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On Hyers-Ulam Stability of a Perturbed Nonlinear Second Order Differential Equations Using Gronwall-Bellman-Bihari Inequality

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Abstract

In this paper, a nonlinear generalisation of Gronwall-Bellman integral inequality is developed and used to investigate Hyers-Ulam stability of a Perturbed non-linear second order ordinary differential equations.

1. Introduction

In 1940, Ulam[21] proposed the following problem concerning the stability of functional equation:"Give conditions in order for a linear mapping near an approximately linear mapping to exist". This problem was also put in this sense "For what metric group G is it true that an ϵ automorphism of G is necessarily near to a strict automorphism"? In 1941, Hyers[3] gave an answer to the problem as follows; "Let E_1 and E_2 be two real Banach spaces and $f : E_1 \to E_2$ be a mapping, if there exists an $\epsilon \geq 0$ such that

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon, \text{ for all } x, y \in E_1. \tag{1.1}$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$||f(x) - T(x)|| \leq \epsilon, \text{ for every } x \in E_1\text{''} \tag{1.2}$$

The result of Hyers called Hyers-Ulam stability has been generalised by several authors in recent years to first order differential equations [1, 4, 5, 6, 10, 11, 19, 20] and second and nth order ordinary differential equations[8, 7, 17, 13, 14, 15, 18, 16].

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Rus[18], Qarawani[13, 14, 15], Qusausy[16] and Ravi[17] investigated the Hyers-Ulam stability of the nonlinear second order differential equations

$$u''(t) + f(t, u(t)) = 0 \tag{1.3}$$

using Gronwall lemma to establish their results.
In this paper, we consider the investigation of the Hyers-Ulam stability of the perturbed nonlinear second order differential equations

$$u''(t) + f(t, u(t), u'(t)) = g(t, u(t)) \tag{1.4}$$

and

$$u''(t) + f(t, u(t), u'(t)) = H(u(t), u'(t)), \tag{1.5}$$

both satisfying the initial conditions

$$u(t_0) = u'(t_0) = 0, \tag{1.6}$$

where $u(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$, $\mathbb{I} = [0, \infty)$, $g(t, 0) = 0$, $g \in C(\mathbb{I} \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{I} \times \mathbb{R}^2, \mathbb{R})$, $H \in C(\mathbb{R}^2, \mathbb{R})$. Using a nonlinear generalisation of the Grownwall-Bellman integral inequality.

2. Preliminaries

In this section, we state the following useful definitions and theorems which will be used in our main results.

Definition 2.1. Equation (1.4) is Hyers-Ulam stable if for every $\epsilon > 0$, $K > 0$ and $t \in \mathbb{I}$ sufficiently large, there exists a solution $u(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ satisfying

$$|u'' + f(t, u(t), u'(t)) - g(t, u(t))| \leq \epsilon \tag{2.1}$$

such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

where $u_0(t) \in C^2(\mathbb{I}, \mathbb{R})$ is the solution of nonlinear differential equation (1.4) with initial condition (1.6) and K is the Hyers-Ulam constant.

Theorem 2.2. [9, 12] (Generalised First Mean Value Theorem). If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbb{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$

Definition 2.3. A function $\omega : [0, \infty) \to [0, \infty)$ is said to belong to a class S if

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$
ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$,
iii there exist a function ϕ, continuous on $[0, \infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Lemma 2.4. [2] Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M\int_{t_0}^{t} f(s)\omega(u(s))ds, \quad t_0 \leq t < b. \tag{2.2}$$

Implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \quad t_0 \leq t \leq b' \leq b, \tag{2.3}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{2.4}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

3. Development of Nonlinear Integral Inequality

In this section, we are going to develop a nonlinear extension of the Grownwall-Bellman-Bihari integral inequality which will be used to investigate the Hyers-Ulam stability of equations (1.3) and (1.4).

Theorem 3.1. Let $u(t), h(t), g(t) \in C(\mathbb{I}, \mathbb{R}_+)$ and $h'(t) \leq 0$. If

$$u(t) \leq c + A\int_{t_0}^{t} f(s)u(s)ds + B\int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} g(\tau)w(u(\tau))d\tau\right)ds, \; t \in \mathbb{I} \tag{3.1}$$

holds, then $u(t)$ is bounded by

$$\begin{aligned} u(t) \leq \Omega^{-1}\Bigg[\Omega(c) + AB\int_{t_0}^{t} f(s)h(s)g(s)\Bigg[c\exp\int_{t_0}^{s} f(\tau)d\tau \\ + \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} f(\sigma)d\sigma\right)d\tau\Bigg]ds\Bigg], \end{aligned} \tag{3.2}$$

where $A, B, c > 0$, Ω is given in equation (2.4) and $t_1 \in [a, \infty)$ is chosen such that

$$\Omega(c) + \int_{t_0}^{t} f(s)h(s)g(s)\left[c\exp\int_{t_0}^{s} f(\tau)d\tau + \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} f(\sigma)d\sigma\right)d\tau\right]ds \in Dom(\Omega^{-1}),$$

for all t lying in the interval $[a, t_1] \in \mathbb{I}$, Ω^{-1} is the inverse of Ω.

Proof. Define a function $v(t)$ as follows

$$v(t) = c + A\int_{t_0}^{t} f(s)u(s)ds + B\int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} g(\tau)\omega(u(\tau))d\tau\right)ds, \tag{3.3}$$

and

$$u(t) \leq v(t), \quad v(a) = c. \tag{3.4}$$

Differentiating equation(3.3) we obtain

$$v'(t) = f(t)\left[Av(t) + Bh(t)\int_{t_0}^{t} g(\tau)\omega(v(\tau))\right]d\tau. \tag{3.5}$$

Let $A, B > 1$, equation (3.5) becomes

$$v'(t) \leq ABf(t)M(t), \tag{3.6}$$

where

$$M(t) = v(t) + h(t)\int_{t_0}^{t} g(t)\omega(v(\tau))d\tau. \tag{3.7}$$

Differentiating equation (3.7) we get

$$M'(t) = v'(t) + h'(t)\int_{t_0}^{t} g(\tau)\omega(u(\tau))d\tau + h(t)g(t)\omega(u(t)). \tag{3.8}$$

Recall that $h'(t) \leq 0$, then

$$\begin{aligned} M'(t) &\leq v'(t) + h(t)g(t)\omega(u(t)) \\ &\leq f(t)M(t) + h(t)g(t)\omega(v(t)) \\ &\leq f(t)M(t) + P(t) \\ P(t) &= h(t)g(t)\omega(v(t)) \\ M'(t) &\leq f(t)M(t) + P(T), \quad P(T) > 1 \end{aligned}$$

Let $T \in [t_0, \infty)$ be any arbitrary number, solving the ordinary differential equation above we have

$$M(t) \leq c\exp\left(\int_{t_0}^{t} f(s)ds\right) + P(T)\left(\int_{t_0}^{t} \exp\left(\int_{s}^{t} f(\tau)d\tau\right)ds\right) \tag{3.9}$$

By equation (3.6), we have

$$\begin{aligned} v'(t) &\leq ABf(t)[c\exp\int_{t_0}^{t} f(s)ds + P(T)\left(\int_{t_0}^{t} \exp\left(\int_{s}^{t} f(\tau)d\tau\right)ds\right) \\ &\leq P(T)ABf(t)\left[c\exp\int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} \exp\left[\int_{s}^{t} f(\tau)d\tau\right]ds\right] \\ &\leq ABh(t)g(t)\omega(v(t))f(t)\left[c\exp\int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} \exp\left(\int_{s}^{t} f(\tau)d\tau\right)ds\right], \quad T = t. \end{aligned}$$

Using inequality (2.4) we obtain

$$\frac{d\Omega(v(t))}{dt} \leq ABf(t)h(t)g(t)\left[c\exp\int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} \exp\left(\int_{s}^{t} f(\tau)d\tau\right)ds\right] \tag{3.10}$$

$$\Omega(v(t)) \leq \Omega(c) + AB\int_{t_0}^{t} f(s))h(s)g(s)\left[c\exp\int_{t_0}^{t} f(\tau)d\tau + \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} f(\sigma)d\sigma\right)d\tau\right]ds$$

$$v(t) \leq \Omega^{-1}\left[\Omega(c) + AB\int_{t_0}^{t} f(s)h(s)g(s)\left[c\exp\int_{t_0}^{s} f(\tau)d\tau + \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} f(\sigma)d\sigma\right)d\tau\right]ds\right]$$

The result follows from (3.4) □

4. Application to Hyers-Ulam Stability

In our first result, we assume that the function $g(t,u(t))$ satisfies $|g(t,u(t))| \leq \alpha(t)|u(t)|$ where $\alpha(t)$ a continuous function on $\mathbb{I}$.

Theorem 4.1. Equation (1.4) is stable in the sense of Hyers-Ulam, if there exists a solution $u(t) \in C^2(\mathbb{I},\mathbb{R}_+)$ of the inequality (2.1) and the functions $f(t,u(t),u'(t))$ and $g(t,u(t))$ in inequality (2.1) satisfying the following conditions:

C_1 $f(t,u(t),u'(t)) \geq 1$ for all $t \geq t_0$,

C_2 $\dfrac{f'(t,u(t),u'(t))u(t)}{f(t,u(t),u'(t))} = q(t,u(t),u'(t),u''(t))$, q a positive, continuous function,

C_3 $|f(t,u(t),u'(t))| \leq h(t)\omega(|u(t)|)|u'(t)|$, where $\omega(u)$ belongs to the class S, and $h \in C(\mathbb{I},\mathbb{R}_+)$,

C_4 (a)$|g(t,u(t))| \leq \alpha(t)|u(t)|$ (b) $f_u(t,u(t),u'(t)) + f_{u'}(t,u(t),u'(t)) \leq 0$ (n) $|u'(t)| \leq \lambda$ where $\lambda > 0$,

then, there exists a solution $u_0(t) \in C^2(\mathbb{I},\mathbb{R}_+)$ of equation (1.4) such that $|u(t) - u_0(t)| \leq K\epsilon$ for any $t \geq 0$, provided $\displaystyle\int_{t_0}^{\infty} h(s) < R < \infty$ and $\displaystyle\int_{t_0}^{t} |u'(s)|ds \leq L$, where

$$K = (L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M)\Omega^{-1}\left(\Omega(1) + \lambda|q((\xi),u(\xi),u'(\xi),u''(\xi))|R\right).$$

Proof. Multiplying inequality (2.1) by $|u'(t)|$ we get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t,u(t),u'(t))u'(t) - g(t,u(t))u'(t) \leq \epsilon|u'(t)| \tag{4.1}$$

for all $t \geq t_0$.
Integrating each term from t_0 to t, we have

$$\begin{aligned}-\epsilon\int_{t_0}^{t}|u(s)|ds \leq \frac{1}{2}u'(t)^2 &+ \int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds \\ &- \int_{t_0}^{t} g(s,u(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}|u'(s)|ds,\end{aligned}$$

for any $t \geq t_0$.
Integrating by part and using C_4(b), we have

$$\begin{aligned}-\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t,u(t),u'(t))u(t) \\ -\int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds - \int_{t_0}^{t} g(s,u(s))u'(s)ds \leq \epsilon L,\end{aligned}$$

for all $t \geq t_0$.
Then,

$$\begin{aligned}f(t,u(t),u'(t))u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 &+ \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds \\ &+ \int_{t_0}^{t} g(s,u(s))u'(s)ds.\end{aligned} \tag{4.2}$$

Using C_1, we get

$$\begin{aligned}
u(t) &\leq \epsilon L + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds + \int_{t_0}^{t} g(s,u(s))u'(s)ds \text{ for } t \geq t_0 \\
&\leq \epsilon L + \lambda^2 + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} \frac{f'(s,u(s),u'(s))u(s)}{f(s,u(s),u'(s))} f(s,u(s),u'(s))ds + \int_{t_0}^{t} g(s,u(s))u'(s)ds \\
&\leq \epsilon L + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} q(s,u(s),u'(s))f(s,u(s),u'(s))ds + \int_{t_0}^{t} g(s,u(s))u'(s)ds \; by \; C_2
\end{aligned}$$

Using the condition , C_3, $C_4(n)$ and theorem 2.2, there exits points $\xi, \rho \in [t_0, t]$ such that

$$\begin{aligned}
|u(t)| &\leq \epsilon L + \frac{1}{2}\lambda^2 + |q(\xi,u(\xi),u'(\xi,u''(\xi)))| \int_{t_0}^{t} |f(s,u(s),u'(s))|ds + |g(\rho,u(\rho))| \int_{t_0}^{t} |u'(s)|ds, \\
&\leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)|u(\rho)| + |q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\omega(|u(s)|)ds, \; by \; C_4, \\
&\leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M + |q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\omega(|u(s)|)ds, \\
&\leq \epsilon(L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M) + |q(\xi,u(\xi),u'(\xi,u''(\xi))|\lambda \int_{t_0}^{t} h(s)\omega(|u(s)|)ds,
\end{aligned}$$

provided $|u(\rho)| \leq M$.
Hence,

$$\frac{|u(t)|}{B} \leq 1 + |q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\omega\left(\frac{|u(s)|}{B}\right) ds \tag{4.3}$$

Where $B = \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M$, setting z = R.H.S of inequality (4.1)
Therefore,

$$z(t) \leq 1 + |q((\xi),u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\omega(z(s)ds, \tag{4.4}$$

where

$$\frac{|u(t)|}{B} \leq z(t). \tag{4.5}$$

Using the Lemma 2.4, we arrive at

$$z(t) \leq \Omega^{-1}\left(\Omega(1) + |q((\xi),u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} h(s)ds\right), \quad t_0 \leq t \leq b' \leq b. \tag{4.6}$$

By inequality (4.5), equation (4.6) becomes

$$\frac{|u(t)|}{B} \leq \Omega^{-1}\left(\Omega(1) + |q((\xi),u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} h(s)ds\right), \quad t_0 \leq t \leq b' \leq b. \tag{4.7}$$

It is cleared from inequality (4.7) that

$$|u(t)| \leq B\Omega^{-1}\left(\Omega(1) + \lambda|q((\xi),u(\xi),u'(\xi),u''(\xi))|R\right), \tag{4.8}$$

provided $\lim_{t_0\to\infty} \int_{t_0}^{t} h(s)ds \le R < \infty$.
Substituting for the value of B in inequality (4.8), we obtain

$$|u(t)| \le \epsilon(L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M)\Omega^{-1}\left(\Omega(1) + \lambda|q((\xi), u(\xi), u'(\xi), u''(\xi))|R\right). \tag{4.9}$$

Therefore,

$$|u(t) - u_0(t)| \le |u(t)| \le \epsilon K,$$

where,

$$K = (L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M)\Omega^{-1}\left(\Omega(1) + \lambda|q((\xi), u(\xi), u'(\xi), u''(\xi))|R\right). \tag{4.10}$$

□

Example 4.2. To investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form

$$u''(t) + (2t)^{-4}u(t)^2\exp(-u'(t))u'(t) = t^{-4}u^2(t) \text{ for all } t > 0$$

We set

$$f(t, u(t), u'(t)) = (2t)^{-4}u^2(t)\exp(-u'(t))u'(t)$$

$$g(t, u(t)) = t^{-4}u^2(t)$$

$\alpha(t)|u(t)| = t^{-2}u^2(t)$, since $|g(t, u(t))| \le \alpha(t)|u(t)|$ and $\omega(u) = u^2(t)$, $h(t) = (2t)^{-2}, \phi(\alpha) = \alpha^2$, where $u(t_0) = u'(t_0) = 0$. By applying the Theorem 4.1, the nonlinear second order differential equation is Hyers-Ulam stable.

Corollary 4.3. Let $|g(t, u(t))| \le A|u(t)|$, $g(t, 0) = 0$, constant $A > 0$ and $\int_{t_0}^{t} |u'(s)|ds \le L$ where $L > 0$. Let the function $f(t, u(t), u'(t))$ satisfies all the conditions on f of Theorem 4.1 Then, equation (1.4) is Hyers-Ulam stable if inequality (2.1) is satisfied and there exists a solution $u_0(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ which satisfies nonlinear differential equation (1.4) above such that

$$|u(t) - u_0(t)| \le K\epsilon \text{ for any } t > 0,$$

Provided $\int_{t_0}^{\infty} h(s)ds \le R < \infty$ and Hyers-Ulam constant

$$K = (L + \frac{1}{2}\lambda^2 + LAM)\Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))|R\right). \tag{4.11}$$

Proof. The proof of this corollary follows the same argument of Theorem 4.1 □

Corollary 4.4. Let all conditions of Theorem 4.1 be satisfied, and define

C_3' $|f(t, u(t), u'(t))| \le h(t)\omega(|u(t)|) + |u'(t)|$, take $u'(t) \le u(t)$

C_5 $\Omega(r) = \int_{r_0}^{r} \frac{ds}{\omega(s) + s}$, $r_0 \ge 0, r \ge r_0$, then, equation (1.4) is Hyers-Ulam stable, if there exists a solution $u(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ which satisfied the inequality (2.1) and there exists a solution $u_0(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ which satisfied differential equation (1.4) such that $|u(t) - u_0(t) \leq K\epsilon$ for any $t > 0$, provided $\int_{t_0}^{\infty} h(s)ds \leq R < \infty$ and Hyers-Ulam constant

$$K = (L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M)\Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))|R\right). \tag{4.12}$$

Proof. Multiplying inequality (2.1) by $|u'(t)|$ and follow the proof of Theorem 4.1 to using the condition C_3', $C_4(n)$ and applying Theorem 2.2, there exits points $\xi, \rho \in [t_0, t]$ such that

$$\begin{aligned}
|u(t)| &\leq \epsilon L + \frac{1}{2}\lambda^2 + |q(\xi, u(\xi), u'(\xi, u''(\xi)))| \int_{t_0}^{t} |f(s, u(s), u'(s))|ds + |g(\rho, u(\rho))| \int_{t_0}^{t} |u'(s)|ds, \\
&\leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)|u(\rho)| + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|) + |u'(s)|ds, \text{ by } C_5, \\
&\leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|) + |u'(s)|ds, \\
&\leq \epsilon(L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M) + |q(\xi, u(\xi), u'(\xi, u''(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|) + |u'(s)|ds,
\end{aligned}$$

provided $|u(\rho)| \leq M$.
Hence,

$$\frac{|u(t)|}{B} \leq 1 + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega\left(\frac{|u(s)|}{B}\right) + \frac{|u'(s)|}{B}ds \tag{4.13}$$

where $B = \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M$, setting $z =$ R.H.S of inequality (4.1)
Therefore,

$$z(t) \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega(z(s)) + z(s)ds, \tag{4.14}$$

where

$$\frac{|u'(t)|}{B} \leq \frac{|u(t)|}{B} \leq z(t), \text{ by } C_3', B > 0. \tag{4.15}$$

Let

$$v(t) = 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega(z(s)) + z(s)ds, \tag{4.16}$$

it follows that

$$z(t) \leq v(t). \tag{4.17}$$

Using inequality (4.17) in equation (4.16) we get

$$v(t) \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)\omega(v(s)) + v(s)ds. \tag{4.18}$$

Differentiating inequality (4.18), we obtain

$$v'(t) \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| h(t)\omega(v(t)) + v(t). \tag{4.19}$$

It is cleared that

$$\frac{v'(t)}{\omega(v(t)) + v(t)} \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| h(t). \tag{4.20}$$

Using condition C_5, we have

$$\frac{d}{dt}\Omega(v(t)) = \frac{v'(t)}{\omega(v(t)) + v(t)} \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))| h(t). \tag{4.21}$$

From equation (4.21) we get

$$v(t) \leq \Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)ds\right), \quad t_0 \leq t. \tag{4.22}$$

By inequalities (4.17) and (4.15) we arrive at

$$\frac{|u(t)|}{B} \leq \Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)ds\right), \quad t_0 \leq t \leq b' \leq b. \tag{4.23}$$

It is cleared from inequality (4.23) that

$$|u(t)| \leq B\Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| R\right), \tag{4.24}$$

provided $\lim_{t_0 \to \infty} \int_{t_0}^{t} h(s)ds \leq R < \infty$.
Substituting for the value of B in inequality (4.24), we obtain

$$|u(t)| \leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M\Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| R\right). \tag{4.25}$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \epsilon K,$$

where,

$$K = (L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M)\Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| R\right). \tag{4.26}$$

□

In the next theorem, we consider the Hyers-Ulam stability of nonlinear second order differential equation (1.5).

Theorem 4.5. Let $u^2(t) \in C(\mathbb{R})$ by any solution of inequality

$$|u''(t) + f(t, u(t), u'(t)) - H(u(t), u'(t))| \leq \epsilon \tag{4.27}$$

and $\epsilon > 0$ be given. Then, equation (1.5) is Hyers-Ulam stable if the conditions C_1, C_2 and C_4 in the Theorem 4.1 hold together with the following properties:

H_1 $\int_{t_0}^{t} |u'(s)|ds \leq L$

H_3 $|f(t,u(t),u'(t))| \leq f(t)|u(t)| + f(t)h(t)\left(\int_{t_0}^{t} g(s)\omega(|u(s)|)ds\right)$, for $\omega(u)$ belongs to the class S and $h, f, g \in C(\mathbb{I}, \mathbb{R}_+)$.

H_3 $\int_{t_0}^{t} f(s)h(s)g(s)ds \leq \frac{1}{1+(t-t_0)}$

H_4 $\lim_{t_0\to\infty} \int_{t_0}^{t} h(s) \leq R < \infty$,

if there exist $K > 0$ and any solution $u_0(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ such that $|u(t) - u_0(t)| \leq K\epsilon$ for any $t \geq 0$, where K is Hyers-Ulam constant given as

$$K = L(1\frac{1}{2}\lambda^2 + |p(u(\rho), u'(\rho))|)\Omega^{-1}\left(\Omega(1) + |q(\xi, u(\xi), u'(\xi), u''(\xi))|^2 \exp R\right)$$

Proof. Multiplying (4.27) by $|u'(t)|$ we get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t,u(t),u'(t))u'(t) - H(u(t),u'(t))u'(t) \leq \epsilon|u'(t)| \tag{4.28}$$

for all $t \geq t_0$. Integrating

$$\begin{aligned} -\epsilon\int_{t_0}^{t}|u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds \\ - \int_{t_0}^{t} H(u(s),u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}|u'(s)|ds \end{aligned} \tag{4.29}$$

Using conditions C_4, H_1 and integrating by part, we obtain

$$\begin{aligned} -\epsilon L \leq \frac{1}{2}u'(t)^2 + f(t,u(t),u'(t))u(t) \\ - \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds - \int_{t_0}^{t}(s,u(s))u'(s)ds \leq \epsilon L \end{aligned}$$

It clear that

$$\begin{aligned} f(t,u(t),u'(t))u(t) \leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds \\ + \int_{t_0}^{t} H(u(s),u'(s))u'(s)ds \end{aligned}$$

$$\begin{aligned} u(t) &\leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds + \int_{t_0}^{t} H((s),u(s))u'(s)ds, \text{ by, } C_1 \\ &\leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} \frac{f'(s,u(s),u'(s))u(s)}{f(s,u(s),u'(s))} f(s,u(s),u'(s))ds + \int_{t_0}^{t} H(u(s),u'(s))u'(s)ds \\ u(t) &\leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} q(s,u(s),u'(s))f(s,u(s),u'(s))ds + \int_{t_0}^{t} H(u(s),u'(s))u'(s)ds, \text{ by} C_2 \end{aligned}$$

By Theorem 2.2 there exist points $\xi, \rho \in (t_0, t)$ such that

$$\begin{aligned} &\leq \epsilon L - \frac{1}{2}u'(t)^2 + q((\xi), u(\xi), u'(\xi), u''(\xi)) \int_{t_0}^{t} f(s, u(s), u'(s)ds \\ &\qquad + H(u(\rho), u'(\rho)) \int_{t_0}^{t} u'(s)ds \end{aligned} \tag{4.30}$$

Taking the absolute value of both sides and setting $|u'(t)| \leq \lambda$ where $\lambda > 0$, we get

$$\begin{aligned} |u(t)| &\leq \epsilon L + \frac{1}{2}\lambda^2 + |q(\xi, u(\xi), u'(\xi, u''(\xi))| \int_{t_0}^{t} |f(s, u(s), u'(s))|ds \\ &\qquad + |H(u(\rho), u'(\rho))| \int_{t_0}^{t} |u'(s)|ds \end{aligned}$$

Since $L > 0$, then we obtain

$$\begin{aligned} |u(t)| &\leq \epsilon L(1 + \frac{1}{2}\lambda^2 + |H(u(\rho), u'(\rho))|) \\ &\quad + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} f(s, u(s), u'(s))|ds, \text{ by } H_1 \end{aligned}$$

Applying H_2, we have

$$\begin{aligned} &\leq \epsilon L + \frac{1}{2}\lambda^2 + L|p(u(\rho), u'(\rho))|) + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \left(\int_{t_0}^{t} f(s)|u(s)|ds \right. \\ &\qquad \left. + \int_{t_0}^{t} f(s)h(s) \left(\int_{t_0}^{s} g(\tau)\omega(|u(\tau)|)d\tau \right) ds \right), \text{ by } H_2 \end{aligned}$$

$$\begin{aligned} |u(t)| &\leq G + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} f(s)|u(s)|ds \\ &\quad + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} f(s)h(s) \left(\int_{t_0}^{s} g(\tau)\omega(|u(\tau)|)d\tau \right) ds \end{aligned} \tag{4.31}$$

$$\begin{aligned} \frac{|u(t)|}{G} &\leq 1 + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} f(s)\frac{|u(s)|}{G}ds \\ &\quad + |q(\xi, u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} f(s)h(s) \left(\int_{t_0}^{s} g(\tau)\omega(\frac{|u(\tau)|}{G})d\tau \right) ds \end{aligned} \tag{4.32}$$

By Theorem 4.1 we have

$$\begin{aligned} &\frac{|u(t)|}{G} \leq \Omega^{-1} \left(\Omega(1) + |q(\xi, u(\xi), u'(\xi), u''(\xi))|^2 \right. \\ &\left. \int_{t_0}^{t} f(s)h(s)g(s) \left(\exp \int_{t_0}^{t} f(\tau)d\tau + \int_{t_0}^{t} \exp \left(\int_{t_0}^{s} f(\delta)d\delta \right) d\tau \right) ds \right) \end{aligned} \tag{4.33}$$

where

$$G = \epsilon L(1 + \frac{1}{2}\lambda^2 + |p(u(\rho), u'(\rho))|) \tag{4.34}$$

By H_3 and H_4 we arrive at

$$|u(t)-u(t_0)| \leq |u(t)| \leq K\epsilon,$$

where

$$K = L(1\frac{1}{2}\lambda^2 + |p(u(\rho), u'(\rho))|)\Omega^{-1}\left(\Omega(1) + |q(\xi, u(\xi), u'(\xi), u''(\xi))|^2 \exp R\right). \tag{4.35}$$

□

Example 4.6. Investigate Hyers-Ulam stability of the second order nonlinear differential equation of the form

$$u''(t) + (t)^{-6}u(t)^2\exp(-u'(t))u'(t) = \cos^2(u(t))\sin^2(u'(t)) \text{ for all } \; t > 0$$

We set

$$f(t, u(t), u'(t)) = (t)^{-6}u^2(t)\exp(-u'(t))u'(t)$$
$$p(u(t), u'(t)) = \cos^2(u(t))\sin^2(u'(t))$$

and $\omega(u) = u^2(t)$, $h(t) = (t)^{-6}, f(t) = t^{-2}$, $g(t) = t^2$, where $u(t_0) = u'(t_0) = 0$. By conditions H_1, H_2, H_3 and H_4 being satisfied, the differential equation above is Hyers-Ulam stable.

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References

[1] Alsina C. & Ger R. (1988). On some inequalities and stability result related to the exponential function. *J. Inequl. Appl.* **2** 373-380.
[2] Bihari I. (1957). Researches of the boundedness and stability of the solutions of nonlinear differential equations. *Acta. Math Acad, Sc. Hung.* **7**, 278-291.
[3] Hyers D. H. (1941). On the stability of the linear functional equation. *Proceedings of the National Academy of Science of the United States of America.* **27**, 222-224.
[4] Jung S. M. (2005). *Hyers-Ulam stability of linear differential equations of first order III.* Journal of Mathematical Analysis and Applications, **311** (1), 137-146.
[5] Jung S. M. (2006). Hyers-Ulam stability of linear differential equations of first order II. *Applied Mathematical Letters.* **19** (9), 854-858.
[6] Jung S. M. (2004). Hyers-Ulam stability of linear differential equations of first order. *Applied Mathematics Letters.* **17** (10), 1135-1140.
[7] Li Y. & Yan S. (2009). Hyers-Ulam Stability of Nonhomogeneous Linear Differential Equations of Second order. *Int. J. of Math and Math Sciences.* **2009**, Article ID576852, 7 pages.
[8] Li Y. & Yan S. (2010). Hyers-Ulam stability of linear differential equations of second order. *Appl. Math. Lett.* **23**, 306-308.
[9] Miura T., Hirasawa G. & Takahasi S. E. (2004). Note on the Hyers-Ulam-Rassias stability of the first order linear differential equation $y' + p(t)y(t) + q(t) = 0$. *Int. J. Mat. Math. Sci.* **22**, 1151-1158.
[10] Miura T. Miyajima S. & Takahasi S. E. (2003). A characterisation of Hyers-Ulam stability of first order linear differential operator. *Journal of Mathematical Analysis and Applications* **286** (1), 136-146.

[11] MIURA T., MIYAJIMA S. & TAKAHASI S. (2003). Hyers-Ulam stability of linear differential operator with constant coefficients. *Mathematisch Nachrichten.* **258**, 90-96.

[12] MURRAY R. S. (1974). *Schum's Outline of Theory and Problem of Calculus.* SI(Metric) Edition, International Edition.

[13] QARAWANI M. N. (2013). On Hyers-Ulam stability for nonlinear differential equations of Nth order. *Intenational Journal of Analysis.* **2**, 71-78.

[14] QARAWANI M. N. (2012). Hyers-Ulam stability of linear and nonlinear differential equations of second order. *Int. Journal of Applied Mathematical Research.* **1** (4), 422-432.

[15] QARAWANI M. N. (2012). Hyers-Ulam stability of a generalised second order nonlinear differential equations. *Applied Mathematics.* **3**, 1857-1861.

[16] QUSAUSY H. ALGFIARY & JUNG S. M. (2014). On the Hyers-Ulam stability of differential equations of second order. *Hindawi Publishing Cooperation Abstract and Appliedd Analysis.* **2014** ID483707, 1-13.

[17] RAVI K., MURALI R., PONMAANNASELLVAN A. & VEERAISIVSJI R. (2016). Hyers-Ulam stability of nth order nonlinear differential equations with initial conditions. *International Journal of Mathematics And Its Applications* bf 4, 121-132.

[18] RUS I. A. (2010). Ulam stability of ordinary differential equation. *Studia Universities Babes-Bolyal Mathematical.* **54** (4), 306-309.

[19] TAKAHASI S., TAKAGI H., MIURA T. & MIYAJIMA S. (2004). The Hyers-Ulam stability constants of first order linear differential operators. *Journal of Mathematical Analysis and Applications.* **296** (2), 403-409.

[20] TAKAHASI S., MIURA T. & MIYAJIMA S. (2002). On the Hyers-UIam stability of the Banach Space-valued differential equation $y' = \lambda y$. *Bulletin of the Korean Mathematical Society.* **392**, 309-315.

[21] ULAM S. M. (1960). *Problems in Modern Mathematics Science Editions.* Chapter 6, Wily, New York. NY, USA.