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Application of Laplace Beltrami Equation to Solve Vortex Flow in a Closed Cylinder with Dirichlet Boundary Condition

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Abstract

The guiding equation here happens to be the Laplacian for Cylinder, the equation was obtained by using the known definition of differential operator for Cylindrical coordinates (rho, theta, z), it was then decomposed into three Partial Differential Equation (PDE) by variable separable methods and solved independently to obtain the general solution. Finally, contour plots of the solution describing vortex movement in the interior region of a closed Cylinder of arbitrary length l were obtained.

1. Introduction

In physics, a fluid is a liquid, gas, or other material that continually deforms (flows) under an applied shear stress, or external force. Fluid, any liquid or gas or generally any material that cannot sustain a tangential, or shearing force when at rest and that undergoes a continuous change in shape when subjected to such a stress. This continuous and irrecoverable change of position of one part of the material relative to another part when under shear stress constitutes flow, a characteristic property of fluids. In contrast, the shearing forces within an elastic

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solid, held in a twisted or flexed position, are maintained; the solid undergoes no flow and can spring back to its original shape. Compressed fluids can spring back to their original shape, too, but while compression is maintained, the forces within the fluid and between the fluid and the container are not shear forces.
The fluid exerts an outward pressure, called hydrostatic pressure that is everywhere perpendicular to the surfaces of the container. Various simplifications, or models, of fluids have been devised since the last quarter of the 18th century to analyze fluid flow. The simplest model, called a perfect or ideal fluid, is one that is unable to conduct heat or to offer drag on the walls of a tube or internal resistance to one portion flowing over another. Thus, a perfect fluid, even while flowing, cannot sustain a tangential force; that is, it lacks viscosity and is also referred to as an inviscid fluid. Some real fluids of low viscosity and heat conductivity approach this behaviour. Fluids of which the viscosity or internal friction, must be taken into account are called viscous fluids and are further distinguished as Newtonian fluids if the viscosity is constant for different rates of shear and does not change with time. The viscosity of non-Newtonian fluids either varies with the rate of shear or varies with time, even though the rate of shear is constant. Fluids in a class in this last category that become thinner and less viscous as they continue to be stirred are called thixotropic fluids (https://www.britannica.com/science/fluid-physics. Accessed 8, August, 2021.)
Interactions of vortex structures play an important role in the understanding of complex evolutions of fluid flow. Incompressible and inviscid flows with pointwise vorticity distributions in two dimensional space, called point 'vortices', have been used as a theoretical model to describe such vortex interactions. The motion of point vortices has been investigated well in unbounded planes with boundaries as well as on a sphere owing to their physical relevance. On the other hand, it is of a theoretical interest to investigate how geometric nature of curved surfaces and the number of holes give rise to
different vortex interactions that are not observed in vortex dynamics in the plane and on the sphere. In physics and engineering, fluid dynamics is a sub discipline of fluid mechanics that describes flow of fluids (liquids and gases). It has several sub discipline, including aerodynamics (the study of air and other gases in motion) and hydrodynamics (the study of liquids in motion). Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space and modelling fission weapon detonation. Fluid dynamics offer a systematic structure which underlines these practical discipline that embraces empirical and semi-empirical laws derived from flow measurement and used to solve practical problems. The solution to a fluid dynamics problem typically involves the calculation of various

properties of the fluid, such as flow velocity, pressure, density and temperature as function of space and time (Eckert M. 2006).

1.1. **The Navier Stokes Equations.** Navier stokes equation in fluid mechanics is a partial differential equation that describes the flow of incompressible fluids. The equation is a generalization of the equation devised by swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids.

1.1.1. *Incompressible Navier stokes Equations.* The flow of fluid inside a given container is governed by the solutions to the following differential equations:

$$\frac{\partial}{\partial t}\overrightarrow{U} + \left(\overrightarrow{U}.\nabla\right)\overrightarrow{U} + \frac{1}{\rho}\nabla P = \overrightarrow{g} + V\Delta\overrightarrow{U}$$

For $\nabla_{\bullet}\overrightarrow{U} = 0$
Here, t denotes time, U the velocity field, the density field, P the pressure field, $\overrightarrow{g}$ the external forces such as gravity, V the viscocity, ∇ the gradient operator, $\nabla_{\bullet}$ the divergence operator and $\Delta = \nabla_{\bullet}\nabla$ the laplacian. Note that these are 4 differential equations at once. The first line is called the momentum equation while the second one is called the incompressibility condition.

1.2. **The Euler Equation.** The Euler equation is just the Navier-stokes equation without viscosity which turns the second order partial differential equation into a first order partial differential equation.

1.2.1. *Incompressible Euler Equation.* The motion of an incompressible, invicid flow with uniform density inside a container without external forces is determined by the following equations:
$\frac{\Delta}{\Delta t}\overrightarrow{U} = -\nabla\rho$ where $\nabla_{\bullet}\overrightarrow{U} = 0$
Vortices form in stirred fluids and may be observed in smoke rings, whirlpools in the wake of a boat, and the winds surrounding a tropical cyclone, tornado or dust devil. Vortices are a major component of turbulent flow. the distribution of velocity, vorticity (the curl of the flow velocity), as well as the concept of circulation are used to characterize vortices. In most vortices, the fluid flow velocity is greatest next to its axis and decreases in inverse proportion to the distance from the axis. In the absence of external forces,viscous friction within the fluid tends to organize the flow into a collection of irrational vortices, possibly superimposed to larger scale flows, including larger scale vortices. Once formed, vortices can move, stretch, twist, and interact in complex ways. A moving vortex carries some angular and linear momentum, energy, and mass with it. (Shigeo, 2001), Vortex flow is used in fluid dynamics, fluid physics, and engineering fields such as mechanical and chemical engineering, as well as powder technology, etc.

The problem of describing point vortex motion dates back to the 19th century, when Helmholtz pioneered two dimensional point vortex models and leonardo da vinci created fascinating sketches of different types of vortex and eddy flows. (Herik ,2007).

Interactions of vortex structures play an important role in the understanding of complex evolutions of fluid flows. Incompressible and inviscid flows with point wise vorticity distributions in two dimensional space, called point vortices, have been used as a theoretical model to describe such vortex interactions. The motion of point has been investigated well in unbounded planes with boundaries as well as on a sphere owing to their physical relevance. On the other hand, it is of a theoretical interest to investigate how geometric nature of curved surfaces and the number of holes give rise to different vortex interactions that are not observed in vortex dynamics in the plane and on the sphere.

The point vortex equation with which we are concerned here are most elegantly stated by considering the two-dimensional flow plane to be the complex z-plane and letting the vortices be represented by time dependent points $Z_\alpha(t)$ in that plane. Many current experimental results on vortex flows do not allow facile explanation in terms of point vortex dynamics since effects of three-dimensionality and/or viscosity are important. (Hassan., 2006). Vortex structures are defined by their vorticity, the local rotation rate of fluid particles. They can be found via the phenomenon known as boundary layer separation which can occur when a fluid moves over a surface and experiences a rapid acceleration from the fluid velocity to zero due to the no-slip condition. This rapid negative acceleration creates a boundary layer which causes a local rotation of fluid at a wall (i.e vorticity) which is referred to as the wall shear rate. The thickness of this boundary layer is proportional to $sqrtvt$ (where v is the free stream fluid velocity and t is time). If the diameter or thickness of the vessel or fluid is less than the boundary layer thickness then the boundary layer will not separate and vortices will not form. However, when the boundary layer does grow beyond this critical boundary layer thickness then separation will occur which generate vortices. Another form of vortex formation on a boundary is when fluid flows perpendicularly into a wall and creates a splash effect. The velocity streamlines are immediately deflected and decelerated so that the boundary layer seperates and forms a toroidal vortex ring (Kheradvar and Pedrizzetti, 2012).

In a stationary vortex, the typical streamline (a line that is everywhere tangent to the flow velocity vector) is a closed loop surrounding the axis, and each vortex line (a line that is everywhere tangent to the vorticity vector) is roughly parallel to this axis. A surface that is everywhere tangent to both flow velocity and vorticity is called a vortex tube. In general, vortex tubes are nested around the axis of rotation. According to Helmholt'z theorems, a vortex line cannot start or end in the fluid except momentarily, in non-steady flow, while the vortex is forming or dissipating. In general, vortex lines (in particular, the axis line) are either closed loops or end at the boundary of the fluid. A whirlpool is an example of the latter, namely a vortex in body of water whose axis ends at the free surface. As long as the effects of viscosity and diffusion are negligible, the fluid in a moving vortex is carried along with it. In particular, the fluid in the core (and matter trapped by it) tends to remain in the core as the vortex moves about. This is a consequence of Helmholtz's second theorem. Thus vortices (unlike surface waves and pressure waves) can transport mass, energy and momentum over considerable distances compared to their size, with surprisingly little dispersion. Vortices contain substantial energy in the circular motion of the fluid. In an ideal fluid, this energy can never be dissipated and the vortex would persist forever. However, real fluids exhibit viscosity and this dissipates energy very slowly from the core of the vortex. It is only through dissipation of a vortex due to viscosity boundary of the fluid. Flow around a circular cylinder is a classical topic in hydrodynamics that is of fundamental importance to many scientific fields with numerous applications (Zdravkorich, 1997).

A point vortex model for the formation of two re circulating, symmetric eddies in the wake of a circular cylinder was first introduced by Foppl. He obtained stationary solutions for a pair of vortices behind the cylinder in a uniform stream and found that the centers of the middlies observed in the experiments lie on the locus of such equilibria- now called foppl curve.

In this paper, the green's function for laplace's equation in a finite length cylinder is considered. The Green's function satisfies a homogeneous mixed boundary condition (a linear combination of the potential and its normal derivative vanishes) on the cylinder surface. The mixed boundary condition for laplace's equation also occurs in many physical problems. It occurs in the description of steady heat flow in a body with heat radiation and convection to the surrounding medium (Luikov, 1968). Takashi (2016) carried out analysis and derive the evolution equation for N-point vortices from Green's function associated with the Laplace Beltrami operator there, and he then formulated it as a Hamiltonian dynamical system with the help of the symplectic geometry and the uniformization theorem. In this paper we are concerned with the vortice problem and and structure of point vortices by using the method of Green's function for a cylinder. Therefore we employ a systematic method using the Method of Separation of Variables to scrutinize the Green function with Dirichlet boundary condition for the interior region of a closed cylinder. Hence we use Green's function to obtain a stream function using cylindrical coordinates which describe point vortice on a cylinder and for the interior region of a cylinder using laplace betrami equation. We also use the variation of parameters to determine the number of point vortices in a cylinder. To present laplacian in cylindrical coordinate (ρ, θ, z) by separating the solution in to three different partial differential equations (PDE). To characterize the Green's function obtained as the stream function for the region describing vortex flow.

2. Result Methodology

In this paper we present the basic material and method needed for the solution to be feasible. In other to obtain stream function for a core rotating vortices in the interior region of a cylindrical coordinates followed by the derivation of Green's function for a cylinder.

2.1. **Laplace equation in cylindrical coordinates.** Suppose that we wish to solve laplace equation $\nabla^2 \phi = 0$ within a cylindrical volume of radius α and height l. We adopt the standard cylindrical coordinates (ρ, θ, z) as follows:

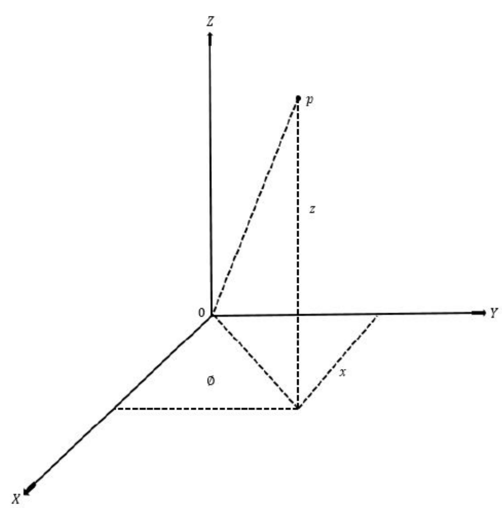

Figure 1: Circular cylindrical coordinates

Suppose that the curved portions of the bounding surface corresponds to $r = \alpha$, while the two at portions corresponds to $z = 0$ and $z = l$ respectively. Suppose finally that the boundary conditions (Dirichlet Boundary Condition) that are imposed at the bounding surface are

$$\phi(\rho, \theta, 0) = 0, \quad \phi\ (\alpha,\ \theta,\ z) = 0, \quad \phi(\rho,\ \theta,\ l) = \phi(\rho,\ \theta)$$

where $\phi(\rho,\ \theta)$ is a given function. In other words, the potential is zero on the curved and bottom surfaces of the cylinder and specified at the top surface.

$$x = \rho\cos\phi\ ,\ \ y = \rho\sin\phi \quad , \quad z = z$$

Since the inverse relations are

$$\rho = \sqrt{x^2+y^2}, tan\phi = \frac{y}{x} \quad , \quad Z = z$$

then,

$$tan\phi = \frac{y}{x}$$

Recall that $tan\phi = \frac{sin\phi}{cos\phi}$, therefore $\frac{sin\phi}{cos\phi} = \frac{y}{x}$ which implies that $\rho = cos\phi + sin\phi$. Similarly, $\phi = cos\phi - sin\phi$, some basis vectors depends on the coordinate according to the rule

$$\frac{\partial \rho}{\partial \phi} = \phi, \ and \ \frac{\partial \phi}{\partial \phi} = -\rho$$

An infinitesimal length dl is

$$dl = \sqrt{\left(d\rho^2\right) + \left(\rho d\phi^2\right) + \left(dz^2\right)}$$

An infinitesimal volume element is;

$$d\rho = \rho d\rho d\phi dz$$

The gradient operator in cylindrical coordinates writes

$$\nabla = \rho\frac{\partial}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial}{\partial \phi} + z\frac{\partial}{\partial z}$$

$$\nabla \mathrm{G} = \rho\frac{\partial G}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial G}{\partial \phi} + z\frac{\partial G}{\partial z}$$

$$\nabla^2 G = \nabla.\left(\nabla G\right) = \left(\rho\frac{\partial}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial}{\partial \phi} + z\frac{\partial}{\partial z}\right)\left(\rho\frac{\partial G}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial G}{\partial \phi} + z\frac{\partial G}{\partial z}\right)$$

$$= \rho\frac{\partial}{\partial \rho}\left(\rho\frac{\partial G}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial G}{\partial \phi} + z\frac{\partial G}{\partial z}\right) + \frac{\phi}{\rho}\frac{\partial}{\partial \phi}\left(\rho\frac{\partial G}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial G}{\partial \phi} + z\frac{\partial G}{\partial z}\right) + z\frac{\partial}{\partial z}\left(\rho\frac{\partial G}{\partial \rho} + \frac{\phi}{\rho}\frac{\partial G}{\partial \phi} + z\right.$$

(1)

$$= \rho.\left(\rho\frac{\partial^2 G}{\partial \rho^2} - \frac{\phi}{\rho^2}\frac{\partial G}{\partial \phi} + \frac{\phi}{\rho}\frac{\partial^2 G}{\partial \phi \partial \rho} + z\frac{\partial^2 G}{\partial z \partial \rho}\right) + \frac{\phi}{\rho}.\left(\phi\frac{\partial G}{\partial \rho} + \rho\frac{\partial^2 G}{\partial \rho \partial \phi} - \frac{\rho}{\rho}\frac{\partial G}{\partial \phi} + \frac{\phi}{\rho}\frac{\partial^2 G}{\partial \phi^2} + z\frac{\partial^2 G}{\partial z \partial \phi}\right)$$

$$= \frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2}$$

On rearranging, we have

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2} \tag{2}$$

Therefore, we can write the laplacian operator as:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

The laplacian operator in (10) above is known to be variable separable and considering ϵ as the source and ϵ^1 as the observation point respectively and knowing that a Green's function for the Drichlet boundary condition (DBC) satisfies the following equation:

(4) $$\nabla^2{?}G\left(\epsilon,\epsilon^1\right) = -4\pi\delta(\epsilon\ -\epsilon^1\ )$$

where $G\left(\epsilon,\epsilon^1\right) = 0$ we can then separate the corresponding Green's function as follows:

(5) $$G\left(\epsilon,\epsilon^1\right) = G\left(\rho,\rho^1\ \phi,\phi^1,z,z^1\ \right) = P\left(\rho,\rho^1\right)Q\left(\ \phi,\phi^1\right)Z(,z,z^1)$$

By substituting (2) and (5) into (4), we obtain

(6) $$\frac{1}{\rho}\frac{P^1}{P} + \frac{P^1}{P} + \frac{1}{\rho^2}\frac{Q^1}{Q} + \frac{Z^1}{Z} = 0\ ,\rho \neq 0$$

Looking closely at (6), it is mathematically valid to assign constant number, say λ^2 to the ordinary differential equation (ODE), $\frac{Z^1}{Z}$, where $\lambda\ \in N$ and $-\mu^2$ to the ordinary differential equation $\frac{Q}{Q^1}$ where $\mu \in \mathbb{N}$, we obtain the following system of ordinary differential equation in (7), (8) and (**??**)

(7)
$Z^1 - \lambda^2 Z = 0$

(8) $$Q^1 + \ \mu^1 Q = 0$$

$\rho^2 P^1 + \ \rho P + \left(\lambda^2\rho^2 -\ \mu^2\right)P = 0$ (**??**)

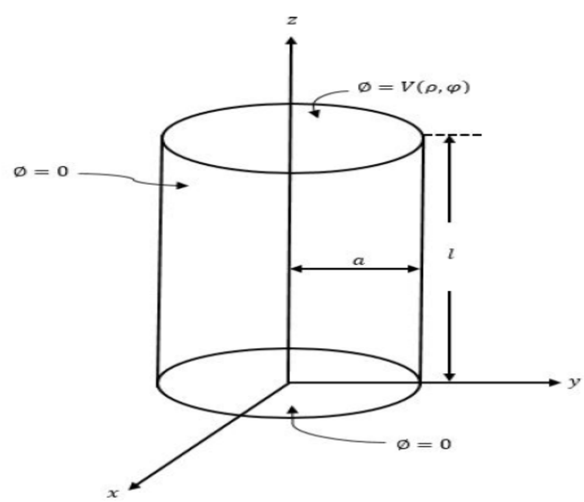

Figure 2: Geometry of a laplace problem for the interior region of a cylinder

3. Main Results

The implementation of three equations in section 2 is carried out independently. Using equation (7) as case for implementation, thus
We can rewrite the equation (7) as:

$$Z_{zz}\left(z, z^1\right) - \lambda^2 Z\left(z, z^1\right) = 0 \tag{9}$$

The general solution of the last equation is:

$$Z_\lambda\left(z, z^1\right) = \begin{cases} \alpha_1\left(z^1\right)\sinh\left(\lambda z\right) + \beta_1\left(z^1\right) cosh\left(\lambda z\right) & 0 < z < z^1 < 1 \\ \alpha_2\left(z^1\right)\sinh\left(\lambda z\right) + \beta_2\left(z^1\right) cosh\left(\lambda z\right) & 0 < z < z^1 < 1 \end{cases} \tag{10}$$

with four unknown coefficients to be determined. The coefficient $\beta_1\left(z^1\right) cosh\left(\lambda z\right)$ will be elimiated applying the boundary condition $Z_\lambda\left(z, z^1\right)|_{z=0} = 0$ we see that $\beta\left(z^1\right) = 0$. On applying the second boundary condition $Z_\lambda\left(z, z^1\right)|_{z=l} = 0$. We have

$$\alpha_2\left(z^1\right) = -\coth(\lambda l)\beta_2\left(z^1\right) \tag{11}$$

Also by the continuity theorem which states that Green's function is continuous around the source point, continuity is applicable to the components of the Green's function hence

$$Z_\lambda\left(z, z^1\right)|_{z=z^{1+}} = Z_\lambda\left(z, z^1\right)|_{z=z^{1-}} \tag{12}$$

With this fact we have
(13)

$$\alpha_1\left(z^1\right) = \beta_2\left(z^1\right)\left(\frac{\cosh\left(\lambda z^1\right)}{\sinh\left(\lambda z^1\right)} - \frac{\cosh\left(\lambda l\right)}{\sinh\left(\lambda l\right)}\right) = \beta_2\left(z^1\right)\left(\frac{\sinh\left(\lambda\left(l - z^1\right)\right)}{\sinh\left(l z^1\right)\sinh\left(\lambda l\right)}\right)$$

In general Green's function between the source and observation points is known to be symmetric. Thus, same holds for its components, hence we can write

$$G_?\left(\epsilon, \epsilon^1\right) = G_?\left(\epsilon, \epsilon^1\right) \tag{14}$$

As a result of this, we apply this rule to the system of solutions, by equating one solution in the system with the other one with its primed and unprimed parameters exchanged, as follows:

$$\beta_2\left(z^1\right)\left(\frac{\sinh\left(\lambda\left(l - z^1\right)\right)}{\sinh(\lambda z^1)\sinh\left(\lambda l\right)}\right) = \left(\frac{\sinh\left(\lambda\left(l - z\right)\right)}{\sinh\left(\lambda z\right)\sinh\left(\lambda l\right)}\right)\beta_2\left(z\right) \tag{15}$$

Above equation (15) implies

$$\beta_2\left(z\right)\sinh(\lambda z^1) = \beta_2\left(z^1\right)\sinh\left(\lambda z\right) \tag{16}$$

and thus

$$\beta_2\left(\epsilon\right)\sinh(\lambda\epsilon^1) = \beta_2\left(\epsilon^1\right)\sinh\left(\lambda\epsilon\right) \tag{17}$$

Substituting (11), (13) and (17) into (10) we have the solution for the ordinary differential equation in (6) as:

$$Z_\lambda\left(z, z^1\right) = \begin{cases} \dfrac{\sinh(\lambda z)\sinh(\lambda(l-z)) \quad 0<z<z^1<1}{\sinh(\lambda l)} \\ \dfrac{\sinh(\lambda(l-z))\sinh(\lambda z^1) \quad 0<z<z^1<1}{\sinh(\lambda l)} \end{cases} \tag{18}$$

This gives us a component of the Green's function which equally describes the component of the vortex flows on the interior region of the closed cylinder of length. To get the contour plots of the flow at different points on l, we have to look at the parameters, l and z in a form of angles that the point of the flow makes with positive $z-$axis on l with a maximum 2π since l is just a straight line.

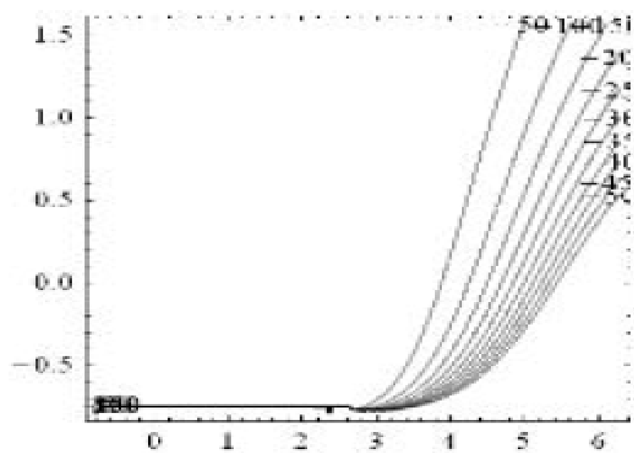

Figure 3a: Angle of rotation at length $0 < z < z^1 < l$ of the cylinder

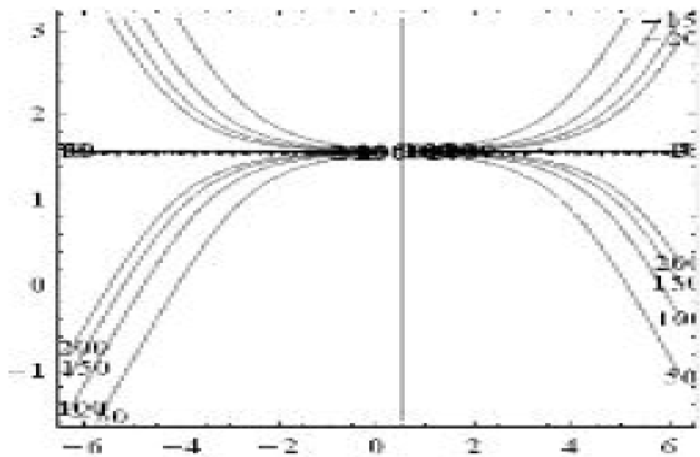

Figure 3b: Angle of rotation at length $0 < z < z^1 < l$ of the cylinder

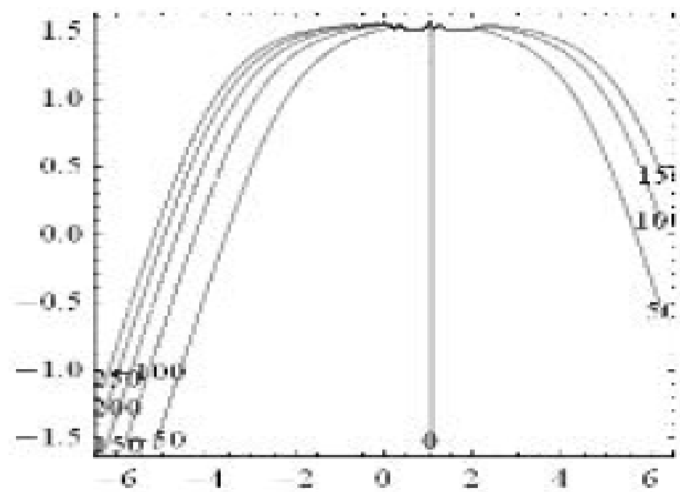

Figure 3c: Angle of rotation at length $0 < z < z^1 < l$ of the cylinder

Figures 1, 2, 3a, 3b and 3c show the contour plots of the component of the stream function, showing component of the vortex flow in the interior region of a closed Cylinder at various points on l where $0 < z < z^1 < l$.

Conclusions: The paper show that vortex flow is an helical motion or circular spiral (such as gas) for fluid. We derived the laplacian equation for a cylinder and presented its solution in the form of a Green's function for such flow (vortex) on the interior region of the closed cylinder and then characterised the solution as the stream function for such flow (vortex) on the interior region of the closed cylinder. The contour plots shown in figures (9), (10), (11) describe the nature of the vortex flow when the angle of rotation is taking at various points on the length l of the cylinder. This clearly underline the result obtained.

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