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Numerical Solution of a Generalized Fractional Burgers Diffusion Equation with Mamadu-Njoseh Polynomials

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Abstract

In this paper, we consider the numerical solution of a generalized fractional Burgers diffusion equation (GFBE). A fractional variational iteration scheme coupled with Mamadu-Njoseh polynomials was developed to effectively solve the GFBE. MAPLE 18 software was used to carry out numerical simulations and approximations. Resulting numerical evidences show that the procedures converges rapidly to the theoretical solution as the order of the fractional derivative v varies.

1. Introduction

The Burgers equation is s partial differential equation arising directly from fluid mechanic. It has major areas of applications in applied mathematics, such as in traffic flow and gas dynamic. The Burgers equation can be viewed as a twin of the Navier-Stokes equations when simplified explicitly, and related to the heat equation via the Hopf-Cole transformation [1]-[3].
In 1974, Burgers was one of the first to employ Burgers equation (SBE) to model the turbulent fluid motion giving rise to the celebrated equation

$$\partial_t u_t(x) = \alpha \partial_x^2 u_t(x) - u(x)\, \partial_x u_t(x),$$

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such $u_t(x)$ denotes the velocity field, and α is the viscosity. Burgers explained in his book " *The nonlinear Diffusion Equation*", stated that the above equation is a simple mathematical model describing the relationship between the inertia and the dissipation in the fluid. As established by [3] that most Burgers equation can be solved explicitly, and that the solution generates shock waves on the disappearance of the viscosity, α. A typical A nonlinear one-dimensional Burger's equation can be seen in [4] and is given as

$$u_t + uu_x = u_{xx},$$

with the condition $u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right)$, where exact solution can be seen in

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}\left(x - \frac{t}{2}\right)\right).$$

Numerical solutions of the Burgers equation in the literature have been obtained using different methods and techniques [5]-[10].

A fractional Burgers equation is derived by using the fractional counter parts to replace the integer order partial space/ or time derivative terms. Here, the generalized time fractional partial derivative defined in the Caputo sense replaces the time partial derivatives constrained to some prescribed initial or boundary conditions. The fractional order Burgers equation has been solved by many authors [10]-[16].

In this paper we seek the numerical approximation of the generalized fractional Burgers equations via the variational iteration method with Mamadu-Njoseh polynomials as basis functions. The method is highly valuable as no guess is needed in seeking the initial approximation, but rather is computed by subjecting the initial approximation to the given prescribed condition. It is explicit and straightforward with no hidden assumptions or transformations. The method is explicit as it converges rapidly to the theoretical solution as the order of the fractional derivative and v varies.

2. Caputo Derivative

Given $f(x)$ *defined in* C^k_{-1}such that $k \in \mathbb{N} \cup \{0\}$. Then the Caputo fractional derivative of the function $f(x)$ is defined as [15].

$$D^\alpha f(x) = \frac{1}{G(k-\alpha)} \int_0^x (x-t)^{k-\alpha-1} r^k(t)\, dt, \tag{1}$$

where $(k-1) < \alpha = k,\ k \in \mathbb{N},\ \ x > 0$.

2.1. Properties of the Caputo Derivative.

i $D^\alpha [I^\alpha r(x)] = f(x)$;

ii. $I^\alpha[D^\alpha f(x)] = f(x) - \sum_{k=0}^{n-1} f^k(0^+)\left(\frac{x^k}{k!}\right)$;

iii. $D^\alpha K = 0$, K is a constant and

iv. $D^{\alpha}x^{a}=\begin{cases} 0, \quad a\in N_a, \quad a<\alpha_a, \\ \frac{G(a+1)}{G(a-\alpha+1)}x^{a-\alpha}, \quad a\in N_a, \quad a=\alpha_a, \end{cases}$
where $\alpha_a=\alpha$ and $N_a=0$.

3. Mamadu-Njoseh Polynomials and Variational Iteration Method

3.1. **Mamadu-Njoseh Polynomials.** The Mamadu-Njoseh polynomials [17]-[18] was constructed in the interval with $x\in[-1,1]$ with respect to the weight function

$$w(x)=x^2+1$$

based on these three properties;
i. $\varphi_n(x)=\sum_{i=0}^{n}C_i^{(n)}x^i$
ii. $<\varphi_m(x),\varphi_n(x)>=0, m?n,$
iii. $\varphi_n(x)=1$,
where $\varphi_n, \quad n=0,\ 1,\ 2,\ 3,\ldots,$ are the orthogonal polynomials.
The first seven Mamadu-Njoseh polynomials as derived from above properties are given below.

$$\left.\begin{array}{c} \varphi_0(x)=1 \\ \varphi_1(x)=x \\ \varphi_2(x)=\frac{1}{3}\left(5x^2-2\right) \\ \varphi_3(x)=\frac{1}{5}\left(14x^3-9x\right) \\ \varphi_4(x)=\frac{1}{648}\left(333-2898x^2+3213x^4\right) \\ \varphi_5(x)=\frac{1}{136}\left(325x-1410x^3+1221x^5\right) \\ \varphi_6(x)=\frac{1}{1064}\left(-460+8685x^2-24750x^4+17589x^6\right) \end{array}\right\}$$

However, if these polynomials are orthogonal in another interval $[a,b]$, then taking a map bijectively from $[-1,1]$ to $[a,b]$, yields

$$\varphi_n^*(x)=\sum_{i=0}^{n}C_i^{(n)}x^i=\left(\frac{2x-a-b}{b-a}\right), x\in[a,b] \tag{2}$$

where $\varphi_i^*(x), \quad i=0,1,2,3,\ldots,$ are called the shifted Mamadu-Njoseh polynomials. For instance, if $a=0, \quad b=1$ in equation (2), then the Mamadu-Njoseh polynomials are termed the shifted Mamadu-Njoseh polynomials;

$$\varphi_0^*(x)=1$$

$$\varphi_1^*(x)=2x-1$$

$$\varphi_2^*(x)=\frac{1}{3}(20x^2-20x+1)$$

$$\varphi_3^*(x)=\frac{1}{5}(112x^3-168x^2+66x-5)$$

$$\varphi_4^*(x) = \frac{1}{9}(238x^4 - 1428x^3 + 910x^2 - 196x + 9)$$

$$\varphi_5^*(x) = \frac{1}{17}(4884x^5 - 12210x^4 + 10800x^3 - 3990x^2 + 550x - 17)$$

$$\varphi_6^*(x) = \frac{1}{133}(140712x^6 - 422136x^5 + 478170x^4 - 252780x^3 + 62010x^2 - 5976x + 333)$$

3.2. **Basic ideas of Variational Iteration Method (VIM).** The variational iteration method (VIM) established by Ji-Huan in [19] is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact solution exists.

Let the generalized form of a differential equation be given as

$$L\left[u\left(x\right)\right] = g\left(x\right), \quad u\left(a_1\right) = a, \quad u\left(a_2\right) = b, \tag{3}$$

where L is considered as differential operator, $u\left(a_1\right) = a, \ \ u\left(a_2\right) = b$, are boundary or initial conditions.

Now, the VIM involves the construction of a correction functional for Equation (3) is as follow:

$$u_{i+1}\left(x\right) = u_i\left(x\right) + \int_0^x \lambda(s)\left(Lu_i(s) - g\left(s\right)\right)ds, \ i = 0, \tag{4}$$

where $\lambda(s)$ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. Also, the Lagrange multiplier $\lambda(s)$ can be estimated using the formula

$$\lambda\left(s\right) = (-1)^n \frac{(s-x)^{(n-1)}}{(n-1)!}, \tag{5}$$

where n in Equation (5) is the

3.2.1. *VIM and Mamadu-Njoseh Polynomials for Generalized Fractional Burgers Equation.* Let the generalized fractional Burgers be given as sated in [5]-[8].

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u\frac{\partial u(x,t)}{\partial t} = v\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad x \in [a,b]\,, \ \alpha \in (0,1)\,, \tag{6}$$

where $u(x)$ is the unknown space-time function, v is the viscosity coefficient. By the fractional variational iteration method (FVIM), we construct a correctional functional for (6) as follows:

(7)

$$u_{k+1}\left(x\right) = u_k\left(x\right) + \frac{1}{G(a)}\int_0^x \lambda(s)\left(\frac{\partial^\alpha u_k(x,s)}{\partial s^\alpha} + u_k(x,s)\frac{\partial u_k(x,s)}{\partial s} - v\frac{\partial^2 u_k(x,s)}{\partial x^2}\right)(ds)^\alpha,$$

where $k = 0$ and $\lambda(s)$ is the general Lagrange multiplier. Solving (7) via the variational theory we obtain

$$\lambda(s) = (s - x)^{(k-1)\alpha}. \tag{8}$$

Thus, our varaitional scheme for a generalized fractional Burgers equation becomes

$$\begin{cases} u_{k+1}(x) = u_k(x) \\ +\frac{1}{G(a)} \int_0^x (s - x)^{(k-1)\alpha} \left(\frac{\partial^\alpha u_k(x,s)}{\partial s^\alpha} + u_k(x,s) \frac{\partial u_k(x,s)}{\partial s} - v \frac{\partial^2 u_k(x,s)}{\partial x^2} \right) (ds)^\alpha \end{cases} \tag{9}$$

where $k = 0$.

3.2.2. *Determination of Initial Approximation.* The approximate solution is defined as

$$u_k(x,t) = \sum_{i=0}^{k} c_i \varphi_i(x), \quad x \in [-1,1], \tag{10}$$

the approximate solution of (10), where $c_i's$ are the constant parameters, and $\varphi_i(x)$, $i = 0(1)\mathbb{N}$, are the Mamadu-Njoseh polynomials. Now, using the necessary constraints from the given generalized boundary condition (6) and (10), we have

$$u_k(x,t) = \sum_{i=0}^{n} a_i \varphi_i(x). \tag{11}$$

Using equal-spaced collocation algorithm on (11) to obtain n systems of linear equations which on solving via the Gaussian elimination method yields the a_i's. The estimated a_i's are substituted into (10) to obtain the required initial approximation to start the iteration scheme (9).

The final result is thus defined by the series

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{12}$$

4. Convergence, Existence and Uniqueness of Solutions

The following theorems are considered:

Theorem 4.1. *For a Banach space y, suppose the non-linear mapping $B : y \to y$ satisfy*

$$||B[u] - B[\overline{u}]|| \leq \alpha || u - \overline{u}||, u, \overline{u} \in y$$

for some positive constant $\alpha < 1$. Then ℓ has a unique fixed point, furthermore, the sequence

$$u_{n+1} = \beta[y_n]$$

with arbitrary choice of $u_0 \in y$, converges to the fixed point B_0 such that

$$||u_i - \overline{u}_i|| \leq ||u_1 - u_0|| \sum_{e-i=1}^{i-2} 2^e$$

According to the theorem, the following is valid for non-linear mapping

$$B[u] = u(x,t) + \int_0^t [l + u(x,t) + Nu(x,t) - f(x,l)\, dl]$$

Proof. For each $u, \overline{u} \in y$, the $\lim_{n\to\infty} \|B[u] - B[\overline{u}]\|$ exist, where B is a non-linear mapping satisfying $B : y \to y$. Now for each $p \in y$, where $p = (u, \overline{u})$, we have

$$||B[u] - B[\overline{u}]||^2 =< B[u] - p, j(B[u] - p) > \tag{13}$$

$$= \alpha_n < u_n - p, j(u_n - p) > + (1 - \alpha_n) < \tau_n u_n - p, j(u_n - p) > \tag{14}$$

$$\leq \alpha_n \|u_{n-1} - p\| \|u_n - p\| + (1 - \alpha_n) ||B[u] - B[\overline{u}]||^2, \; n > 0. \tag{15}$$

Simplifying, we have that,

$$\|u_n - p\| \leq \|u_{n-1} - p\| \Rightarrow u_{n+1} = \beta[y_n].$$

Thus, the limit $\lim_{n\to\infty} \|B[u] - B[\overline{u}]\|$ exists, and so the sequence $\{u_n\}$ is bounded. We next show that for some fixed constant the sequence $\{u_n\}$ converges to a definite fixed point

$$||u_i - \overline{u}_i|| \leq ||u_1 - u_0|| \sum_{e-i=1}^{i-2} 2^e.$$

In view of (14) and (15), we have

$$\|u_n - \tau_n u_n\| = \alpha_n \|u_n - \tau_n u_n\| \to 0 \tag{16}$$

as n tends to infinity. We now show that $\{u_n\}$ converges to some point in y. In fact, it follows from (16) that there exists a subsequence $\{u_r\} \subset \{u_n\}$ such that $\|u_r - \tau_r u_r\| \to 0$ as $n_r \to \infty, \tau_r u_r \to p$ and $u_r \to p$ (some point u_0). Consequently,

$$\|p - T_p\| \leq \|p - u_r\| + \|u_n - pu_n\| + \|\tau_n u_n - T_n \tau_n\|$$

$$\leq (1 + L)\|p - u_r\| + \|u_n - pu_n\| + \|\tau_n u_n - T_n \tau_n\|$$

$$= \|u_n - pu_n\| \leq \|u_{n+1} - u_n\| \sum_{j-i=1}^{i-2} 2^j \to 0.$$

This implies that $T_p = p$. Since $u_n \to p$ and the $\lim_{n\to\infty} \|B[u] - B[\overline{u}]\|$ exists, we have that $u_n \to p$. □

Theorem 4.2. *Let $L(X)$ denote equation (9), then, the fractional variational iterative scheme converges if the following conditions are satisfied:*

(1) $(L(X) - L(Y),\ X - Y) \geq \alpha||X - Y||^2,\ \alpha > 0,\ u, v \in H$

(2) For $\beta > 0$, there exists $\alpha(\beta)$ such that $||X|| \leq \beta, ||Y|| \leq \beta$,

Then,

$$(L(X) - L(Y),\ u - v) \geq \alpha(\beta)||X - Y||||\Omega||,\ \Omega \in H.$$

Proof. Let $\alpha > 0$ and $X, Y \in H$ such that

$$(L(X) - L(Y),\ X - Y) = \int_0^t \lambda(s)([L(X) - L(Y)]ds, X - Y)$$

Applying the Schwartz inequality, we get

$$\left(\int_0^t \lambda(s)([L(X) - L(Y)]ds, X - Y)\right) \leq \alpha_1||L(X) - L(Y)||||X - Y||$$

By the conventional use of the mean value theorem, we obtain

$$\left(\int_0^t \lambda(s)([L(X) - L(Y)]ds, X - Y)\right) \geq K_2||X - Y||^2$$

where $K_2 = 0.5\alpha_2\beta^2$ Hence,

$$(\mathrm{L}\,(\mathrm{X}) - \mathrm{L}\,(\mathrm{Y})\,,\ \mathrm{X}{-}\mathrm{Y}) = a\|\mathrm{X}{-}\mathrm{Y}\|^2,$$

holds with $\alpha = 0.5\alpha_2\beta^2$
Similarly, for $\beta > 0,\ \exists\ \alpha(\beta) > 0$ such that $||X|| \leq \beta,\ ||Y|| \leq \beta$ and $X, Y \in H$. Then,

$$(L(u) - L(v), q) = \left(\int_0^t \lambda(s)([L(X) - L(Y)]ds, X - Y)\right) \leq \beta^2||X - Y||||q||$$

This complete the proof. □

4.1. **Numerical Illustrations.** We consider numerical results for just one example to illustrate our numerical scheme. To this end, we vary the order of the fractional derivative and the viscosity coefficient.

Example 4.3. Consider the generalized fractional Burgers equation of the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + u\frac{\partial u(x,t)}{\partial t} = v\frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \quad x \in [0,1]\,, \ \alpha \in (0,1)\,, \tag{17}$$

with the boundary conditions: $u(x,0) = \sin(6\pi x) + 2x(1-x) - xsin(6\pi)$, $u(0,t) = u(1,t) = 0$.

Using the above methodology, we have our initial approximation as:

$$u_0 := -6.44815784 - 2.90303829x + 42.68894700\left(\frac{x}{2}+\frac{1}{2}\right)^2 - 33.79263132\left(\frac{x}{2}+\frac{1}{2}\right)^3$$

Using (9) for , $k \geq 0$ with help of MAPLE 18, we obtain the following approximations:

$$u_1 := -6.44815784 - 2.90303829x + 42.68894700\left(\frac{x}{2}+\frac{1}{2}\right)^2 - 33.79263132\left(\frac{x}{2}+\frac{1}{2}\right)^3$$

$$+\frac{1}{\Gamma(1-\alpha)}\left(-8.921421340t^6 – 8.448157809t^5 + 22.36954960t^4 + 11.53839688t^3\right)$$

$$+\frac{1}{\Gamma(1-\alpha)}\left(0.5000000000(-33.28365095 + 25.34447349v)t^2 - \frac{1399.181450t^{5-1.\alpha}t}{\Gamma(6.-1.\alpha)}\right)$$

$$+\frac{1}{\Gamma(1-\alpha)}\left(-\frac{220.8262795t^{4-1.\alpha}t}{\Gamma(5.-1.\alpha)} + \frac{116.9434692t^{3-1.\alpha}t}{\Gamma(4.-1.\alpha)} + \frac{15.08009978t^{2-1.\alpha}t}{\Gamma(3.-1.\alpha)}\right)$$

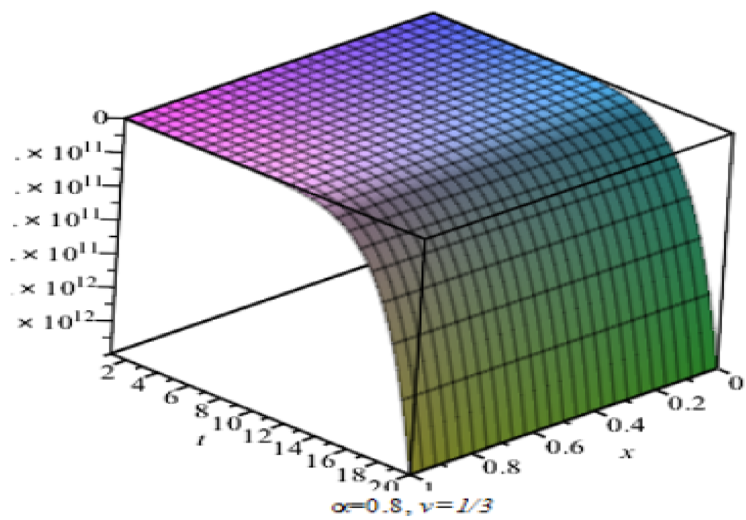
$$+\frac{1}{\Gamma(1-\alpha)}\left(-\frac{5.409788412t^{1-1.\alpha}t}{\Gamma(2.-1.\alpha)} - \frac{12.14544540t^{1.200000000-1.\alpha}t}{\Gamma(2.200000000-1.\alpha)} - \frac{1.04555944t^{0.2000000000-1.\alpha}t}{\Gamma(1.200000000-1.\alpha)}\right)$$

$$+\frac{1}{\Gamma(1-\alpha)}(2.88459923210t^{-8} + 3.999999990vt)$$

and so on. These solutions are as shown below.

$\alpha = 0.8,\ v = \frac{1}{3}$: The approximate solution is given as:

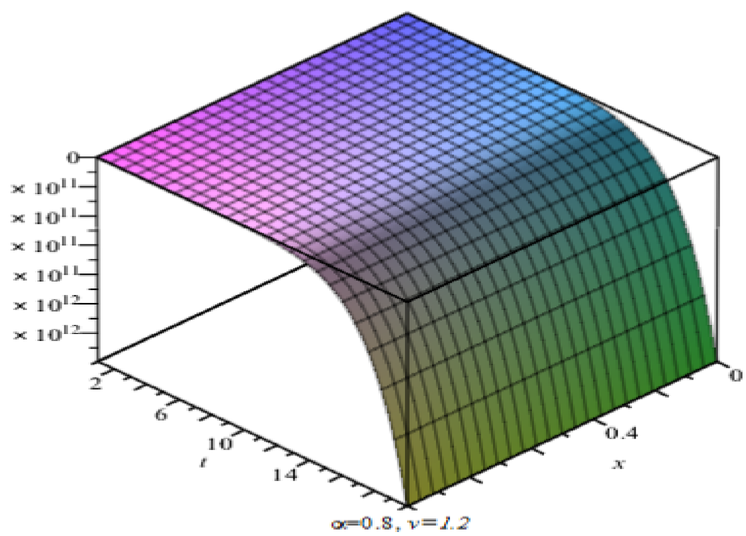
$$\begin{aligned} u = {} & 85.37789400\left(\frac{1}{2}x+\frac{1}{2}\right)^2 - 67.58526264\left(\frac{1}{2}x+\frac{1}{2}\right)^2 - 1.788055757t^9 \\ & - 2.539804472t^8 + 6.123781581t^7 + 0.8712995530t - 0.036878113x \\ & - 0.1026746993t^{0.4000000000} - 4.060284247t^{1.4000000000} - 4.224078912x^3 \\ & - 12.89631568 - 0.5417584552t^{0.600000000} + \frac{0.005330070176}{t^{0.4000000000}} \\ & - 36.80169429t^{4.400000000} + 18.72426483t^{6.2} - 8.607862989t^{8.2} - 9.781479724t^{7.2} \\ & - 16.15470623t^{3.400000000} + 17.99654299t^{2.400000000} - 12.95144132t^{5.2} \\ & - 31.69219408t^{4.2} + 27.22102180t^{3.2} + 10.55677557t^{2.2} - 3.850228653t^{1.2} \\ & - 0.2415367992t^{0.4000000000} - 8.479662740t^2 + 11.06574993t^3 + 0.917958119t^6 \\ & - 13.03784605t^5 + 10.54831342t^4 - 1.999999989x^2 \end{aligned}$$



Figures 1: Approximate solution for $\alpha = 0.8,\ v = \frac{1}{3}$

$\alpha = 0.8,\ v = 1.2$: The approximate solution is given as:

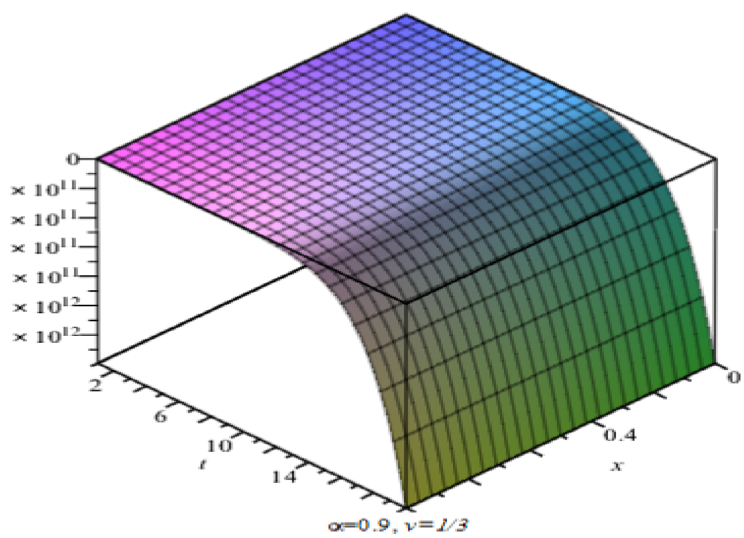
$$
\begin{aligned}
u = {} & 0.8712995531t - 0.036878113x + \frac{0.005330070176}{t^{0.4000000000}} - 31.69219410t^{4.2} \\
& - 1.999999989x^2 - 0.2415367992t^{0.400000000} - 12.95144132t^{5.2} + 27.22102180t^{3.2} \\
& + 10.55677557t^{2.2} - 3.850228653t^{1.2} - 36.80169429t^{4.400000000} \\
& - 0.5417584552t^{0.600000000} + 17.99654299t^{2.400000000} - 16.15470623t^{3.400000000} \\
& - 13.03784605t^5 + 10.54831341t^4 + 11.06574993t^3 + 0.917958119t^6 \\
& - 0.1026746993t^{0.4000000000} - 4.224078912x^3 - 8.479662740t^2 + 6.123781581t^7 \\
& - 4.060284247t^{1.400000000} - 1.788055757t^9 - 2.539804472t^8 + 18.72426483t^{6.2} \\
& - 8.607862989t^{8.2} - 9.781479724t^{7.2} - 67.58526264\left(\frac{x}{2} + \frac{1}{2}\right)^3 \\
& + 85.37789400\left(\frac{x}{2} + \frac{1}{2}\right)^2 - 12.89631568
\end{aligned}
$$



Figures 2: Approximate solution for $\alpha = 0.8,\ v = 1.2$

$\alpha = 0.9$, and $v = \frac{1}{3}$: The approximate solution is given as:

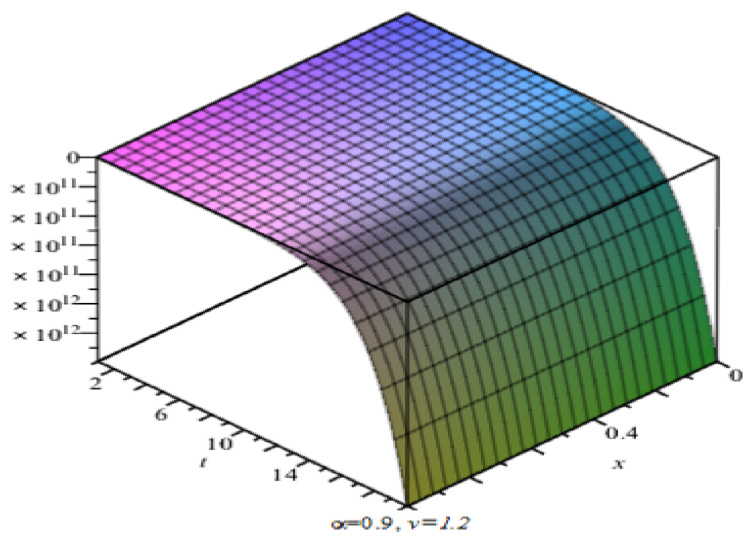
$$u = 0.8712995531t - 0.036878113x + \frac{0.005330070176}{t^{0.4000000000}} - 12.89631568 - 12.95144132t^{5.2} - 31.69219410t^{4.2} - 1.999999989x^2 - 3.850228653t^{1.2} - 0.2415367992t^{0.400000000} + 27.22102180t^{3.2} + 10.55677557t^{2.2} - 36.80169429t^{4.400000000} - 0.5417584552t^{0.600000000} + 17.99654299t^{2.400000000} - 16.15470623t^{3.400000000} + 0.917958119t^6 - 13.03784605t^5 + 10.54831341t^4 - 8.479662740t^2 + 11.06574993t^3 - 0.1026746993t^{0.4000000000} - 4.224078912x^3 + 6.123781581t^7 - 4.060284247t^{1.400000000} - 9.781479724t^{7.2} - 1.788055757t^9 - 2.539804472t^8 + 18.72426483t^{6.2} - 8.607862989t^{8.2} + 85.37789400\left(\frac{x}{2} + \frac{1}{2}\right)^2 - 67.58526264\left(\frac{x}{2} + \frac{1}{2}\right)^3$$



Figures 3: Approximate solution for $\alpha = 0.9,\ v = \frac{1}{3}$

$\alpha = 0.9,\ v = 1.2$: The approximate solution is given as:

$$u = 0.8712995531t - 0.036878113x + \frac{0.005330070176}{t^{0.4000000000}} - 31.69219410t^{4.2}$$
$$- 1.999999989x^2 - 0.2415367992t^{0.400000000} - 12.95144132t^{5.2} + 27.22102180t^{3.2}$$
$$+ 10.55677557t^{2.2} - 3.850228653t^{1.2} - 36.80169429t^{4.400000000}$$
$$- 0.5417584552t^{0.600000000} + 17.99654299t^{2.400000000} - 16.15470623t^{3.400000000}$$
$$- 13.03784605t^5 + 10.54831341t^4 + 11.06574993t^3 + 0.917958119t^6$$
$$- 0.1026746993t^{0.4000000000} - 4.224078912x^3 - 8.479662740t^2 + 6.123781581t^7$$
$$- 4.060284247t^{1.400000000} - 1.788055757t^9 - 2.539804472t^8 + 18.72426483t^{6.2}$$
$$- 8.607862989t^{8.2} - 9.781479724t^{7.2} - 67.58526264\left(\frac{x}{2} + \frac{1}{2}\right)^3$$
$$+ 85.37789400\left(\frac{x}{2} + \frac{1}{2}\right)^2 - 12.89631568$$



Figures 4: Approximate solution for $\alpha = 0.9,\ v = 1.2$

Discussion of Result: From the Figures above and other approximations, we make the following observations.

(1) The response is zero and diffuses accordingly.
(2) Comparing Figure 1 and 2, we observe that the response tends smoothly in both time and space. This shows convergence which agrees with the exact solution.
(3) Comparing the Figures 1, 2 and 3, as the fractional order increases the rate of diffusion slow.
(4) In like manner, comparing the Figures 1 to 4, we observe that as v increases the rate of diffusion increases.

Conclusions: In this paper, we have considered the numerical solution of a generalized fractional Burgers diffusion equation (GFBE). A fractional variational iteration scheme coupled with Mamadu-Njoseh polynomials was developed to effectively solve the GFBE. MAPLE 18 software was used to carry out numerical simulations and approximations. Resulting numerical evidences show that the procedures converges rapidly to the theoretical solution as the order of the fractional derivative v varies.

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