x^{37}\\
&+\frac{15916781603}{350479266373118449837670400000}x^{38}-\frac{78574925549}{451100733170562133823324160000000}x^{39}\\
&+\frac{4239542936071}{821185552795132428170390323200000}x^{40}-\frac{766033493301221}{665707754799254021770129755340800000}x^{41}\\
&+\frac{44275313739178013}{916327144841326124083590369116160000000}x^{42}-\frac{609433978801}{126263687325271435334815580160000000}x^{43}\\
&+\frac{10370474269511}{411268888215592458537790970265600000}x^{44}-\frac{105829324787}{519679886829109459717812387840000000}x^{45}
\end{aligned}
$$

Similarly, the remaining approximations can computed. The graphical comparison between the exact solutions solutions $u\,(x)=1+x+x^2,\ v\,(x)=1-x+x^2$, and the MVIM solution of the system in Example 1 is shown in Figure 1.

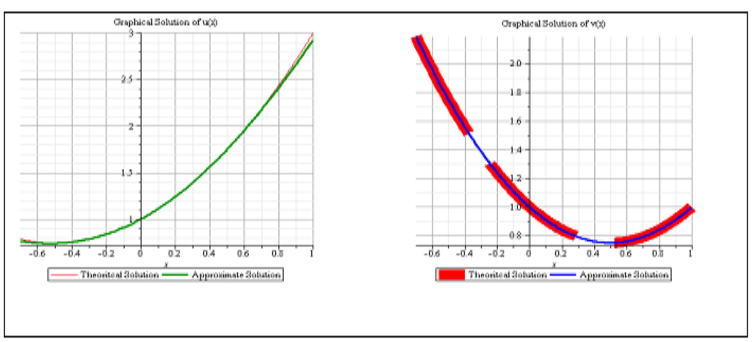


Figure 1: Approximate of u (x) and $v\,(x)$at some selected values by using 3-iterate of MVIM

Example 3.2. Consider the first-order nonlinear system of two Volterra integro-differential equations that occur in physical phenomena such as the glass formation and nano-hydrodynamics.

(37)
$$\left.\begin{array}{l} u^{'}(x) = e^{x} - \frac{1}{4}e^{2x} - \frac{1}{6}x^4 + x + \frac{3}{2} + \int_0^x \left((x-s)\,u^2(s) + (x-s)\,v^2(s)\right) ds, \;\; u(0) = 1 \\ v^{'}(x) = 7e^{x} - 4xe^{x} - 4x - 7 + \int_0^x \left((x-s)\,v^2(s) - (x-s)\,u^2(s)\right) ds, \;\; v(0) = -1 \end{array}\right\}$$

The functional correction for equation (37) is given as

$$\left.\begin{array}{l} u_{n+1}(x) = u_n(x) + \int_a^x \lambda(\tau)[Lu_n(\tau) - e^{\tau} + \frac{1}{4}e^{2\tau} + \frac{1}{6}\tau^4 - \tau + \frac{3}{2}]d\tau \\ + \int_a^x \lambda(\tau)[-\int_a^x \mathcal{L}^{-1} \sum_{k=0}^{i} ((\tau - r)\breve{u}_k^2(r) + (\tau - r)\breve{v}_k^2(r))dr]d\tau \\ v_{n+1}(x) = v_n(x) + \int_a^x \lambda(\tau)[Lu_n(\tau) - 7e^{\tau} + 4\tau e^{\tau} + 4\tau + 7]d\tau \\ + \int_a^x \lambda(\tau)[-\int_a^x \mathcal{L}^{-1} \sum_{k=0}^{i} ((\tau - r)\breve{u}_k^2(r) - (\tau - r)\breve{v}_k^2(r)\,dr]d\tau \end{array}\right\} \quad (38)$$

In order to obtain optimal $\lambda(\tau)$, calculate variation with respect to u_n we get the following stationary conditions:
$\delta u_n : 1 + \lambda|_{\tau=x} = 0$ and $\delta u_n : \lambda^{'}\Big|_{\tau=x} 0.$
It follows that the Lagrange multiplier, $\lambda = -1$. Hence,

$$\left.\begin{array}{l} u_{n+1}(x) = u_n(x) - \int_a^x [Lu_n(\tau) - e^{\tau} + \frac{1}{4}e^{2\tau} + \frac{1}{6}\tau^4 - \tau + \frac{3}{2}]d\tau \\ - \int_a^x [-\int_a^x \mathcal{L}^{-1} \sum_{k=0}^{i} ((\tau - r)\breve{u}_k^2(r) + (\tau - r)\breve{v}_k^2(r)\,dr]d\tau \\ v_{n+1}(x) = v_n(x) - \int_a^x [Lu_n(\tau) - 7e^{\tau} + 4\tau e^{\tau} + 4\tau + 7]d\tau \\ - \int_a^x [-\int_a^x \mathcal{L}^{-1} \sum_{k=0}^{i} ((\tau - r)\breve{u}_k^2(r) - (\tau - r)\breve{v}_k^2(r)\,dr]d\tau \end{array}\right\} \quad (39)$$

Using the initial conditions to obtain the zeroth approximation, we have
$u_0 = 1;\; v_0 = -1$

$$u_1 = 1 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4 - \frac{11}{120}x^5 - \frac{1}{48}x^6 - \frac{31}{5040}x^7 - \frac{1}{640}x^8$$

$$v_1 = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 - \frac{3}{40}x^5 - \frac{13}{720}x^6 - \frac{17}{5040}x^7 - \frac{1}{1920}x^8 - \frac{1}{10080}x^9$$

$$\begin{aligned} u_2 = {} & 1 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4 + \frac{3}{40}x^5 + \frac{7}{48}x^6 + \frac{53}{5040}x^7 + \frac{1217}{40320}x^8 + \frac{11}{2520}x^9 + \frac{1}{540}x^{10} \\ & + \frac{7}{25920}x^{11} + \frac{311}{1663200}x^{12} + \frac{323}{39916800}x^{13} + \frac{5167}{518918400}x^{14} + \frac{31}{10319400}x^{15} + \frac{1033}{141523200}x^{16} \\ & + \frac{569}{2438553600}x^{17} + \frac{7519}{207277056000}x^{18} + \frac{191}{35533209600}x^{19} + \frac{2749}{4725916876800}x^{20} \\ & + \frac{1}{66189312000}x^{21} + \frac{1}{810819072000}x^{22} \end{aligned}$$

$$
\begin{aligned}
v_2 = & -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5 - \frac{13}{720}x^6 + \frac{13}{1680}x^7 - \frac{1301}{40320}x^8 - \frac{1}{630}x^9 - \frac{23}{9072}x^{10} \\
& - \frac{223}{226800}x^{11} - \frac{677}{4989600}x^{12} - \frac{2963}{39916800}x^{13} - \frac{1}{102960}x^{14} - \frac{101}{73382400}x^{15} - \frac{311}{4953312000}x^{16} \\
& + \frac{547}{6096384000}x^{17} + \frac{1}{70502400}x^{18} + \frac{11}{4441651200}x^{19} + \frac{61}{236295843840}x^{20} \\
& - \frac{1}{66189312000}x^{21} - \frac{1}{810819072000}x^{22}
\end{aligned}
$$

$$
\begin{aligned}
u_3 = & 1 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4 + \frac{3}{40}x^5 + \frac{19}{240}x^6 + \frac{53}{5040}x^7 + \frac{13}{8064}x^8 + \frac{1}{420}x^9 + \frac{11}{5040}x^{10} \\
& + \frac{103}{129600}x^{11} + \frac{73}{199584}x^{12} + \frac{371}{5702400}x^{13} + \frac{1943}{20217600}x^{14} + \frac{211}{12700800}x^{15} + \frac{148483}{7264857600}x^{16} \\
& + \frac{150251}{80472268800}x^{17} + \frac{3487943}{1097800704000}x^{18} + \frac{1545727}{3952082534400}x^{19} + \frac{1284432101}{2764661372928000}x^{20} \\
& + \frac{1372813243}{17888985354240000}x^{21} + \frac{15290311129}{354798209525760000}x^{22} + \frac{166661051}{13693965981696000}x^{23} \\
& + \frac{52696601861}{14688645874366464000}x^{24} + \frac{628168703867}{527592178344591360000}x^{25} + \frac{2272413640577}{7344322937183232000000}x^{26} \\
& + \frac{102369966371}{127727355429273600000}x^{27} + \frac{9046204664347}{4109627660936878080000000}x^{28} + \frac{31182425600119}{736445276839888551936000}x^{29} \\
& + \frac{92473481183}{8960942529380462100480 0}x^{30} + \frac{1738100618417}{94416061133319045120000000}x^{31} \\
& + \frac{1812932271119}{60628599256324158259200000}x^{32} + \frac{24497388449}{41467266519001989912000000}x^{33} \\
& + \frac{559189489081}{70245549483189369569280000 0}x^{34} + \frac{77628925111}{38521752942394170408960000 0}x^{35} \\
& + \frac{5180159}{9903782636362137600000 0}x^{36} + \frac{9097152467}{70713008023625662464000000 0}x^{37} \\
& + \frac{113681454049}{35569105212406738452480000000}x^{38} + \frac{9002560759}{148954701828291286630400000}x^{39} \\
& + \frac{53187082931}{5334095456674853384355840000 00}x^{40} + \frac{591921709}{43172329176729554780160000000}x^{41} \\
& + \frac{242943768401}{1710442237424996824016486400 00000}x^{42} + \frac{879805609}{708467198933430637166592000000 0}x^{43} \\
& + \frac{2138261633}{2894088507643064152825528320 00000}x^{44} + \frac{2059}{96665144144819310821376000 00}x^{45} \\
& + \frac{48073}{326244861488765174022144000000 0}x^{46} + \frac{1}{12220102622372482252800000 00}x^{47}
\end{aligned}
$$

$$\begin{aligned}
v_3 = & -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 + \frac{11}{120}x^5 + \frac{7}{144}x^6 + \frac{13}{1680}x^7 - \frac{149}{40329}x^8 + \frac{1}{420}x^9 + \frac{5}{4536}x^{10} \\
& + \frac{17}{18900}x^{11} + \frac{863}{4989600}x^{12} + \frac{4819}{39916800}x^{13} + \frac{1153}{31135104}x^{14} + \frac{2687}{283046400}x^{15} + \frac{374167}{54486432000}x^{16} \\
& + \frac{400781}{100590336000}x^{17} + \frac{3119911}{1587890304000}x^{18} + \frac{14519639}{26676557107200}x^{19} + \frac{51010147}{447224633856000}x^{20} \\
& + \frac{801660623}{17888985354240000}x^{21} + \frac{49811941}{7884404656128000}x^{22} - \frac{13498967}{4279364369280000}x^{23} \\
& + \frac{55485784619}{161575104618031104000}x^{24} - \frac{1715452427363}{369314524841213952000}x^{25} - \frac{2253838787}{417291075976320000000}x^{26} \\
& - \frac{16904224033}{976738600341504000000}x^{27} - \frac{1410708625943}{41096276609368780800000}x^{28} \\
& - \frac{10500819019}{1292009257613839564800000}x^{29} + \frac{422373682403}{352037027939946725376000}x^{30} \\
& + \frac{21554572493197}{27380657728662523084800000}x^{31} + \frac{105383170035127}{394085895166107028684800000}x^{32} \\
& + \frac{92333691509}{1179225391634119065600000}x^{33} + \frac{6162490913147}{304397381093820601466880000}x^{34} \\
& + \frac{23648390053}{5058412002536608235520000}x^{35} + \frac{10864406637803}{1095462349299334221004800000}x^{36} \\
& + \frac{174147827999}{919269104307133612032000000}x^{37} + \frac{1064122282753}{3201219469116606460723200000}x^{38} \\
& + \frac{18258933467}{347560970932660130021376000}x^{39} + \frac{13327353097}{1778031818891617794785280000}x^{40} \\
& + \frac{63404461181}{660536636403962188136448000000}x^{41} + \frac{3239372873}{300077585513157337546752000000}x^{42} \\
& + \frac{83385137}{787185776592700707962880000000}x^{43} + \frac{275086457}{321565389738118239202836480000000}x^{44} \\
& + \frac{5443}{96665144144819310821376000000}x^{45} + \frac{1}{41099125911912972288000000}x^{46}
\end{aligned}$$

with the exact solutions $u(x) = x + e^x$, $v(x) = x - e^x$

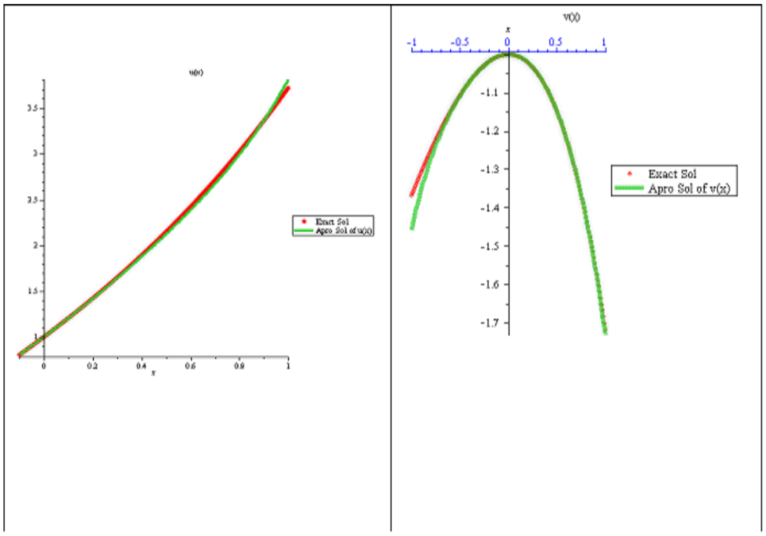


Figure 2: Graphical comparisons of systems VIDE Example (2)

Table 1: Results and Comparison of Absolute Error Table for Example (2)

	Exact Solution		MVIM		Errors	
x_i	$u(x)$	$v(x)$	$u(x)$ Approxi Sol. $n = 3$	$v(x)$ Approxi Sol. $n = 3$	E_u	E_v
0.00000	1.00000000	-1.0000000	1.00000000	-1.00000000	0.00E+00	0.00E+00
0.10000	1.20517092	-1.0051709	1.20514286	-1.00533333	2.81E-05	1.62E-04
0.20000	1.42140276	-1.0214028	1.42114286	-1.02266667	2.60E-04	1.26E-03
0.30000	1.64985881	-1.0498588	1.64885714	-1.05400000	1.00E-03	4.14E-03
0.40000	1.89182470	-1.0918247	1.88914286	-1.10133333	2.68E-03	9.51E-03
0.50000	2.14872127	-1.1487213	2.14285714	-1.16666667	5.86E-03	1.79E-02
0.60000	2.42211880	-1.2221188	2.41085714	-1.25200000	1.13E-02	2.99E-02
0.70000	2.71375271	-1.3137527	2.69400000	-1.35933333	1.98E-02	4.56E-02
0.80000	3.02554093	-1.4255409	2.99314286	-1.49066667	3.24E-02	6.51E-02
0.90000	3.35960311	-1.5596031	3.30914286	-1.64800000	5.05E-02	8.84E-02
1.00000	3.71828183	-1.7182818	3.64285714	-1.83333333	7.54E-02	1.15E-01

Example 3.3. Consider a system of three nonlinear second-order Volterra Integro-Differential equations

(40)
$$\left.\begin{array}{r} u''(x) = x + 2x^3 + 2\mathrm{v}''^2(x) - \int_0^x \left(\mathrm{v}'^2(s) + \mathrm{u}(\mathrm{t})\mathrm{w}''(s)\right)\mathrm{ds}, \quad u(0) = 1,\ u'(0) = 0, \\ v''(x) = 3x^2 - x\mathrm{u}(x) + \int_0^x \left(sx\mathrm{v}'^{(\mathrm{s})}\mathrm{u}''(\mathrm{s}) - \mathrm{w}''(s)\right)\mathrm{ds}, \quad v(0) = 0,\ v'(0) = 1, \\ w''(x) = 2 - \frac{4}{3}x^3 + \mathrm{u}''^2(x) - 2\mathrm{u}^2(x) + \int_0^x \left(x^2\mathrm{v}(\mathrm{s}) + \mathrm{u}'^2(s) - s^3\mathrm{w}''(s)\right)\mathrm{ds}, w(0) = 0,\ w'(0) = 0, \end{array}\right\}$$

whose exact solutions is give as $u(x) = x^2$, $v(x) = x$ *and* $w(x) = 3x^2$
The functional correction for equation (27) is as follows:

(41)
$$\left.\begin{array}{c} u_{n+1}(x) = u_n(x) + \int_a^x \lambda(\tau)[Lu_n(\tau) - \tau - 2\tau^3 - 2v'^2{}_n(\tau]d\tau \\ + \int_a^x \lambda(\tau)[+\int_a^\xi L^{-1}\sum_{k=0}^i (\breve{v}_k''^2(r) + \breve{u}_k(r)\breve{w}_k''(r))dr]d\tau \\ v_{n+1}(x) = v_n(x) + \int_a^x \lambda(\tau)[Lv_n(\tau) - 3\tau^2 - \tau u_n(\tau)]d\tau \\ + \int_a^x \lambda(\tau)[-\int_a^\tau L^{-1}\sum_{k=0}^i (\tau r\breve{v}_k'(r)\breve{u}_k''(r) - \breve{w}_k'(r))dr]d\tau \\ w_{n+1}(x) = w_n(x) + \int_a^x \lambda(\tau)[Lv_n(\tau) - 2 + \frac{4}{3}\tau^3 - u_n''^2(\tau) + 2u_n^2(\tau)]d\tau \\ + \int_a^x \lambda(\tau)[-\int_a^\tau L^{-1}\sum_{k=0}^i (\tau^2\breve{v}_k(r) + \breve{u}_k'^2(r) - r^3\breve{w}_k''(r))dr]d\tau \end{array}\right\}$$

The stationary conditions:
$\delta u_n : \lambda''\big|_{\tau=x} = 0$ $\delta u_n : 1 - \lambda'\big|_{\tau=x} = 0$ and $\delta u_n : \lambda|_{\tau=x} = 0.$
The Lagrange multiplier $\lambda = \tau - x$, so

(42)
$$\left.\begin{array}{c} u_{n+1}(x) = u_n(x) + \int_a^x (\tau - x)[Lu_n(\tau) - \tau - 2\tau^3 - 2v'^2{}_n(\tau)]d\tau \\ + \int_a^x (\tau - x)[+\int_a^\tau L^{-1}\sum_{k=0}^i (\breve{v}_k''^2(r) + \breve{u}_k(r)\breve{w}_k''(r))dr]d\tau, \\ v_{n+1}(x) = v_n(x) + \int_a^x (\tau - x)[Lv_n(\tau) - 3\tau^2 - \tau u_n(\tau)]d\tau \\ + \int_a^x (\tau - x)[-\int_a^\tau L^{-1}\sum_{k=0}^i (\tau r\breve{v}_k'(r)\breve{u}_k''(r) - \breve{w}_k'(r))dr]d\tau, \\ w_{n+1}(x) = w_n(x) + \int_a^x (\tau - x)[Lv_n(\tau) - 2 + \frac{4}{3}\tau^3 - u_n''^2(\tau) + 2u_n^2(\tau)]d\tau \\ + \int_a^x (\tau - x)[-\int_a^\tau L^{-1}\sum_{k=0}^i (\tau^2\breve{v}_k(r) + \breve{u}_k'^2(r) + r^3\breve{w}_k''(r))dr]d\tau. \end{array}\right\}$$

Taking zero*th* approximations as $\mathrm{u}_0 = 1$, $\mathrm{v}_0 = x$, and $\mathrm{w}_0 = 0$. We then have,

$$u_1 = x^2 + \frac{1}{10}x^5,\ v_1 = x + \frac{1}{4}x^4,\ w_1 = x^2 - \frac{1}{15}x^5 + \frac{1}{60}x^6$$

$$u_2 = x^2 + \frac{4}{15}x^5 - \frac{1}{60}x^6 + \frac{197}{5040}x^8 - \frac{1}{336}x^9 + \frac{1}{7425}x^{11} + \frac{1}{26400}x^{12}$$

$$v_2 = x + \frac{1}{6}x^4 + \frac{1}{360}x^7 + \frac{41}{3360}x^8 + \frac{1}{440}x^{11} + \frac{1}{4752}x^{12}$$

$$w_2 = 3x^2 + \frac{2}{5}x^5 - \frac{1}{30}x^6 + \frac{13}{168}x^8 - \frac{227}{30240}x^9 + \frac{1}{1440}x^{10} + \frac{1}{3960}x^{11} - \frac{1}{6600}x^{12}$$

The remaining primary approximations can be computed in a similar manner. In order to demonstrate the efficiency and high precision of MVIM, the results are shown graphically in Figure 3 on the interval [-1,1].

Table 2: Absolute Error comparison Table (3-iterates of Example (3))

x_i	Exact Solution			E_u n=3	E_v n=3	E_w n=3
	$u(x)$	$v(x)$	$w(x)$			
-1.0	1.00000	-1.00000	3.00000	1.54E-01	2.33E-02	3.48E-01
-0.8	0.64000	-0.80000	1.92000	4.88E-02	4.80E-03	1.26E-01
-0.6	0.36000	-0.60000	1.08000	1.11E-02	5.78E-04	3.13E-02
-0.4	0.16000	-0.40000	0.48000	1.42E-03	2.84E-05	4.18E-03
-0.2	0.04000	-0.20000	0.12000	4.34E-05	1.72E-07	1.30E-04
0.0	0.00000	0.00000	0.00000	0.00E+00	0.00E+00	0.00E+00
0.2	0.04000	0.20000	0.12000	4.19E-05	7.21E-08	1.26E-04
0.4	0.16000	0.40000	0.48000	1.31E-03	2.80E-06	4.01E-03
0.6	0.36000	0.60000	1.08000	9.70E-03	7.16E-05	3.08E-02
0.8	0.64000	0.80000	1.92000	3.93E-02	1.64E-03	1.34E-01
1.0	1.00000	1.00000	3.00000	1.13E-01	1.53E-02	4.37E-01

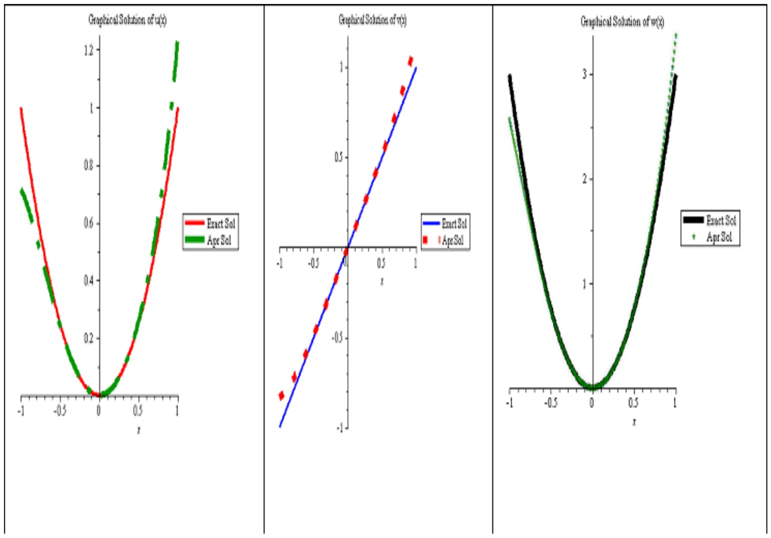
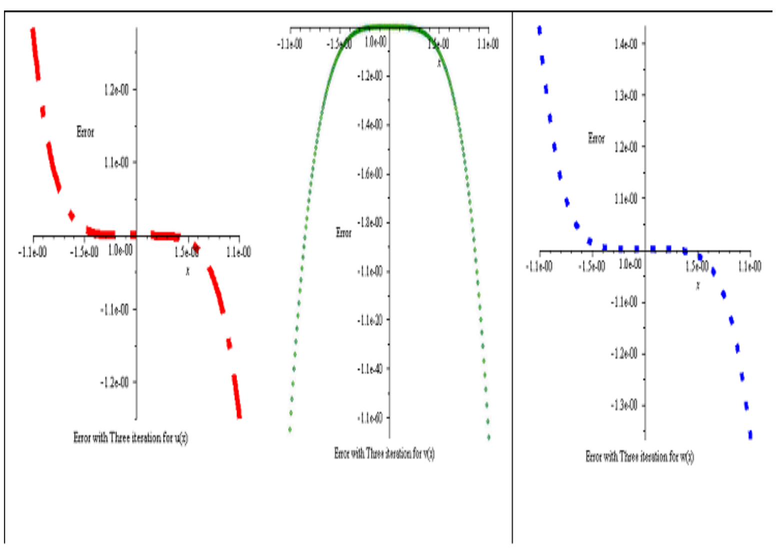


Figure 3: Graphical comparison using 3-iterate of systems of Volterra-Integro-Differential Equations (VIDE) of Example 3



Figures 4: Results showing error using 3-iterate of Example (3)

Conclusion: The modified variational iteration method is used to solve systems of non-linear Volterra Integro-differential equations of the second kind. Both the implementation and the numerical results demonstrate that our method is accurate and efficient for approximating many nonlinear applied problems. This approach reduces the computational complexity involved in other conventional methods since simple manipulations can be carried out in all calculations.

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