Nigerian Journal of Mathematics and Applications
Volume **32**, (2022), 216 − 233. *Printed by Unilorin press*
©*Nig. J. Math. Appl.* *http://www.njmaman.com*

Collocation Approach for the Solution of Fredhlom Integro-Differential Equation of Fractional Order

Ajileye Ganiyu[1*], James A. Adewale[2*] and Ayinde M. Abdullahi[3]

Abstract

In this work, the collocation method is developed to solve the fractional integro-differential equations using the Caputo sense. The integral form of the problem is obtained and transformed into a system of linear algebraic equations. Computed results are solved using Maple 18 in order to show the efficiency and accuracy of the method.

1. Introduction

Over the years, there has been a drastic increase in the use of fractional differential models to simulate the dynamics of many different anomalous processes, especially those involving ultra-slow diffusion. There are many different methods and different basic functions that have been used to estimate the solution of fractional integro-differential equations or Abel's integral equations such as the Adomian decompositions method ([11]; [12]; [14]), collocation method by [1] and [8], Laplace decomposition method by ([23]; [17]), Homotopy perturbation method ([9]; [16]; [21]), Bernoulli matrix method [6] (Chebyshev-Galarkin method [10], Differential transform method [4] ([18]), Taylor matrix method ([15]) least

Received: 18/08/2022, Accepted: 20/09/2022, Revised: 15/10/2022. * Corresponding author.
2015 *Mathematics Subject Classification.* 26A33 & 35R11.
Keywords and phrases. Fredholm, Integro-differential equations; Collocation method; Fractional derivatives; Polynomial
[1]Department of Mathematics and Statistics, Federal University Wukari, Taraba State, Nigeria
[2]Department of Mathematics, American University of Nigeria, Yola, Adamawa State, Nigeria
[3]Department of Mathematics, Moddibo-Adama University, Yola, Nigeria
E-mail of the corresponding author: adewale.james@aun.edu.ng; +2348077092831
ORCID of the corresponding author: 0000-0003-2257-5596

square method ([13]). ([19]) presented two numerical methods for solving fractional integro-differential equations. The proposed methods reduce the equation into systems of linear algebraic equations. [3] considered approximate analytical solutions for fractional integro-differential equations by using the Sumudu transform method and Hermite spectral collocation method. The two methods were developed for differential equations.
This research considers Fredholm integro-differential equations with a fractional derivative of the form

$$D^{\beta}y(x) = \sum_{j=0}^{N} q_j(x)D^{\alpha_j}y(x) + h(x) + \int_0^b k(x,t)y(t)dt,\ 0 \le x \le b \tag{1}$$

subject to the initial condition

$$y^{(j)}(a_j) = \lambda_j,\ j = 0, 1, ..., n-1,\ n \in \mathbb{N} \tag{2}$$

where $y(x)$ is the unknown function, D^{β} and D^{α_j} are the Caputo's derivative, $k(x,t)$ is the Fredholm integral kernel function. $h(x)$ *and* $q_j(x)$ are the known functions, a_j and λ_j are known constants.

2. Basic Definitions

In this section, we present some definitions and basic concepts of fractional calculus for the formulation of the given problem.

Definition 2.1. ([22]) The Caputo derivative with order $\alpha > 0$ of the given function $f(x),\ x \in (a,b)$ is defined as

$${}_{x}^{C}D_a^{\alpha}y(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-s)^{m-\alpha-1}y^{(m)}(s)ds \tag{3}$$

where $m-1 \le \alpha \le m, m \in N, x > 0$

Definition 2.2. ([5]) Let (a_n), $n \ge 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression.

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_Nx^N = \sum_{n=0}^{N} a_nx^n = \ \phi(x)\ \mathbf{A} \tag{4}$$

where $\phi(x) = [1\ x\ \ x^2\ \cdots\ \ x^N],\ \mathbf{A} = [a_0\ a_1\ \ \cdots\ \ a_N]^T$
then $y(x,n) = x^n\mathbf{A},\ n = 0(1)N,\ n \in Z^+$

Definition 2.3. ([1]). Standard Collocation Method (SCM). This method is used to determine the desired collocation points within an interval. i.e [a,b] and is given by

$$x_i = a + \frac{(b-a)i}{N}, i = 1, 2, 3, ...N \tag{5}$$

Definition 2.4. ([7]). For $\alpha > 0$, Let $y(x)$ be a continuous function, then

$$ {}_0I_x^{\beta}\left({}_0^C D_x^{\beta} y(x)\right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k \tag{6}$$

where $m - 1 < \beta < 1$

Definition 2.5. ([5]) Let $p(s)$ be an integrable function, then

$$ {}_0I_x^{\beta}\left(p(s)\right) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} p(s) ds \tag{7}$$

Definition 2.6. The Riemann -Liouville derivative of order $\alpha > 0$ with $n - 1 < \alpha < n$ of the power function $f(t) = t^{p-\alpha}$ is given by

$$D^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \tag{8}$$

3. Methodology

In this section, we develop numerical solution of fredholm integro-differential equation. This method is based on collocation approach and also consider power series polynomial as our approximated solution.

Theorem 3.1. *(Banach Contraction Principle) Let (X, d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X, such that $T(x) = x$.*

Lemma 3.2. *Let $y(x)$ be the solution to (1) subject to (2), the integral form*

$$\begin{aligned} y(x) &= W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \\ &\quad \int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt\right] ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} \left(\int_0^b k(s,t) y(t) dt\right) ds \end{aligned} \tag{9}$$

where

$$W(x)=\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}h(s)ds$$

Proof. Multiply equation (1) by ${}_0I_x^{\beta}(.)$ gives

$$\begin{aligned}{}_0I_x^{\beta}\left(D^{\beta}y(x)\right) &= {}_0I_x^{\beta}\left(\sum_{j=0}^{N}q_j(x)D^{\alpha_j}y(x)\right)+_0 I_x^{\beta}\left(h(x)\right)\\ &\quad + {}_0I_x^{\beta}\left(\int_0^b k(x,t)y(t)dt\right)\end{aligned} \tag{10}$$

using(6) on equation (9)gives

$$\begin{aligned}y(x) &= \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+ {}_0I_x^{\beta}\left(\sum_{j=0}^{N}q_j(x)D^{\alpha_j}y(x)\right)\\ &\quad + {}_0I_x^{\beta}\left(h(x)\right)+ {}_0I_x^{\beta}\left(\int_0^b k(x,t)y(t)dt\right)\end{aligned} \tag{11}$$

subsituting equation (4) into equation (11) gives

$$\begin{aligned}y(x) &= \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k+\frac{1}{\Gamma(\beta)}\int_0^x(x-s)^{\beta-1}\\ &\quad\times\left(\sum_{j=0}^{N}q_j(x)\frac{1}{\Gamma(m_j-\alpha_j)}\int_0^s(s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)\ dt\mathbf{A}\right)ds\\ &\quad+\frac{1}{\Gamma(\beta)}\int_0^x(x-s)^{\beta-1}h(s)ds+\frac{1}{\Gamma(\beta)}\int_0^x(x-s)^{\beta-1}\\ &\quad\times\left(\int_0^b k(x,t)\phi(t)dt\right)ds\mathbf{A}\end{aligned} \tag{12}$$

□

3.1. **Method of Solution.**
Collocatiing at x_i in equation (12) gives

$$\begin{aligned}y(x_i) &= W(x_i)+\sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}q_j(s)\\ &\quad\times\left(\int_0^s(s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)\ dt\right)ds\ \mathbf{A}\\ &\quad+\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}\left(\int_0^b k(s,t)\phi(t)dt\right)ds\mathbf{A}\end{aligned} \tag{13}$$

simplifying gives

$$\phi(x_i)\mathbf{A} = W(x_i) + \begin{bmatrix} \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \\ \times \left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t))\, dt \right) ds \\ + \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} \left(\int_0^b k(s,t)\, (\phi(t))\, dt \right) ds \end{bmatrix} \mathbf{A} \tag{14}$$

where

$$W(x_i) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x_i^k + \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} h(s) ds$$

Factorise the values of $\mathbf{A}$ from equation(14) gives

$$\begin{bmatrix} \phi(x_i) - \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \\ \times \left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t))\, dt \right) ds - \\ \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} \left(\int_0^b k(s,t)\, (\phi(t))\, dt \right) ds \end{bmatrix} \mathbf{A} = W(x_i) \tag{15}$$

Equation (15) can be in the form

$$V(x_i)\mathbf{A} = W(x_i) \tag{16}$$

where

$$\begin{aligned} V(x_i) \quad = \quad & \phi(x_i) - \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \\ & \left(\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) \;\; dt \right) ds \\ & - \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} \left(\int_0^b k(s,t)\, (\phi(t))\, dt \right) ds \end{aligned}$$

and

$$\mathbf{A} = [a_0 \ a_1 \ \ \cdots \ \ a_N]^T$$

Multiply both sides of equation (16) by $V^{-1}(x_i)$ gives

$$\mathbf{A} = V^{-1}(x_i) W(x_i) \tag{17}$$

Substituting $\mathbf{A}$ into the approximate solution (4) gives

$$y(x) = \phi(x_i) V^{-1}(x_i)\; W(x_i) \tag{18}$$

Lemma 3.3. *Let $q \in C([0,1], \mathbb{R})$ be defined as $q(s) = s^p$ and Let $y(t)$ be approximated by (9), if*

$$L(x) = {}_0I_x^\beta \left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right) \tag{19}$$

then, it is equivalent to

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j} \mathbf{A} \tag{20}$$

Proof.
Applying equation (3) and (7) into equation(19)gives

$$(21)\; {}_0I_x^\beta \left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right) = \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt \right] ds$$

substitute(8) into (21) gives

$$= \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x} (x-s)^{\beta-1} S^{p_j} \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left(\frac{\Gamma(n+1)}{\Gamma(n-m_j+1)} t^{n-m_j} \right) dt \right] ds \; \mathbf{A} \tag{22}$$

Let $s - t = (1-v)s$, then $t = vs \Longrightarrow \dfrac{dt}{dv} = s \Longrightarrow dt = sdv$,
substitute into (22) gives

$$= \sum_{j=0}^{N} \frac{\Gamma(n+1)}{\Gamma(m_j-\alpha_j)\Gamma(n-m_j+1)} \frac{1}{\Gamma(\beta)} \int_0^{x} (x-s)^{\beta-1} S^{p_j} \left[S^{n-\alpha_j} \int_0^1 (1-v)^{m_j-\alpha_j-1} V^{n-m_j} dt \right] ds \; \mathbf{A} \tag{23}$$

Simplify(23), we get

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x^{\beta+n-\alpha_j+p_j} \mathbf{A} \tag{24}$$

□

Lemma 3.4. *Let* $k \in C([0,1] \times [0,1], \mathbb{R})$ *and Let* $y(t)$ *be approximated by* (9) *,if*

$$E(x) = {}_0I_x^{\beta}\left[\int_0^b k(s,t)y(t)dt\right] \tag{25}$$

then, it is equivalent to

$$\mathbf{E}(x;n) = \left(\frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)}x^{\beta+r}\right)\mathbf{A} \tag{26}$$

Proof. Applying equation (7) into (9) gives

$${}_0I_x^{\beta}\left[\int_0^b k(s,t)y(t)dt\right] = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)_0^{\beta-1}\left[\int_0^b k(s,t)y(t)dt\right]ds$$

Substituting for $k(s,t)$

$$= \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left(\int_0^b s^r t^{\sigma} y(t)dt\right)ds \tag{27}$$

Apply (4) to (27) and simplify gives

$$= \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left[s^r\frac{b^{\sigma+n+1}}{\sigma+n+1}\right]\mathbf{A}\ ds \tag{28}$$

Let $x-s=(1-u)x$,then $s=ux \Longrightarrow ds = xdu$, substitute into(28) gives

$$= \left(\frac{b^{\sigma+n+1}}{\Gamma(\beta)\ \ (\sigma+n+1)}\int_0^1 ((1-u)\,x)^{\beta-1}(ux)^r x\ du\right)\mathbf{A} \tag{29}$$

Solving the equation (29), gives

$$\mathbf{E}(x;n) = \left(\frac{b^{\sigma+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)}x^{\beta+r}\right)\mathbf{A}$$

□

Lemma 3.5. *Let* $h \in C([0,1], \mathbb{R})$ *be defined as* $h(s) = s^m$ *and Let* $y(t)$ *be approximated by* (9)*, if*

$$C(x) = {}_0I_x^{\beta}(h(x)) \tag{30}$$

then, it is equivalent to

$$C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)}x^{\beta+m}$$

Proof.
Applying equation (7) into (30)gives

$$ {}_0I_x^{\beta}\left(h(x)\right) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s)\ ds $$

Substituting for $h(s)$ gives

$$ = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} s^m ds $$

Let $x - s = (1-u)x,\ s = ux \Longrightarrow \dfrac{ds}{du} = x \Longrightarrow ds = xdu.$

$$ C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m} \tag{31} $$

□

Lemma 3.6. *Let $y(x)$ be the solution of (1)and (2)then the solution is*

$$ y(x) = \phi(x_i) V^{-1}(x_i)\ W(x_i) \tag{32} $$

where

$$ \begin{aligned} V(x_i) &= \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j} \\ &+ \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)} x^{\beta+r} \end{aligned} $$

and

$$ W(x_i) = -\sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x_i^k + \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x_i^{\beta+m} $$

Proof. Approximate solution of equation (9) is

$$ y(x) = \phi(x)\ \mathbf{A} $$

From equation (32) $\mathbf{A} = V^{-1}(x_i)\ W(x_i)$
where

$$ \begin{aligned} V(x_i) &= \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j} \\ &+ \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)} x^{\beta+r} \end{aligned} $$

Substituting for $\mathbf{A}$ in the approximate solution gives

$$ y(x) = \phi(x_i) V^{-1}(x_i)\ W(x_i) $$

□

3.2. **Uniqueness of the Method.** In order to establish the uniqueness of the

method, we introduce the following hypothesis

$$\begin{aligned} H_1 &: q^* = \max_{x\in[0,1]} |q(x)| \\ H_2 &: k^* = \max_{x\in[0,1]} \int_0^b |k(x,t)|\, dt \\ H_3 &: d\left(y_n^{(m_j)}, y^{(m_j)}\right) \leq L_{n_j} d\left(y_n, y\right) \end{aligned}$$

Before the establishment of uniqueness of solution, it must satisfy the contraction principle. Thus, the proof of Banach contraction principle and Lemma 3.7 are not new results and could be source from [2] & [20].

Lemma 3.7. *(q-contraction) Let $T : X \longrightarrow X$ be a mapping defined by theorem (3.0) for $y_1,\ y_2\ \in X$. T is q-contraction if and only if*

$$\left[\frac{q_j^*}{\Gamma(m_j-\alpha_j+1)}\left(\frac{\Gamma(\beta+m_j-\alpha_j+1)-\beta\Gamma(\beta+m_j-\alpha_j)}{\Gamma(\beta+m_j-\alpha_j+1)}\right)+k^*\right]\frac{L}{\Gamma(\beta+1)}<1$$

then there exist a unique solution of T.

Proof.

$$\begin{aligned} (Ty_1)(x) &= W(x)+\sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}q_j(s) \\ &\quad \left[\int_0^s (s-t)^{m_j-\alpha_j-1}y_1^{(m_j)}(t)dt\right]ds+\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left[\int_0^b k(s,t)y_1(t)dt\right]ds \end{aligned}$$

and

$$\begin{aligned} (Ty_2)(x) &= W(x)+\sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}q_j(s) \\ &\quad \left[\int_0^s (s-t)^{m_j-\alpha_j-1}y_2^{(m_j)}(t)dt\right]ds+\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \\ &\quad \left[\int_0^b k(s,t)y_2(t)dt\right]ds \end{aligned}$$

$$
\begin{aligned}
((Ty_1)(x) - (Ty_2)(x)) &= \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} q_j(s) \\
&\left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left(y_1^{(m_j)}(t) - y_2^{(m_j)}(t)\right) dt\right] ds \\
&+\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\int_0^b k(s,t)\left(y_1(t)-y_2(t)\right) dt\right] ds
\end{aligned}
$$

Taking maximum of both sides we get

$$
\begin{aligned}
d\left(Ty_1(x), Ty_2(x)\right) &\leq \sum_{j=0}^{N} \frac{q_j^*}{\Gamma(\beta)\Gamma(m_j-\alpha_j)} \int_0^x (x-s)^{\beta-1} \\
&\left[\int_0^s (s-t)^{m_j-\alpha_j-1} dt\right] ds \; Ld\left(y_1, y_2\right) \\
&+\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\int_0^b k(s,t) dt\right] ds \; Ld\left(y_1, y_2\right)
\end{aligned}
$$

$$
\begin{aligned}
d\left(Ty_1(x), Ty_2(x)\right) &\leq \left[\frac{q_j^*}{\Gamma(m_j-\alpha_j+1)} \left(\frac{\Gamma(\beta+m_j-\alpha_j+1) - \beta\Gamma(\beta+m_j-\alpha_j)}{\Gamma(\beta+m_j-\alpha_j+1)}\right) + k^* \right] \\
&\frac{L}{\Gamma(\beta+1)} d\left(y_1, y_2\right)
\end{aligned}
$$

Since T is a contraction map, By Banach contraction principle, we can conclude that T has a unique solution. □

3.3. **Convergence Analysis.** We establish the convergence of the method.

$$
\begin{aligned}
y_N(x) &= W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \\
&\int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y_N^{(m_j)}(t) dt\right] ds \\
&+\frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} \left(\int_0^b k(s,t) y_N(t) dt\right) ds
\end{aligned}
\tag{33}
$$

Substract (9) from (33) gives

$$
E_N(x) = y_N(x) - y(x)
$$

hence

$$
\begin{aligned}
|E_N(x)| &= \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \\
&\int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left| E_N^{(m_j)}(t)dt \right| \right] ds \\
&+\frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} \left| \int_0^b k(s,t) y_N(t)dt \right| ds \qquad (34)
\end{aligned}
$$

Therefore

$$
\frac{\|E_N(x)\|_\infty}{\|E_N(t)\|_\infty} \le \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\begin{array}{c} \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left| E_N^{(m_j)}(t)dt \right| \right] \\ + \left| \int_0^b k(s,t) y_N(t)dt \right| \end{array} \right] ds
$$

The method converges.

4. Numerical Examples

In this section, we present numerical examples to test the efficiency and simplicity of the method. All computations are done with the aid of program written in MAPLE 18. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions respectively. $\text{Error}_N = |y_n(x) - y(x)|$.

Example 4.1. ([15] Consider fractional fredholm integro-differential equation of the form

$$
D^2 y(x) = -D^{1.5} y(x) - y(x) + \int_0^1 y(t)dt + x - \frac{1}{2}
$$

with the condition $y(0) = y^{'}(0) = 1$ and exact solution is $y(x) = x + 1$

Solution 4.2. Comparing with equation(1) and equation (2), $\beta = 2$, $\alpha =$

$1.5, h(x) = x - \frac{1}{2}$

Using $N = 3$ for illustration

Writing in the integral form

$$
\begin{aligned}
y(x) &= W(x) - \frac{1}{\Gamma(2-1.5)}\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\
&\left[\int_0^s (s-t)^{2-1.5-1}\frac{\Gamma(n+1)}{\Gamma(n-2+1)}t^{n-2}dt\right] ds\ \mathbf{A} - \\
&\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}s^n ds\mathbf{A} + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\
&\left[\int_0^1 t^n dt\right] ds\ \mathbf{A}
\end{aligned} \tag{35}
$$

where

$$
W(x) = \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\left(s-\frac{1}{2}\right) ds \tag{36}
$$

$$
\begin{aligned}
y(x) &= W(x) + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\
&\left[\frac{\Gamma(n+1)}{\Gamma(n-0.5)}s^{n+0.5}\right] ds\ \mathbf{A} - \\
&\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}s^n ds\mathbf{A} + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\
&\left[\int_0^1 t^n dt\right] ds\ \mathbf{A}
\end{aligned}
$$

Using Lemma (3.2), and Lemma (3.3) gives

$$
\begin{aligned}
y(x) &= W(x) + \frac{\Gamma(n+1)\Gamma(n+1.5)}{\Gamma(n-0.5)\Gamma(n+3.5)}x^{n+4.5}ds\ \mathbf{A} - \\
&\frac{\Gamma(n+1)}{\Gamma(n+3)}x^{n+4}\mathbf{A} + \frac{\Gamma(n+1)}{\Gamma(n+4)}1^{n+5}\ \mathbf{A}
\end{aligned} \tag{37}
$$

$$
W(x) = \frac{\Gamma(2)}{\Gamma(4)}x^5 - \frac{\Gamma(1)}{2\Gamma(3)}x^4 \tag{38}
$$

for $n = 0(1)N$

Applying Lemma (3.5) on equatiion (37) and (38) gives

$$
y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i)
$$

We obtained the result

$$y_3(x) = \begin{pmatrix} 1.0000000000 \times x^0 + 1.0000000000 \times x + \\ 8.8817841970 \times 10^{-16} \times x^2 + 2.2204460493 \times 10^{-16} x^3 \end{pmatrix}$$

Example 4.3. ([15] considered fractional fredholm integro-differential equations of the form

$$D^{1.5}y(x) = D^{0.5}y(x) + \int_0^1 e^x y(t)dt + f(x)$$

where $f(x) = e^x - e^{x+1}$ with the condition $y(0) = 0$ and exact solution $y(x) = e^x$

Solution 4.4. Comparing with equation (1) and equation (2), $\beta = 1.5, \alpha = 0.5,\ k(x,t) = e^x,\ f(x) = e^x - e^{x+1}$

Writing in the integral form

$$\begin{aligned} y(x) \quad = \quad & W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ & \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{\Gamma(n+1)}{\Gamma(n-m_j+1)} t^{n-m_j} dt \right] ds\ \mathbf{A} - \\ & \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\int_0^1 e^s\ t^n dt \right] ds\ \mathbf{A} \end{aligned} \tag{39}$$

where

$$W(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s) ds \tag{40}$$

Using $N = 3$ for illustration
substituting for $\beta = 1.5, \alpha = 0.5, f(x) = e^x - e^{x+1}$ in equation (39) and (40) gives

$$\begin{aligned} y(x) \quad = \quad & W(x) + \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} \\ & \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt \right] ds\ \mathbf{A} + \\ & \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} \left[\int_0^1 e^s\ t^n dt \right] ds\ \mathbf{A} \end{aligned}$$

$$W(x) = \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} \left(e^s - e^{s+1} \right) ds$$

for $n = 0(1)N$
Applying Lemma (3.2) gives

$$(41)\qquad y(x) = W(x) + \frac{\Gamma(n+1)}{\Gamma(n+0.5)}\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}s^{n+1.5}ds\ \mathbf{A} + \frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\left[\int_0^1 e^s\ t^n dt\right] ds\ \mathbf{A}$$

where

$$(42)\qquad W(x) = \frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\frac{s^n}{\Gamma(n+1)}ds - \frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\frac{(s+1)^n}{\Gamma(n+1)}ds$$

Applying Lemma (3.5) on equation(41) and(42) gives

$$y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i)$$

We obtained the result

$$y_3 = \begin{pmatrix} 0.9990233401 \times x^0 + 1.0116982759 \times x + \\ 0.4050677749 \times x^2 + 0.3003042742 \times x^3 \end{pmatrix}$$

Table 1: Exact and approximate values, Example 1

x	Exact	N=3	N=5	N=6
0.2	1.2214027580	1.2199681400	1.2213960720	1.2214031090
0.4	1.4918246980	1.4877329680	1.4918178290	1.4918249590
0.6	1.8221188000	1.8167324280	1.8221150110	1.8221190520
0.8	2.2255409280	2.2213811250	2.2255348760	2.2255409420
1.0	2.7182818280	2.7160936650	2.7182828500	2.7182817820

Table 2: Absolute Error for Example 1

x	$\mathbf{ERR}_3$	$\mathbf{ERR}_5$	$\mathbf{ERR}_6$	[18] $_{\mathbb{N}=18}$
0.2	1.4346180e-3	6.686e-6	3.51e-7	0.21823e-7
0.4	4.09173e-3	6.869e-6	2.61e-7	0.21586e-7
0.6	5.386372e-3	3.789e-6	2.52e-7	0.86325e-7
0.8	4159803e-3	6.052e-6	1.40e-8	0.12423-5
1.0	2.1881634e-3	1.022e-6	4.60e-8	0.83792e-5

Example 4.5. ([13]) considered fractional fredholm integro-differential equations of the form

$$D^{0.5}y(x) = f(x) + \int_0^1 xty(t)dt$$

where $f(x) = \dfrac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \dfrac{x}{12}$ with the condition $y(0) = 0$ and exact solution $y(x) = x^x - x$

Comparing with equation (1) and equation (2), $\beta = 0.5, \alpha = 0,\ k(x,t) = xt$
Writing in the integral form

$$y(x) = W(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\int_0^1 xt\ t^n dt \right] ds\ \mathbf{A} \tag{43}$$

where

$$W(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds \tag{44}$$

Using $N = 4$ for illustration
substituting for $\beta = 0.5,\ f(s)$ in equation (43) and (44) gives

$$y(x) = W(x) + \frac{1}{\Gamma(0.5)} \int_0^x (x-s)^{0.5-1} \left[\int_0^1 x\ t^{n+1} dt \right] ds\ \mathbf{A} \tag{45}$$

$$W(x) = \frac{1}{\Gamma(0.5)} \int_0^x (x-s)^{0.5-1} \left(\frac{\frac{8}{3} s^{\frac{3}{2}} - 2 s^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{s}{12} \right) ds \tag{46}$$

Applying Lemma (3.3), Lemma (3.4) and Lemma (3.5) on equation(45) and(46) gives

$$y(x) = \phi(x_i) V^{-1}(x_i)\ W(x_i)$$

We obtained the result

$$y_4 = \begin{pmatrix} 4.996003611 \times 10^{-16} - 0.999933822618743x + 0.999941067872896x^2 \\ -0.288451178676041e - 5x^3 + 8.253709893e - 7x^4 \end{pmatrix}$$

Table 3: Exact and approximate values, Example 3

x	Exact	N=4	N=6
0.2	-0.1600	-0.1599891436	-0.1599907726
0.4	-0.2400	-0.2399831216	-0.2399889081
0.6	-0.2400	-0.2399820253	-0.2399889081
0.8	-0.1600	-0.1599859134	-0.1600041982
1.0	0	0.000005186159202	-0.1719383090e-4

Table 4: Absolute Error for Example 3

x	$\mathbf{ERR_4}$	$\mathbf{ERR_6}$
0.2	1.08564e-5	9.2274e-6
0.4	1.68784e-5	1.10919e-6
0.6	1.79747e-5	6.1422e-6
0.8	1.40866e-5	4.1982e-6
1.0	5.186159202e-6	1.719383090e-6

Discussion of Results. We established the uniqueness of the method using theorem (3.0). The method of solution satisfies that T is q-contraction using some hypothesis. This shows that the method is unique and converges.
In example 1, The approximate solution obtained at N=3 gives $y_3(x) = 1.00000000 \times x^0 + 1.0000000000 \times x + 8.8817841970 \times 10^{-16} \times x^2 + 2.2204460493 \times 10^{-16}x^3$ which shows that the result converges to the exact solution.
In example 2, The approximate solution at $N = 3$ gives $y_3(x) = 0.9990233401x^0 + 1.0116982759x$
$+ 0.4050677749x^2 + 0.3003042742x^3$ and for solving N=5 and 7, we obtained Table 1 which displays the results obtained from example 2 at $x = 0.2\ to\ 1.0$ for various values of N and the exact solution. Error of example 2 as shown in Table 2 indicates that as the values of N increases, the error becomes smaller. For instance, the least error in [15] at N=18 is 0.21823e-7 while the least error in our method at N=6 is 1.40e-8. This shows that the numerical method developed is consistent and converges faster.
In example 3, The approximate solution obtained as N=4 gives $y_4 = 4.996003611 \times 10^{-16} - 0.999933822618743x + 0.999941067872896x^2 - 0.288451178676041e - 5x^3 + 8.253709893e - 7x^4$ and solving for N=6 and N=10. We obtained Table 3 which shows the results obtained at $x = 0.2\ to\ 1.0$ fot the values of N and the exact solution. Table 4 shows the error of example 3 and it indicates that as the values of N increase, the error becomes smaller and more consistent across all values of x.

Conclusion. An enhanced numerical method is developed for the solution of Fredholm integro-differential equations of fractional order with initial conditions using the collocation method. The numerical method derived is consistent, efficient and reliable and all computations are done with MAPLE 18.

Acknowledgement. The authors will like to appreciate the reviewers for their helpful suggestions and comments which have improved the quality of this paper.

Competing interests: The manuscript was read and approved by all the authors. They therefore declare that there is no conflicts of interest.

Funding: The authors received no financial support for the research, authorship, and/or publication of this article.

References

[1] Agbolade O. A. & Anake T. A. (2017). Solutions of first-order Volterra type linear integro- differential equations by collocation method. *Journal of Applied Mathematics.* **2017**, Article ID 1510267, 5 pages. http://doi.org/10.1155/2017/1510267.

[2] Ahmed A. H., Kirtiwant P. G. & Shakir M. A. (2019). The approximate solutions of fractional integro-differential equations by using modified adomian decomposition method. *Khayyan J. Math.* **5** (1), 21-39.

[3] Amer Y. A., Mahdy A. M. S. & Youssef E. S. M. (2018). Solving fractional integro-differential equations by using Sumudu transform method and Hermite spectral collocation method. *Tech. Science press (CMC).* **54** (2), 161-180.

[4] Arikoglu A. & Ozkol I. (2009). Solution of fractional integro-differential equations by using fractional differential transform method. *Chaos, Solitons & Fractals.* **40** (2), 521-529.

[5] Berinde V. (2002). *Iterative approximation of fixed points.* Romania. Editura Efemeride, Baia Mare.

[6] Bhraway A. H., Tohidi E. & Soleymani F. (2012). A new Bernoulli matrix method for solving high order linear and nonlinear Fredholm integro differential equations with piecewise interval. *Appl Math. Comput.* **219**, 482-497.

[7] Bragdi A., Frioui A. & Lakoud A. G. (2020). Existence of solutions for nonlinear fractional integro differential equations, Advances in Difference Equations. **2020**, doi: 10/1186/s13662-020-02874-9.

[8] Cem O., Mehmet S. & Arzu D. O. (2015). Chelyshkov collocation approach to solving the systems of linear functional differential equations. *New Trend in Mathematical Sciences. (NTMSCI).* **3** (4), 83-97.

[9] Elbeleze A. A., Klman A. & Taib B. M. (2016). Approximate solution of integro-differential equation of fractional (arbitrary) order. *Journal of King Saud University- Science.* **28**, 61-68.

[10] Issa K. & Saleh F. (2017). Approximate solution of perturbed Volterra Fredholm integro differential equation by Chebyshev-Galerkin method. *Journal of Mathematics.* **2017**, Article ID 8213932, 6 pages. http://doi.org/10.1155/2017/8213932.

[11] Khan R. H. & Bakodah H. O. (2013). Adomian decomposition method and its modifcation for nonlinear Abel.s integral equations. *Computers and Mathematics with Applications.* **45-48** (7), 2349-2358.

[12] Mittal R. C. & Nigam R. (2008). Solution of fractional integro-differential equations Adomian decomposition method. *The International Journal of Applied Mathematics and Mechanics.* **4** (2), 87-94.

[13] Mohammed D. S. (2014). Numerical solution of fractional integro-differential equations by least square method and shifted Chebyshev polynomial. *Mathematical Problems in Engineering.* **2014**, Article ID 431965, 5 pages. http://doi.org/10.1155/2014/431965.

[14] Momani S. & Noor M. A. (2006). Numerical methods for fourth-order fractional integro-differential equations. *Applied Mathematics and Computation.* **182** (1), 754-760.

[15] Mustafa G., Yalcin O & Ayse A. (2013). Numerical approach for solving fractional Fredholm integro-differential equation. *International Journal of Computer Mathematics.* **90** (7), 1413-1434. http://dx.doi.org/10.1080/00207160.2012.750720.

[16] Nawaz Y. (2011). Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. *Computers & Mathematics with Applications.* **6** (8), 2330-2341.

[17] Nazari D. & Jahanshahi M. (2012). Some notes on multi-order frational integro-differential equations. *43^{rd} Annual Iranian Mathematics Conference.* 27-30.

[18] Nazari D. & Shahmorad S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. *Journal of Computational and Applied Mathematics.* **234** (3), 883-891.

[19] Oyedepo T., Akinduko O. B., Faniyi O. E. & Okperhie E. P. (2019). Bernstein Least-Square Technique for Solving Fractional Integro-Differential Equations. *Science World Journal.* **3**, 1597-6343.

[20] Palais R. S. (2007). Asimple proof of the Banach contraction principle. *Journal of Fixed Point Theory and Applications.* **2**, 221-223. doi:10.1007/s11784-007-0041-6.

[21] Saeed R. K. & Sdeq H. M. (2010). Solving a system of linear fredholm fractional integro-differential equations using homotopy perturbation method. *Australian Journal of Basic and Applied Sciences.* **4** (4), 633-638.

[22] Yang C. (2011). Numerical solution of Nonlinear Fredholm integro differential equations of fractional order by using hybrid of block pulse functions and chebyshev polynomials. *Mathematical Problems in Engineering.* **2011**, Article ID 341989, 1-11. doi:10.1155/2011/341989.

[23] Yang C. & Hou J. (2013). Numerical solution of Volterra Integro-differential equations of fractional order by Laplace decomposition method, World Academy of Science, Engineering and Technology. *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering.* **7**, 863-867.