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A Class of Two-Step Obrechkoff-Type of Hybrid Block Method for Solving Initial Value Problems in Ordinary Differential Equations

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Abstract

In this paper, we derived A Class of Two-Step Obrechkoff-Type of Hybrid Block Method for Solving Initial Value Problem (IVP) in Ordinary differential equations (ODEs) by collocation and interpolation techniques. The analysis of the proposed scheme shows that it is zero stable, convergent and consistent. Numerical examples are also presented to show that the method compared favorably with results of the other methods considered in the literature.

1. Introduction

Mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called Differential equation (D.E) [1]. A Differential equation is an equation which includes the derivative of a function as well as the function itself, and the independent variable. Differential equation is divided into two parts, ordinary and partial differential equation. An equation is called ordinary differential equation if it has one independent variable involve while partial differential equation is an equation that has two or more independent variables. An Ordinary Differential Equations of order with initial condition is given in the form the linear multistep methods have the advantage of been self-starting and permitting easy change of step

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length.of the form;

$$y^{'}(x) = f(x, y(x)); y(x_0) = y_0 \tag{1}$$

where f is a continuous function over an interval of integration. However in most cases, these first order problems (IVP) cannot be solved analytically and hence the need for numerical methods. These numerical methods are adopted to obtain an approximate solution to the initial value problem under consideration [6]-[11]. To that extent, several numerical methods such as one Step Method, Linear Multistep Methods, Hybrid Method and and Block Method have been developed base on the nature and type of the differential equation to be solved [3]-[5].

Several authors have all proposed Linear Multistep methods (LMMs) to generate numerical solutions to initial value problems (IVP). Linear multistep method is unlike the one step where only single value y_n is used to compute the next approximation y_{n+1}, Linear Multistep Method need two or more preceding values to be able to calculate y_{n+1}. These proposed methods in which the approximate solution ranges from Power series, Chebychev's, Lagrange's and Laguerre's polynomials.

One distinct family of methods for the numerical approximation of (1) above is the Obrechkoff methods. This family of methods is regarded to be distinct due to the presence of higher derivatives in the method. The general form of the K-step obrechkoff method with L derivatives of y is given by [2] as shown below.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} \tag{2}$$

where $\alpha_k = +1$.

However, this research work is aimed at developing A Class of Two-Step Obreachkoff-Type Hybrid Block Method for Solving Initial Value Problems in Ordinary Differential Equations.

2. Derivation of the Method

Consider the general form in equation (2) for implicit method when $k = 2$, $l = 2$. Equation (2) can be written as;

$$y_{n+2} = \sum_{j=0}^{k-2} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} \tag{3}$$

Then equation (3) can be express as:

$$\begin{aligned} y_{n+2} = \alpha_0 y_0 + h\Big[&\beta_{10} y_n^{(1)} + \beta_{1\frac{1}{2}} y_{n+\frac{1}{2}}^{(1)} + \beta_{11} y_{n+1}^{(1)} + \beta_{1\frac{3}{2}} y_{n+\frac{3}{2}}^{(1)} + \beta_{12} y_{n+2}^{(1)}\Big] \\ + h^2\Big[&\beta_{20} y_n^{(2)} + \beta_{2\frac{1}{2}} y_{n+\frac{1}{2}}^{(2)} + \beta_{21} y_{n+1}^{(2)} + \beta_{2\frac{3}{2}} y_{n+\frac{3}{2}}^{(2)} + \beta_{22} y_{n+2}^{(2)}\Big] \end{aligned} \tag{4}$$

Using Taylor's series expansion to expand individual terms in equation (4) and upon substitution of the expansions back equation (4), the matrix form $Ax = B$ obtained from the expansion;

$y_{n+a} = y(x_n + ah) = y(x_n) + ahy^{'}(x_n) + \frac{(ah)^2}{2!}y^{''}(x_n) + ...$

Here, Taylor series expansion is used to expand each term in equation (4) with collocation and interpolation techniques.

$y_{n+a} = y(x_n + ah) = y(x_n) + ahy^{'}(x_n) + \frac{(ah)^2}{2!}y^{''}(x_n) + ...$

and the coefficient of $y^{(i)}(x_n)$ are equated to be;

(5)

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{1}{2} & \frac{\left(\frac{3}{2}\right)^2}{2!} & \frac{2^2}{2!} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{1}{3!} & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{2^3}{3!} & 0 & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{1}{2!} & \frac{\left(\frac{3}{2}\right)^2}{2!} & \frac{2^2}{2!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{1}{4!} & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{2^4}{4!} & 0 & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{1}{3!} & \frac{\left(\frac{3}{2}\right)^3}{3!} & \frac{2^3}{3!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{1}{5!} & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{2^5}{5!} & 0 & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{1}{4!} & \frac{\left(\frac{3}{2}\right)^4}{4!} & \frac{2^4}{4!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{1}{6!} & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{2^6}{6!} & 0 & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{1}{5!} & \frac{\left(\frac{3}{2}\right)^5}{5!} & \frac{2^5}{5!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^7}{7!} & \frac{1}{7!} & \frac{\left(\frac{1}{2}\right)^7}{7!} & \frac{2^7}{7!} & 0 & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{1}{6!} & \frac{\left(\frac{3}{2}\right)^6}{6!} & \frac{2^6}{6!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^8}{8!} & \frac{1}{8!} & \frac{\left(\frac{1}{2}\right)^8}{8!} & \frac{2^8}{8!} & 0 & \frac{\left(\frac{1}{2}\right)^7}{7!} & \frac{1}{7!} & \frac{\left(\frac{3}{2}\right)^7}{7!} & \frac{2^7}{7!} \\
0 & 0 & \frac{\left(\frac{1}{2}\right)^9}{9!} & \frac{1}{9!} & \frac{\left(\frac{1}{2}\right)^9}{9!} & \frac{2^9}{9!} & 0 & \frac{\left(\frac{1}{2}\right)^8}{8!} & \frac{1}{8!} & \frac{\left(\frac{3}{2}\right)^8}{8!} & \frac{2^8}{8!}
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\ \beta_{10} \\ \beta_{1\frac{1}{2}} \\ \beta_{11} \\ \beta_{1\frac{3}{2}} \\ \beta_{12} \\ \beta_{20} \\ \beta_{2\frac{1}{2}} \\ \beta_{21} \\ \beta_{2\frac{3}{2}} \\ \beta_{22}
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 2 \\ \frac{2^2}{2!} \\ \frac{2^3}{3!} \\ \frac{2^4}{4!} \\ \frac{2^5}{5!} \\ \frac{2^6}{6!} \\ \frac{2^7}{7!} \\ \frac{2^8}{8!} \\ \frac{2^9}{9!} \\ \frac{2^{10}}{10!}
\end{bmatrix}$$

Solving equation (5) using matrix inverse we have the following parameters;

$$\left(\alpha_0, \beta_{1\frac{1}{2}}, \beta_{11}, \beta_{1\frac{3}{2}}, \beta_{20}, \beta_{2\frac{1}{2}}, \beta_{21}, \beta_{2\frac{3}{2}}, \beta_{22}\right)^T$$

$$= \left(1, \frac{1601}{8505}, \frac{4096}{8505}, \frac{208}{315}, \frac{4096}{8505}, \frac{1601}{8505}, \frac{29}{2835}, -\frac{128}{2835}, 0, \frac{128}{2835}, -\frac{29}{2835}\right)^T$$

Substituting the values of $\alpha_0, \beta_{1\frac{1}{2}}, \beta_{11}, \beta_{1\frac{3}{2}}, \beta_{20}, \beta_{2\frac{1}{2}}, \beta_{21}, \beta_{2\frac{3}{2}}$, and β_{22} into equation (4), we have;

$$y_{n+2} = y_n + h\Big[\frac{1601}{8505}y_n^{(1)} + \frac{4096}{8505}y_{n+\frac{1}{2}}^{(1)} + \frac{208}{315}y_{n+1}^{(1)} + \frac{4096}{8505}y_{n+\frac{3}{2}}^{(1)} + \frac{1601}{8505}y_{n+2}^{(1)}\Big]$$

(6)

$$+ h^2\Big[\frac{29}{2835}y_n^{(2)} - \frac{128}{2835}y_{n+\frac{1}{2}}^{(2)} + \frac{128}{2835}y_{n+\frac{3}{2}}^{(2)} - \frac{29}{2835}y_{n+2}^{(2)}\Big]$$

In other to implement the derived method in equation (6), additional methods are needed. These methods are obtained by similar approach to have the following:

(7)

$$y_{n+1} = y_n + h\Big[\frac{24463}{136080}y_n^{(1)} + \frac{3308}{8505}y_{n+\frac{1}{2}}^{(1)} + \frac{104}{315}y_{n+1}^{(1)} + \frac{788}{8505}y_{n+\frac{3}{2}}^{(1)} + \frac{1153}{136080}y_{n+2}^{(1)}\Big]$$

$$+ h^2\Big[\frac{421}{54360}y_n^{(2)} - \frac{38}{567}y_{n+\frac{1}{2}}^{(2)} - \frac{1}{10}y_{n+1}^{(2)} - \frac{62}{2835}y_{n+\frac{3}{2}}^{(2)} - \frac{43}{45360}y_{n+2}^{(2)}\Big]$$

(8)

$$y_{n+\frac{1}{2}} = y_n + h\Big[\frac{1539551}{8709120}y_n^{(1)} + \frac{89371}{544320}y_{n+\frac{1}{2}}^{(1)} + \frac{103}{1260}y_{n+1}^{(1)} + \frac{38341}{544320}y_{n+\frac{3}{2}}^{(1)} + \frac{59681}{8709120}y_{n+2}^{(1)}\Big]$$

$$+ h^2\Big[\frac{26051}{2903040}y_n^{(2)} - \frac{31207}{362880}y_{n+\frac{1}{2}}^{(2)} - \frac{81}{1280}y_{n+1}^{(2)} - \frac{1243}{72576}y_{n+\frac{3}{2}}^{(2)} - \frac{2237}{2903040}y_{n+2}^{(2)}\Big]$$

(9)

$$y_{n+\frac{3}{2}} = y_n + h\Big[\frac{6501}{35840}y_n^{(1)} + \frac{921}{2240}y_{n+\frac{1}{2}}^{(1)} + \frac{81}{140}y_{n+1}^{(1)} + \frac{711}{2240}y_{n+\frac{3}{2}}^{(1)} + \frac{411}{35840}y_{n+2}^{(1)}\Big]$$

$$+ h^2\Big[\frac{339}{35840}y_n^{(2)} - \frac{279}{4480}y_{n+\frac{1}{2}}^{(2)} - \frac{81}{1280}y_{n+1}^{(2)} - \frac{183}{4480}y_{n+\frac{3}{2}}^{(2)} - \frac{9}{7168}y_{n+2}^{(2)}\Big]$$

Hence, Equation (6), (7), (8) and (9) present the desired hybrid block method for the solution of equation (1).

3. Order and Consistency

To check for the order and the convergence of this hybrid block method, the following theorem and definition are adopted.

Theorem 3.1. *According to [7]: A linear multistep method is convergent if and only if, it is consistent and zero-stable.*

Definition 3.2. [4]: A linear multistep method is consistent if it has order $p \geq 1$.

Definition 3.3. [5]: The linear operator associated with Equation (2) is defined as:

$$L\Big[y(x);h\Big] = \sum_{j=0}^{k} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)} \tag{10}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$, expanding $y(x_n + jh)$ and $y^{'}(x_n + jh)$ in Taylor series about x_n and collecting like terms in h and y gives;

$$L\Big(y(x);h\Big) = C_0 y(x_n) + C_1 h y^{(1)}(x_n) + C_2 h^2 y^{(2)}(x_n) + \cdots + C_p h^p y^{(p)}(x_n) + C_{p+1} h^{p+1} y^{(p+1)}(x_n)$$

Definition 3.4. The differential operator (10) and then associated LMM, are said to be order P if;

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = C_{10} = 0$$

and the error constant

$$C = \Big(\frac{551}{643778150400}, \frac{1}{1005903360}, \frac{1}{883097600}, \frac{1}{502951680}\Big)$$

This implies that the hybrid block method has order $p = 10$. Hence, the hybrid block method is consistent.

Definition 3.5. [1]: The hybrid block method is said to be zero-stable if the roots R of the characteristic polynomial

$$P(R) = Det\Big[RA^0 - A^{'}\Big] \tag{11}$$

Satisfies $|R| \leq 1$ and every root with $|R_0| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{3}{2}} \\ y_{n-2} \end{bmatrix}$$

$$+h \begin{bmatrix} \frac{89371}{54432} & \frac{103}{1260} & \frac{38241}{544320} & \frac{59681}{8709120} \\ \frac{3308}{8505} & \frac{104}{315} & \frac{788}{8505} & \frac{1153}{1366080} \\ \frac{921}{2240} & \frac{81}{140} & \frac{711}{2240} & \frac{411}{35840} \\ \frac{4096}{8505} & \frac{208}{315} & \frac{4096}{8505} & \frac{1601}{8505} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & \frac{1539551}{8709120} \\ 0 & 0 & 0 & \frac{24463}{136080} \\ 0 & 0 & 0 & \frac{6501}{35840} \\ 0 & 0 & 0 & \frac{1601}{8505} \end{bmatrix} \begin{bmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_{n-2} \end{bmatrix}$$

$$+h^2 \begin{bmatrix} -\frac{31207}{362880} & -\frac{81}{1280} & -\frac{1243}{72576} & -\frac{2237}{2903040} \\ -\frac{58}{567} & -\frac{1}{10} & -\frac{62}{2835} & -\frac{43}{45360} \\ -\frac{279}{4480} & -\frac{81}{1280} & -\frac{183}{4480} & -\frac{9}{7168} \\ -\frac{128}{2835} & \frac{128}{2835} & 0 & -\frac{29}{2835} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 0 & \frac{26051}{2903040} \\ 0 & 0 & 0 & \frac{421}{54360} \\ 0 & 0 & 0 & \frac{339}{35840} \\ 0 & 0 & 0 & \frac{421}{54360} \end{bmatrix} \begin{bmatrix} g_{n-\frac{1}{2}} \\ g_{n-1} \\ g_{n-\frac{3}{2}} \\ g_{n-2} \end{bmatrix}$$

$$A^0 y_{n+1} = A^{(')} y_{n-1} + h[B^{(')} f_{n-1} + B^{'} f_n] + h^2 [C^{(')} g_{n-1} + C^{'} g_n]$$

The first characteristics polynomial of the above matrix is given by

$P(R) = det[RA^0 - A']$

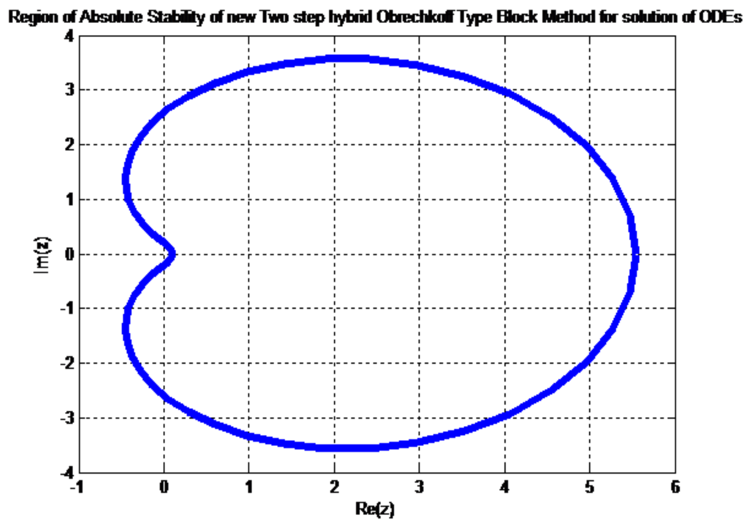
$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P(R) = det\left[R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right] = det \begin{bmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{bmatrix}$$

$P(R) = det[R^3(R-1)]$

Therefore, $R = 0$ and $R = 1$. The hybrid block method is zero-stable.
Therefore, since the hybrid block method is consistent and zero-stable, it is likewise convergent. The region of absolute stability is determine by obtaining the stability polynomial of the form:

$$\frac{3(3z^8 + 75z^7 + 1045^6 + 9950z^5 + 68190z^4 + 336000z^3 + 1142400z^2 + 2419200z + 2419200)}{4(3z^8 - 75z^7 + 1045^6 - 9950z^5 + 68190z^4 - 336000z^3 + 1142400z^2 - 2419200z + 2419200)}$$



Figures 1: Region of Absolute Stability of the new Step HOTBM for Solution of ODEs

The Figure 1 shows the stability nature of the method, the interior part shows the area in which the method is unstable while the external region of the circle depicts the stable region.

4. Implementation of the New Method

We now illustrate the self-starting scheme with numerical examples below. All calculations and programs were carried out with the aid of maple and MATLAB. The algorithms are coded using MAT-LAB environment. The package has 3 subroutines. The first subroutine is for the exact solution to the problem, the second subroutine is for the problem under consideration and the last subroutine are for the codes. The program has been written in such a way that the user can use any of the independent solution in the block or solve the problem using all the solutions in the block simultaneously. The following notations are adopted in the tables below.

2SBM: Two-Step Implicit Obrechkoff Method

2SHBM: New Two-Step Hybrid Block Method

2SEM: Two-Step Implicit Obrechkoff Method

Error: | Computed Solution-Exact Solution |.

Problem 1.1 [7].

$y^{'} = 0.5(1-y), \quad y(0) = 0.5, \quad h = 0.1$, with exact solution $y(x) = 1 - 0.5e^{-0.5x}$

The numerical solution is shown in Tables 1 and 2.

Problem 2.2 [7].

$y^{'} = -y, \quad y(0) = 1, \quad h = 0.1$, with exact solution $y(x) = e^{-x}$

Problem 3.3 [7].
In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In the preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of $40galmin^{-1}$ the well-mixed solution is pumped out at a rate of $40galmin^{-1}$. Using a numerical integrator, how much of the additive is in the tank 0.1, 0.5 and 1 min after the pumping process begins?. Let y be the amount (in pounds) of additive in the tank at time t. we know that $y = 100$ when $t = 0$. Thus, the initial value problem modeling the mixture process is;

$$y^{'} = 80 - \frac{45}{2000-5t}, \quad y(0) = 100, \quad h = 0,1.$$

with theoretical solution:

$$y(t) = 2(2000-5t) - \frac{3900}{(2000)^9}(2000-5t)^9.$$

The numerical solutions are shown in the tables below:

TABLE 1. Comparison of computed results for solving tested problem 1.1

X	Exact Solution	Computed Solution (2SEM)[7]	Computed Solution [9]
0.1	0.52438528774964299546	0.52439024390243902439	0.5243852877552174
0.2	0.54758129098202021343	0.54758601685347485034	0.5475812909859664
0.3	0.56964601178747109641	0.56965499173623217471	0.5696460117956543
0.4	0.59063462346100907069	0.59064317588795690893	0.5906346234953703
0.5	0.61059960846429756591	0.61061180145439803533	0.6105996086572718
0.6	0.62959088965914106700	0.62960249743238951881	0.66295908898470451
0.7	0.64765595514064328286	0.64767066828934612765	0.6476559553183269
0.8	0.66483997698218034967	0.66485398110580363236	0.6648399771546479
0.9	0.68118592418911335348	0.68120256739332540639	0.6811859243738679
1.0	0.69673467014368328825	0.69675050937978129410	0.6967346704442603

TABLE 2. Comparison of computed results for solving tested problem 1.1 continue

X	Computed Solution [7]	Computed Solution (2SHBM)
0.1	0.52438528774960472804	0.52438528774964299546
0.2	0.54758129098194536511	0.54758129098202021342
0.3	0.56964601178736527269	0.56964601178747109638
0.4	0.59063462346087361956	0.59063462346100907066
0.5	0.61059960846413739010	061059960846429756588
0.6	0.62959088965895722513	0.62959088965914106696
0.7	0.64765595514044005788	0.64765595514064328282
0.8	0.66483997698195855368	0.66483997698218034963
0.9	0.68118592418887672320	0.68118592418911335343
1.0	0.69673467014343242661	0.69673467014368328820

TABLE 3. Comparison of error for solving tested problem 1.1

X	Error (2SEM) [7]	Error [9]	Error [7]	Error (2SHBM)
0.1	4.956150E-06	5.574430E-12B	3.826740E-14	0.0
0.2	4.725970E-06	3.946177E-12	7.484830E-14	1.10E-20
0.3	8.979940E-06	8.183232E-12	1.058240E-14	2.10E-20
0.4	8.552430E-06	3.436118E-11	1.354510E-13	3.10E-20
0.5	1-219300E-05	1.92949473E-10	1.601760E-13	3.10E-20
0.6	.160780E-05	1.879040E-10	1.838420E-13	4.10E-20
0.7	1.471310E-05	1.776835E-10	2.032250E-13	4.10E-20
0.8	1.400410E-05	1.724676E-10	2.217960E-13	4.10E-20
0.9	1.664320E-05	1.847545E-10	2.366300E-13	5.10E-20
1.0	1.583920E.05	3.005770E-10	2.508620E-13	5.10E-20

TABLE 4. Comparison of computed results for solving tested problem 2.2

X	Exact Solution	Computed Solution [7]	Computed Solution [8]
0.1	0.90483741803595957317	0.90476190476190476190	0.904837417881202
0.2	0.81873075307798185868	0.81866206899176567085	0.818730752939751
0.3	0.74081822068171786608	0.74069425289731179744	0.740818220548903
0.4	0.67032004603563930076	0.67020758320587849512	0.670320045918305
0.5	0.60653065971263342362	0.60637828956722340034	0.606530659599218
0.6	0.54881163609402643265	0.54867352672129543193	0.548811635994641
0.7	0.49658530379140951473	0.49641890512879110507	0.496585303694640
0.8	0.44932896411722159146	0.44917820458666454624	0.449328964033219
0.9	0.40656965974059911191	0.40639932795936316088	0.406569659658082
1.0	0.36787944117144232162	0.36772515831292540605	0.367879441100594

Table 5. Comparison of computed results for solving tested problem 2.2 continue

X	Computed Solution [7]	Computed Solution (2SHBM)
0.1	0.90483741804503260091	0.90483741803595957316
0.2	0.81873075309534995788	0.81873075307798185867
0.3	0.74081822070486153894	0.74081822068171786607
0.4	0.67032004606407889464	0.67032004603563930074
0.5	0.60653065974444846468	0.60653065971263342360
0.6	0.54881163612895298782	0.54881163609402643263
0.7	0.49658530382799175192	0.49658530379140951470
0.8	0.44932896415534885121	0.44932896411722159143
0.9	0.40656965977917485733	0.40656965974059911191
1.0	0.36787944121046227174	0.36787944117144232160

Table 6. Comparison of error for solving tested problem 2.2

X	Error [7]	Error [8]	Error [7]	Error (2SHBM)
0.1	7.5513E-05	1.5476E-10	9.0730E-12	1.10^{-20}
0.2	6.8684E-05	1-3823E-10	1.1768E-11	1.10^{-20}
0.3	1.2397E-04	1.3282E-10	2.3144E-11	1.10^{-20}
0.4	1.1246E-04	1.1733E-10	2.8440E-11	2.10^{-20}
0.5	1.5237E-04	1.1342E-10	3.1815E-11	2.10^{-20}
0.6	1.3811E-04	9.9385E-11	3.4927E-11	2.10^{-20}
0.7	1.6640E-04	9.6770E-11	3.6582E-11	3.10^{-20}
0.8	1.5076E-04	8.4003E-11	3.8127E-11	3.10^{-20}
0.9	1.7033E-04	8.2517E-11	3.8576E-11	3.10^{-20}
1.0	1.5428E-04	7.0848E-11	3.9020E-11	2.10^{-20}

TABLE 7. Comparison of computed result for solving tested problem 3.3

X	Exact Solution	Computed Solution [7]	Computed Solution (2SHBM)
0.1	107.7662301168309485	107.76623267141251405	107.76623011260318238
0.2	115.5149409193028512	115.51494346840455900	115.51494090305853318
0.3	123.2461630508845220	123.24616814117862409	123.24616302194271446
0.4	130.9599271090910725	130.95993218819786255	130.95992706677669876
0.5	138.6562636455413534	138.65627125250773431	138.65626358918532018
0.6	146.3352031660153395	146.33521075612409816	146.33520309495466018
0.7	153.9967761305114567	153.99678623520317743	153.99677604408937590
0.8	161.6410129533038516	161.64102303550463010	161.64101285086997114
0.9	169.2679440029996050	169.26795658656269977	169.26794388391000992
1.0	176.8775996025958865	176.87761215807155490	176.87759946621327280

TABLE 8. Comparison of error for solving tested problem 3.3

X	Error [7]	Error (2SHBM)
0.1	2.554000E-06	4.22776612E-09
0.2	2.549000E-06	1.624431802E-08
0.3	5.090000E-06	2.894180754E-08
0.4	5.079000E-06	4.231437374E-08
0.5	7.607000E-06	5.635603322E-08
0.6	7.590000E-06	7.106067932E-08
0.7	1.010000E-05	8.642208080E-08
0.8	1.008000-05	1.0243388046E-07
0.9	1.258000E-05	1.1908959508E-07
1.0	1.256000E-05	1.3638261370E-07

Conclusion: We have considered problems ranging from Linear, stiff and application problem namely mixture model. Table 1 shows the comparison of the exact solution, computed solution of the present method with that of other existing methods for problem 1.1. On the other hand, Table 2 shows the comparison of the absolute error of the present method and other similar method in the literature. it could be observed that the absolute error in the present method gives minimal error compared to other methods presented in [7] and [9]. Similarly, Table 3 presents the comparison of the computational results of the present method against that of other existing method for problem 1.2. In the same vein, we compared the errors generated from the present method and other existing method in the literature in Table 4 using h=0.1. It is very clear that the present method performs better in terms of accuracy and convergence that the existing methods [7] and [8]. Finally, to determine the application, suitability and accuracy of the present method, we applied the present method to a mixture model presented in Problem 1.3, Table 5 shows the comparison of the exact solution, computed solution of the present method and that of computed solution of existing method [7]. It could be seen that there is a good agreement between exact solution and the computed solution of the present method. In Table 6, the comparison of the present method with other existing method in the literature namely [7] is demonstrated. There is no doubt that the present method outperforms the existing method in the literature. Hence, we conclude that the method accurate, absolutely stable and computational dependable.

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