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A Three-Step Algorithm for Solving Second Order Ordinary Differential Equations

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Abstract

A linear Multistep Method (LMM) of step length of three with six integrators and uniform order four through interpolation and collocation procedures were developed. The integrators are implemented on a Vanderpol's oscillator Problem, stiff problem and, exponential problems. The results acquired compares satisfactorily with the existing method in the literatures. The properties of the integrators are fully examined and confirmed to be computationally reliable with the numerical experiments tested.

1. Introduction

This article presents numerical solution of second order ordinary differential equation. Among the furthermost significant mathematical tools used in producing models in the physical sciences, Biological sciences and Engineering are differential equations. On the other hand, most of these differential equations do not possess closed form or analytical solution or finite solutions.

In many actual situations of real-life, the differential equation that models the problem is too complicated to solve exactly. Hence there is need to develop an accurate algorithm for obtaining an approximating solution to the unique

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problems. Most recent researchers have developed some different methods to solve problem of the form:

$$y^{''}(x) = f(x, y, y^{'}), y(x_0) = y_0, y^{'}(x_0) = y_1 \tag{1}$$

Amongst such researchers are ([2], [4], [5], [8], [9], [10] & [11]) to mention but a few. This article is therefore motivated by the success story of the block methods. Hence, a direct algorithm for solving second order ODEs in IVPs at step length of three which are of uniform order four.

2. Methodology

The exact solution $y(x)$ to (1) is approximated by:

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j \tag{2}$$

with the second derivative given as:

$$y''(x) = \sum_{j=2}^{c+i-1} j(j-1) a_j x^{j-2} \tag{3}$$

Here, c is the number of collocation points and i is the number of interpolation points. (2) is called interpolation equation while (3) is called collocation equation. Imposing the condition (2) and (3) at a strategic points give the following equations:

$$h^5 a_5 + h^4 a_4 + h^3 a_3 + h^2 a_2 + h a_1 + a_0 = y_{n+1} \tag{4}$$

$$32h^5 a_5 + 16h^4 a_4 + 8h^3 a_3 + 4h^2 a_2 + 2h a_1 + a_0 = y_{n+2} \tag{5}$$

$$2a_2 = f_n \tag{6}$$

$$20h^3 a_5 + 12h^2 a_4 + 6h a_3 + 2a_2 = f_{n+1} \tag{7}$$

$$160h^3 a_5 + 48h^2 a_4 + 12h a_3 + 2a_2 = f_{n+2} \tag{8}$$

$$540h^3 a_5 + 108h^2 a_4 + 18h a_3 + 2a_2 = f_{n+3} \tag{9}$$

Combining equations (4) to (9) and solve simultaneously gives the following values of $a_0, a_1, a_2, a_3, a_4, a_5$

$$a_0 = \frac{1}{12}h^2 f_n + \frac{5}{6}h^2 f_{n+1} + \frac{1}{12}h^2 f_{n+2} + 2y_{n+1} - y_{n+2} \tag{10}$$

(11)

$$a_1 = -\frac{1}{360}\ \frac{127h^2 f_n + 414h^2 f_{n+1} - 9h^2 f_{n+2} + 8h^2 f_{n+3} + 360y_{n+1} - 360y_{n+2}}{h}$$

(12) $$a_2 = \frac{1}{2} f_n$$

(13) $$a_3 = -\frac{1}{36}\ \frac{11f_n - 18f_{n+1} + 9f_{n+2} - 2f_{n+3}}{h}$$

(14) $$a_4 = \frac{1}{24}\ \frac{2f_n - 5f_{n+1} + 4f_{n+2} - f_{n+3}}{h^2}$$

(15) $$a_5 = -\frac{1}{120}\ \frac{f_n - 3f_{n+1} + 3f_{n+2} - f_{n+3}}{h^3}$$

Substituting (10) to (15) into (2) gives a continuous coefficient of the form:

(16) $$y\,(t) = \alpha_1\,(t)\,y_{n+1} + \alpha_2\,(t)\,y_{n+2} + h^2\,(\beta_0(t)+\beta_1(t)+\beta_2(t)+\beta_3(t))$$

where $a_1(t), a_2(t), \cdots$ and $\beta_0(t), \beta_1(t), \beta_2(t)$ & $\beta_3(t)$ are continuous coefficients. See ([4], [7], [10] & [11]). The continuous method (16) is used to generate the required method for solving (1). That is, evaluating (16) at $t = 3$ and $t = 0$ gives:

(17) $$\frac{1}{12}h^2 f_{n+1} + \frac{5}{6}h^2 f_{n+2} + \frac{1}{12}h^2 f_{n+3} - y_{n+1} + 2y_{n+2} = y_{n+3}$$

(18) $$\frac{1}{12}h^2 f_n + \frac{5}{6}h^2 f_{n+1} + \frac{1}{12}h^2 f_{n+2} + 2y_{n+1} - y_{n+2} = y_n$$

Also evaluate the first derivative of (16) at $t = 0, 1, 2,$ and 3 gives

(19)

$$-\frac{1}{360}\ \frac{127h^2 f_n + 414h^2 f_{n+1} - 9h^2 f_{n+2} + 8h^2 f_{n+3} + 360y_{n+1} - 360y_{n+2}}{h} = y'_n$$

(20)

$$\frac{1}{360}\ \frac{8h^2 f_n - 129h^2 f_{n+1} - 66h^2 f_{n+2} + 7h^2 f_{n+3} - 360y_{n+1} + 360y_{n+2}}{h} = y'_{n+1}$$

(21)

$$-\frac{1}{360}\ \frac{7h^2 f_n - 66h^2 f_{n+1} - 129h^2 f_{n+2} + 8h^2 f_{n+3} + 360y_{n+1} - 360y_{n+2}}{h} = y'_{n+2}$$

(22)

$$\frac{1}{360}\ \frac{8h^2 f_n - 9h^2 f_{n+1} + 414h^2 f_{n+2} + 127h^2 f_{n+3} - 360y_{n+1} + 360y_{n+2}}{h} = y'_{n+3}$$

Combining (16) to (21) together and solve simultaneously gives

(23)

$$y_{n+1} = \frac{97}{360}h^2 f_n + \frac{19}{60}h^2 f_{n+1} - \frac{13}{120}h^2 f_{n+2} + \frac{1}{45}h^2 f_{n+3} + y'_n h + y_n$$

$$y_{n+2} = \frac{28}{45}h^2 f_n + \frac{22}{15}h^2 f_{n+1} - \frac{2}{15}h^2 f_{n+2} + y_n + \frac{2}{45}h^2 f_{n+3} + 2y'_n h \tag{24}$$

$$y_{n+3} = \frac{39}{40}h^2 f_n + \frac{27}{10}h^2 f_{n+1} + \frac{27}{40}h^2 f_{n+2} + \frac{3}{20}h^2 f_{n+3} + y_n + 3y'_n h \tag{25}$$

$$y'_{n+1} = \frac{3}{8}hf_n + \frac{19}{24}hf_{n+1} - \frac{5}{24}hf_{n+2} + \frac{1}{24}hf_{n+3} + y'_n \tag{26}$$

$$y'_{n+2} = \frac{1}{3}hf_n + \frac{4}{3}hf_{n+1} + \frac{1}{3}hf_{n+2} + y'_n \tag{27}$$

$$y'_{n+3} = \frac{3}{8}hf_n + \frac{9}{8}hf_{n+1} + \frac{9}{8}hf_{n+2} + \frac{3}{8}hf_{n+3} + y'_n \tag{28}$$

3. Analysis of the Block Methods

3.1. Order and error Constants of the Block Methods. According to ([1], [3], [4], [7] & [11]), the order of the new hybrid block methods (23) to (25) is obtained by using the Taylor series and it has uniformly of order four, with an error constants vector

$$C_{p+2} = \left[-\frac{7}{480}, -\frac{1}{45}, -\frac{9}{160}\right]^T \tag{29}$$

3.2. Consistency.

Definition 3.1. ([1], [3] & [7]). The hybrid block method (23) to (25) is said to be consistent if it has an order more than or equal to one i.e. $p \geq 1$.

Therefore, the method is consistent.

3.3. Zero Stability.

Definition 3.2. ([11] & [12]). The hybrid block method (23) to (25) is said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1$ and if $|r_z| = 1$, then the multiplicity of r_z must not be greater than six.

In order to find the zero-stability of hybrid block method (23) to (25), we only consider the first characteristic polynomial of the method as follows:

$$\pi(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = r^2(r-1) \tag{30}$$

which implies $r = 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

3.4. **Convergence.**

Theorem 3.3. *([1], [6] & [11]). Consistency and zero stability are sufficient condition for linear multistep method to be convergent.*

Since the method (23) to (25) are consistent and zero stable, it implies that the method is convergent for all points.

4. Implementation of the Block Methods

In this section, the derived method (23) to (25) is implemented and its derivatives (26) to (28) with the aid of MATLAB coding to solve second order problems in order to show the level of accuracy and efficiency of the method.

4.1. **Numerical Examples.** The method is tested on Vanderpol oscillator, exponential and stiff second order problems to test the accuracy of the proposed methods and our results are compared with the results obtained using existing methods. The following problems are taken as test problems:

Example 4.1. (Vanderpol oscillator)
$y'' = 2\cos x - \cos^3 x - y' - y - y^2 y'$
$y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = 0,\ \ h = 0.1$
Exact solution : $y(x) = \sin x$

Source [4].

Example 4.2. (Exponential)
$y'' = y' + 2e^x(x + 1)$
$y(0) = 1,\ \ y'(0) = 1,\ \ y''(0) = 0,\ \ h = 0.01$
Exact solution : $y(x) = e^x(x^2 + 1)$

Source [7].

Example 4.3. (Stiff)
$y'' - 100y = 0$
$y(0) = 1,\ \ y'(0) = -10,\ \ y''(0) = 0,\ \ h = 0.01$
Exact solution : $y(x) = e^{-10x}$

Source [11].

Table 1: Showing the result of Problem 1 with $h = 0.1$

X	y-exact solution	y-computed solution	Error $K = 3, P = 4$	Error in [8] $K = 1, P = 6$
0.1	0.099833416646828155	0.099833416691809868	$4.4982e^{-11}$	$3.307291e^{-10}$
0.2	0.198669330795061130	0.198669331622222580	$8.2716e^{-10}$	$2.315513e^{-9}$
0.3	0.295520206661339600	0.295520211025704750	$4.3644e^{-9}$	$6.161694e^{-9}$
0.4	0.389418342308650690	0.389418356431192710	$1.4123e^{-8}$	$1.192381e^{-8}$
0.5	0.479425538604203500	0.479425573589082890	$3.4985e^{-8}$	$1.934444e^{-8}$
0.6	0.564642473395035930	0.564642546677376680	$7.3282e^{-8}$	$2.775449e^{-8}$

Table 2: Showing the result of Problem 1 with $h = 0.01$

X	y-exact solution	y-computed solution	Error $K = 3, P = 4$
0.01	0.009999833334166665	0.009999833334166481	$1.8388e^{-16}$
0.02	0.019998666693333080	0.019998666693332671	$4.0939e^{-16}$
0.03	0.029995500202495660	0.029995500202494946	$7.1471e^{-16}$
0.04	0.039989334186634161	0.039989334186632350	$1.8111e^{-15}$
0.05	0.049979169270678331	0.049979169270676402	$1.9290e^{-15}$
0.06	0.059964006479444595	0.059964006479451083	$6.4879e^{-15}$
0.07	0.069942847337532754	0.069942847368077044	$3.0544e^{-11}$
0.08	0.079914693969172695	0.079914694029952382	$6.0780e^{-11}$
0.09	0.089878549198011040	0.089878549288762599	$9.0752e^{-11}$
0.1	0.099833416646828141	0.099833416869428440	$2.2260e^{-10}$

Table 3: Problem 2 with $h = 0.01$

X	y-exact solution	y-computed solution	Error $K = 3, P = 4$
0.01	1.010151172100876500	1.010151172101339000	$4.6252e^{-13}$
0.02	1.020609420562766500	1.020609420563829900	$1.0634e^{-12}$
0.03	1.031381943034074900	1.031381943035875700	$1.8008e^{-12}$
0.04	1.042476071431096100	1.042476071433893200	$2.7971e^{-12}$
0.05	1.053899274116964200	1.053899274110861900	$6.1022e^{-12}$
0.06	1.065659158112922900	1.065659158037588900	$7.5334e^{-11}$

Table 4: Results of Problem 3 using $h = 0.01$

X	y-exact solution	y-computed solution	Error $K = 3, P = 4$
0.01	0.904837418035959520	0.904837430763888890	$1.2728e^{-8}$
0.02	0.818730753077981820	0.818730782222222290	$2.9144e^{-8}$
0.03	0.740818220681717880	0.740818268124999960	$4.7443e^{-8}$
0.04	0.670320046035639330	0.670320112940006040	$6.6904e^{-8}$
0.05	0.606530659712633420	0.606530124252597220	$5.3546e^{-7}$
0.06	0.548811636094026500	0.548806820609242130	$4.8155e^{-6}$
0.07	0.496585303791409530	0.496572236683406890	$1.3067e^{-5}$
0.08	0.449328964117221560	0.449306428523652770	$2.2536e^{-5}$
0.09	0.406569659740599170	0.406531037370746310	$3.8622e^{-5}$
0.1	0.367879441171442330	0.367817516384073950	$6.1925e^{-5}$

4.2 Discussion of Results. Tables 1 shows the comparison of error in new method with another error in the literature. Table 2-4 above show the tabular display of the exact solution, computed solution and error of problem 2-3 on the implementation of the newly established method with $h = 0.1$ and 0.01. It is obvious that the block method is more efficient in terms of error when compared with existing methods of higher order of accuracy.

Conclusion: In this article, the derivation of the new block method for solving second order ordinary differential equations directly is examined. The method is of order $p = 4$ which shows that it is consistent. The plus of the method over the existing numerical methods is its ability to outperforms another method despite that the proposed method here is of order $p = 4$ while the one in the literature is $p = 6$. This has been shown in Table 1. This method is highly recommended.

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